## Section 3: Measures and Measure Spaces

Intuitively, in $\mathbb{R}^2$, we expect the area of a disjoint union of sets $A \dot\cup B$ to be the sum of area of $A$ and area of $B$ (i.e. total area = sum of smaller areas).

What if $(A_n)_{n=1}^{\infty}$ is a sequence of disjoint subsets of $\mathbb{R}^2$? We would hope that the area of $\bigcup_{n=1}^{\infty} A_n$ would equal $\sum_{n=1}^{\infty}$ (area of $A_n$).

In $\mathbb{R}$ the equivalent notion is that of length. We want to define a function $\lambda\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$ to measure the length of as many sets as possible, such that

$$\lambda((a,b]) = b-a \quad \text{and} \quad \lambda(A \dot\cup B) = \lambda(A)+\lambda(B), \quad \lambda\left(\dot\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n) \quad \text{etc.}$$

Unfortunately this cannot be done for all subsets of $\mathbb{R}$.

Area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$ have the same problems. But we will succeed in defining our measurements of size on at least all the Borel sets.

**Definition 3.1**

Let $X$ be a set, let $\mathcal{C} \subseteq \mathcal{P}(X)$ s.t. $\varnothing \in \mathcal{C}$, and let $\mu\colon \mathcal{C} \to [0,\infty]$.

Then $\mu$ is a *measure* on $\mathcal{C}$ if

(i) $\mu(\varnothing) = 0$,

(ii) whenever $A_1, A_2, \ldots,$ is a sequence of pairwise disjoint sets in $\mathcal{C}$ s.t. $\dot\bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{C}$, then $\mu\left(\dot\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

**Examples**

(i) $X = \mathbb{R}, \qquad \mathcal{C} = \mathcal{P}(\mathbb{R})$,

define

$$\mu(E) = \begin{cases} \infty & \text{if } E \text{ has infinitely many elements} \\ n & \text{if } E \text{ has exactly } n \text{ elements} \end{cases}$$

**Easy exercise**: check $\mu$ is a measure. This measure $\mu$ is called *counting measure* on $\mathbb{R}$.

[Counting measure is usually used on $\mathbb{N}$ rather than on an uncountable set.]

(ii) 'point mass' measures. Let $X$ be a set, $\mathcal{C} = \mathcal{P}(X)$. Let $x$ be any fixed point in $X$. Define

$$\mu(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Certainly $\mu(\emptyset) = 0$.

If $A_1, A_2, \ldots,$ are disjoint subsets of $X$, then either $x \in \dot{\bigcup}_{n=1}^{\infty} A_n$, in which case $x$ is in exactly one set $A_n$ or $x \notin \bigcup_{n=1}^{\infty} A_n$, in which case $x$ is in none of the $A_n$. In both cases $\mu\left(\dot{\bigcup}_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$. This measure $\mu$ is called the *point mass at x*, and is often denoted by $\delta_x$.

If $a$ and $b \geqslant 0$, $\mu, \nu$ are measures on $\mathscr{C}$, then so is $a\mu + b\nu$ defined by

$$(a\mu + b\nu)(E) = a\mu(E) + b\nu(E).$$

In the examples above (i) and (ii) $\mathscr{C}$ was a $\sigma$-field.

**Definition 3.2**

A *measurable space* is a pair $(X, \mathscr{F})$ where $X$ is a set and $\mathscr{F}$ is a $\sigma$-field of subsets of $X$.

A *measure space* is a triple $(X, \mathscr{F}, \mu)$ where $\mathscr{F}$ is a $\sigma$-field on $X$, and

$$\mu \colon \mathscr{F} \to [0, \infty] \quad \text{is a measure.}$$

By abuse of terminology, $X$ is a measurable space and $\mu$ is a ‘measure on $X$’, provided we know which $\sigma$-field we are working with.

Our aim: with $\mathscr{B}$ = Borel subsets of $\mathbb{R}$, we wish to find a measure $\lambda \colon \mathscr{B} \to [0, \infty]$ s.t.

$$\lambda((a, b]) = b - a \quad \forall\ a \leqslant b \text{ in } \mathbb{R}.$$

Is this possible?

The first problem. Suppose $(a, b] = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n]$. We would need $\lambda((a, b]) = \sum_{n=1}^{\infty} \lambda((a_n, b_n])$, i.e. we need $b - a = \sum_{n=1}^{\infty} (b_n - a_n)$.

Is this last equality true? Yes! (See later.)

**General Results about Measures on Rings**

**Proposition 3.3**

Let $X$ be a set, $R$ be a ring of subsets of $X$, and let $\mu \colon R \to [0, \infty]$ be a measure.

(i) If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in $R$ then

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

(ii) If $A, B \in R$ then

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B).$$

**Proof**

(i) To see this, set $A_{n+1} = A_{n+2} = \ldots = \emptyset$. Then

$$\dot{\bigcup_{k=1}^{\infty}} A_k = \dot{\bigcup_{k=1}^{n}} A_k \in R.$$

Thus

$$\mu\left(\dot{\bigcup_{k=1}^{n}} A_k\right) = \mu\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

$$= \sum_{k=1}^{n} \mu(A_k),$$

since $\mu(\emptyset) = 0$.

(ii) $$\mu(A) = \mu(A \cap B) + \mu(A \setminus B)$$

because $A = (A \cap B) \,\dot{\cup}\, (A \setminus B)$.

Is it true that $\mu(A \cap B) = \mu(A) - \mu(A \setminus B)$? Not necessarily! (May have $\infty - \infty$.) [Remember $\infty - \infty$ is not defined.]

e.g. work with counting measure on $\mathbb{N}$.

Set

$$A = \{2, 4, 6, \ldots\}$$

$$B = \{\text{primes}\}$$

$A \cap B = \{2\}$, $\mu(A \cap B) = 1$, $\mu(A) = \mu(B) = \mu(A \setminus B) = \infty$ so $\mu(A) - \mu(A \setminus B)$ is not defined.

**Proposition 3.4**

Let $\mu$ be a measure on a ring $R$ of subsets of a set $X$.

(i) If $A, B \in R$ with $A \subseteq B$, then

$$\mu(A) \leqslant \mu(B). \quad \text{(Monotonicity)}$$

(ii) If $A \in R$, $B_1, B_2, \ldots \in R$ and $A \subseteq \bigcup_{n=1}^{\infty} B_n$, then

$$\mu(A) \leqslant \sum_{n=1}^{\infty} \mu(B_n). \quad \text{(Countable subadditivity)}$$

**Proof**

(i) $\mu(B) = \mu(A) + \mu(B \setminus A) \geqslant \mu(A)$.

(ii) (N.B. The $B_n$ are NOT assumed disjoint, and we do not assume $\bigcup_{n=1}^{\infty} B_n \in R$.)

Set $C_n = B_n \cap A$. Then

$$A = A \cap \left( \bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} C_n .$$

Set $D_1 = C_1$ and $D_n = C_n \setminus \bigcup_{k=1}^{n-1} C_k$ $(n > 1)$. We then have

$D_n$ are in $R$,
$D_n \subseteq C_n \subseteq B_n \quad \forall n$,
$D_n$ are pairwise disjoint.

Also, for each $n$,

$$\bigcup_{k=1}^{n} D_k = \bigcup_{k=1}^{n} C_k .$$

We then have

$$A = \dot{\bigcup_{n=1}^{\infty}} D_n ,$$

and so

$$\mu(A) = \sum_{n=1}^{\infty} \mu(D_n) \leqslant \sum_{n=1}^{\infty} \mu(B_n). \qquad \square$$

The property that

$$\mu\left( \dot{\bigcup_{n=1}^{\infty}} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$$

is '$\mu$ is *countably additive*'.

('$\mu$ is *finitely additive*' means $\mu\left( \dot{\bigcup_{n=1}^{N}} A_n \right) = \sum_{n=1}^{N} \mu(A_n)$.)

The property that $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$ is called *monotonicity* ($\mu$ is monotone).

$\mu\left( \bigcup_{n=1}^{\infty} A_n \right) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$ means '$\mu$ is *countably subadditive*'.

If $\mathscr{F}$ is $\sigma$-field on $X$, then $(X, \mathscr{F})$ is a measurable space. If $\mu: \mathscr{F} \to [0, \infty]$ is a measure, then $(X, \mathscr{F}, \mu)$ is a measure space.

**Definition 3.5**.

If $\mu(X) < \infty$ then $\mu$ is a *finite measure*.

If $\mu(X) = 1$ then $\mu$ is a *probability measure* (informally, for $A \in \mathscr{F}$, $\mu(A)$ represents the probability that a random point chosen from $X$ will be in $A$).

We say that a measure is **$\sigma$-finite** if there are countably many sets $E_n \in \mathscr{F}$ with $\mu(E_n) < \infty$ all $n$, and s.t.

$$X = \bigcup_{n=1}^{\infty} E_n.$$

**Examples**.

The point mass measures are all probability measures (and hence finite measures).

Counting measure $\mu$ on a set $X$

$$\mu(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

is a finite measure if and only if $X$ is finite.

If $\mu$ is a counting measure on $\mathbb{N}$, then $\mu(\mathbb{N}) = \infty$, but

$$\mathbb{N} = \bigcup_{n=1}^{\infty} \{1, 2, 3, \ldots, n\}$$

so that $\mu$ is $\sigma$-finite.

But counting measure on $\mathbb{R}$ (or on any uncountable set) is not $\sigma$-finite.

## More Standard Properties of Measures

**Proposition 3.6**

Let $R$ be a ring of subsets of a set $X$. Suppose $\mu: R \to [0, \infty]$ is a measure and let $A_1, A_2, A_3, \ldots \in R$.

(i) If $\bigcup_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} A_k\right).$$

(ii) If $\mu(A_1) < \infty$ and $\bigcap_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{k=1}^{n} A_k\right).$$

**Proof**

(i) If $\bigcup_{n=1}^{\infty} A_n \in R$, set $A = \bigcup_{n=1}^{\infty} A_n$, set $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for $n > 1$. Then each $B_n \in R$, the sets $B_n$ are pairwise disjoint,

$$\bigcup_{k=1}^{n} A_k = \dot{\bigcup_{k=1}^{n}} B_k \quad \forall n,$$

and $A = \dot{\bigcup_{k=1}^{\infty}} B_k$. Thus

$$\begin{aligned}\mu(A) &= \sum_{k=1}^{\infty} \mu(B_k)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \mu(B_k)\right)\\ &= \lim_{n\to\infty}\left(\mu\left(\dot{\bigcup_{k=1}^{n}} B_k\right)\right)\\ &= \lim_{n\to\infty}\left(\mu\left(\bigcup_{k=1}^{n} A_k\right)\right).\end{aligned}$$

(ii) Now suppose that $\mu(A_1) < \infty$.

If $\bigcap_{n=1}^{\infty} A_n$ is in $R$, then set

$$C_n = A_1 \setminus A_n \quad \forall n.$$

Then $C_n \in R$ and

$$\begin{aligned}\bigcap_{n=1}^{\infty} A_n &= A_1 \setminus \bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\\ &= A_1 \setminus \bigcup_{n=1}^{\infty} C_n.\end{aligned}$$

$\bigcup_{n=1}^{\infty} C_n \subseteq A_1$ by definition of $C_n$ and so $\bigcup_{n=1}^{\infty} C_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n \in R$. Thus

$$\mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} C_k\right)$$

(by the first part).

Now note

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcup_{n=1}^{\infty} C_n\right) \qquad \text{[this holds because } \mu(A_1) < \infty \text{ and } \bigcup_{n=1}^{\infty} C_n \subseteq A_1.\text{]}$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} C_k\right)$$

$$= \lim_{n\to\infty}\left(\mu(A_1) - \mu\left(\bigcup_{k=1}^{n} C_k\right)\right)$$

$$= \lim_{n\to\infty}\left(\mu\left(A_1 \setminus \bigcup_{k=1}^{n} C_k\right)\right)$$

$$= \lim_{n\to\infty}\left(\mu\left(\bigcap_{k=1}^{n} A_k\right)\right) \quad \text{as required.} \qquad \square$$

**Properties which hold almost everywhere**

**Definition 3.7**

Let $(X, \mathcal{F}, \mu)$ be a measure space. To say that a property holds almost everywhere (with respect to $\mu$) (a.e. $(\mu)$) means that there is a set $E \in \mathcal{F}$ with $\mu(E) = 0$ such that the property holds $\forall\ x \in X \setminus E$.

**For example:**

Using Lebesgue measure (see Chapter 5 for the construction) on $\mathbb{R}$ we can say

$\chi_{\mathbb{Q}}(x) = 0$ almost everywhere $(\lambda)$.

**OR** alternatively

$\chi_{\mathbb{Q}}(x) = 0$ for almost all $x$ $(\lambda)$.

[“$(\lambda)$” means “with respect to $\lambda$”.]

This is because $\lambda(\mathbb{Q}) = 0$ (see question sheet 5).

**Definition 3.8**

Given two functions $f, g\colon X \to Y$ where $Y$ is some set, we say $f$ and $g$ are *equivalent* if

$$f(x) = g(x) \quad \text{a.e. } (\mu)$$

(this depends on the measure $\mu$).

Check: this really is an equivalence relation (make sure your sets are really in $\mathcal{F}$).

Note that if you use *counting measure*, a.e. means *everywhere*! (Because $\mu(E) = 0 \Rightarrow E = \emptyset$ when $\mu$ is counting measure.)

[Warning! When working with counting measure, some authors say instead that something holds almost everywhere if it is true for all but finitely many points. This does NOT agree with our usage.]