Section 4: The Integral

The abstract theory of integration with respect to a measure goes through just as easily in general as it does in special cases. You should think of the following examples:

(a) Lebesgue measure on $\mathbb{R}$, or on an interval $[a, b]$

(b) counting measure on $\mathbb{N}$.

The Riemann Integral Revisited

With Riemann integration we attempt to approximate our function from below and from above by step functions.

A step function is a finite linear combination of characteristic functions of intervals $\sum_{k=1}^{n} \alpha_k \chi_{I_k}$ where $I_1, I_2, \ldots, I_n$ are disjoint intervals, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real numbers. These functions are Riemann integrable, with integral

$$\sum_{k=1}^{n} \alpha_k \times \text{length of } I_k = \sum_{k=1}^{n} \alpha_k \lambda(I_k).$$

The beginning of the theory of Lebesgue is to generalise by replacing I_k by A_k, where $A_1, \ldots, A_n$ are disjoint Borel sets (or, more generally, **Lebesgue measurable sets**: see Section 5).

Then we will define

$$\int \left(\sum \alpha_i \chi_{A_i}\right) \mathrm{d}\lambda = \sum \alpha_i \lambda(A_i).$$

Note that this will already be enough to integrate $\chi_{\mathbb{Q}}$, since $\chi_{\mathbb{Q}} = 1 \times \chi_{\mathbb{Q}}$, so the above gives

$$\int \chi_{\mathbb{Q}} \, \mathrm{d}\lambda = 1 \times \lambda(\mathbb{Q}) = 0.$$

Simple Functions

Definition 4.1. Let X be a non-empty set. Then a *simple function* from X is a function $s \colon X \to \mathbb{R}$ such that s takes only finitely many different values.

Note that simple functions are real-valued. Writing $\alpha_1, \alpha_2, \ldots, \alpha_n$ for the distinct values taken by s, we can set

$$A_k = \{x \in X \colon s(x) = \alpha_k\}.$$

Then

$$X = \dot{\bigcup_{k=1}^{n}} A_k$$

and

$$s(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x) \quad \text{all} \quad x \in X,$$

i.e. $s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$.

The following two results are obvious.

Proposition 4.2. If s, t are simple functions from a set X, and a, b are real numbers, then $s+t$, st and $as+bt$ are all simple functions from X.

Corollary 4.3. Let X be a set. For any real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and any subsets $A_1, A_2, \ldots, A_n$ of X,

$$\sum_{k=1}^{n} \alpha_k \chi_{A_k}(x)$$

is a simple function on X.

Continuous Functions and Measurable Functions

Let X, Y be metric spaces, and let $f\colon X \to Y$ be a function. Then f is continuous if

$$\forall\, x \in X \quad \forall\, \varepsilon > 0 \quad \exists \quad \delta > 0 \quad \text{s.t.} \quad \text{for } z \in X$$

$$d_X(z, x) < \delta \Rightarrow d_Y(f(z), f(x)) < \varepsilon.$$

Equivalently: $f\colon X \to Y$ is continuous if, whenever $x_n \to x$ is a convergent sequence in X then

$$f(x_n) \to f(x) \quad \text{in} \quad Y.$$

Recall: for $E \subseteq X$,

$$f(E) = \{f(x)\colon x \in E\}$$

$$= \{y \in Y\colon \exists\, x \in E \quad \text{with} \quad f(x) = y\}.$$

For $F \subseteq Y$, $f^{-1}(F) = \{x \in X\colon f(x) \in F\}$.

Note: $f(E_1 \cup E_2) = f(E_1) \cup f(E_2)$ but $f(E_1 \cap E_2)$ need not equal $f(E_1) \cap f(E_2)$. But f^{-1} behaves better.

$$f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2)$$

$$f^{-1}(F_1 \cap F_2) = f^{-1}(F_1) \cap f^{-1}(F_2)$$

$$f^{-1}(Y \setminus F) = X \setminus f^{-1}(F).$$

Similar results hold for infinite intersections and unions

The following result is standard except for condition (iv), whose equivalence to the other conditions is an optional exercise.

Proposition 4.4 Let X, Y be metric spaces, and let $f\colon X \to Y$. Then the following four conditions are equivalent:

(i) f is continuous,

(ii) for every open set $U \subseteq Y$, $f^{-1}(U)$ is open in X,

(iii) for every closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in X,

(iv) $\forall\, A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

We now begin to introduce the class of functions which we intend to integrate.

Definition 4.5 Let $(X,\mathcal{F}_1)$, $(Y,\mathcal{F}_2)$ be measurable spaces, and let $f\colon X \to Y$ be a function. Then f is $\mathcal{F}_1$-$\mathcal{F}_2$ *measurable* (or simply *measurable* if the σ-fields involved are unambiguous) if, for all $E \in \mathcal{F}_2$, $f^{-1}(E) \in \mathcal{F}_1$.

Proposition 4.6 Let $(X,\mathcal{F})$ be a measurable space, and let Y be a metric space. Let $\mathcal{B}_Y$ be the set of Borel subsets of Y. Let $f\colon X \to Y$ be a function. Then f is $\mathcal{F}$-$\mathcal{B}_Y$ measurable if and only if

(*) $f^{-1}(U) \in \mathcal{F}$ for all open subsets U of Y.

Proof. The "only if" part is trivial, so we prove the "if" part. Suppose that condition (*) above holds. From Exercise Sheet 3, $\{F \subseteq Y\colon f^{-1}(F) \in \mathcal{F}\}$ is in fact a σ-field. By (*) this σ-field includes all the open sets and hence all the Borel sets. The result follows.

For similar reasons,

f is measurable $\Leftrightarrow \forall$ closed sets $F \subseteq Y$, $f^{-1}(F) \in \mathcal{F}$.

Given a metric space Y we will usually use the Borel sets on Y to make Y into a measurable space. However, on $\mathbb{R}$ we will sometimes use the Lebesgue sets.

Corollary 4.7 Using the Borel sets on $\mathbb{R}$, every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is measurable.

Note that we should really consider separately the σ-field used on $\mathbb{R}$ as domain and on $\mathbb{R}$ as range. The result remains true if we change to the Lebesgue sets on $\mathbb{R}$ as domain, and keep the Borel sets on $\mathbb{R}$ as range.

Proposition 4.8 Let $(X,\mathcal{F})$ be a measurable space and let f be a function either from X to $\mathbb{R}$ or from X to $\overline{\mathbb{R}}$. Then the following five conditions are equivalent:

(i) f is measurable;

(ii) $\forall\ a \in \mathbb{R}$,

$$\{x \in X:\ f(x) \leqslant a\} \in \mathcal{F};$$

(iii) $\forall\ a \in \mathbb{R}$,

$$\{x \in X:\ f(x) > a\} \in \mathcal{F};$$

(iv) $\forall\ a \in \mathbb{R}$,

$$\{x \in X:\ f(x) \geqslant a\} \in \mathcal{F};$$

(v) $\forall\ a \in \mathbb{R}$,

$$\{x \in X:\ f(x) < a\} \in \mathcal{F}.$$

Remark. Here we use the Borel sets on $\mathbb{R}$ or on $\overline{\mathbb{R}}$ as appropriate.

Proof. We prove the equivalence of (i) and (ii). The rest is similar. Let us consider condition (ii). For $f\colon X \to \overline{\mathbb{R}}$ this means

$$f^{-1}([-\infty, a]) \in \mathcal{F} \quad \forall\ a \in \mathbb{R};$$

For $f\colon X \to \mathbb{R}$ it means

$$f^{-1}((-\infty, a]) \in \mathcal{F} \quad \forall\ a \in \mathbb{R}.$$

But the Borel sets on $\overline{\mathbb{R}}$ are generated by

$$\{[-\infty, a]:\ a \in \mathbb{R}\}$$

and the Borel sets on $\mathbb{R}$ are generated by

$$\{(-\infty, a]:\ a \in \mathbb{R}\}.$$

Thus, by the same reasoning as in Proposition 4.6, (i) and (ii) are equivalent.

Example Suppose

$$f\colon \mathbb{N} \to [0,\infty].$$

Unless otherwise specified we will use counting measure on $\mathbb{N}$, using the σ-field $\mathcal{P}(\mathbb{N})$.

In this case every such function is measurable. Writing a_n for $f(n)$, we will see later that

$$\int_{\mathbb{N}} f\,\mathrm{d}\mu = \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} a_n\ ,$$

where μ is a counting measure.

To make it very clear when we are using the Borel sets on the domain of our functions, we sometimes use the following definition.

Definition 4.9. Let X, Y be metric spaces. Use the Borel sets on X and on Y to make them measurable spaces. Then a measurable function from X to Y is said to be *Borel measurable*.

With this terminology, corollary 4.7 can be rephrased as the following proposition.

Proposition 4.10. Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is Borel measurable.

Let

$$f\colon X \to \overline{\mathbb{R}}$$

then we can define $(-f)$ by

$$(-f)(x) = -f(x).$$

Proposition 4.11. If $(X, \mathcal{F})$ is a measurable space and $f\colon X \to \overline{\mathbb{R}}$ is measurable then so is $-f$.

Proof. For all $a \in \mathbb{R}$

$$f^{-1}([-\infty, a]) \in \mathcal{F}$$

and so

$$f^{-1}((a, \infty]) \in \mathcal{F}$$

i.e.

$$\{x \in X\colon (-f)(x) < -a\} \text{ is in } \mathcal{F}.$$

But this last set is just $(-f)^{-1}([-\infty, -a))$. The rest is easy.

In the next few propositions, $(X,\mathcal{F})$ is a measurable space.

Proposition 4.12 Suppose $f_1, f_2, f_3, \ldots X \to \overline{\mathbb{R}}$ are all measurable. Define

$$f(x) = \sup\{f_n(x)\colon n \in \mathbb{N}\} \in \overline{\mathbb{R}}.$$

Then f is measurable.

Proof

Let $a \in \mathbb{R}$. We show that $f^{-1}([-\infty, a])$ is in $\mathcal{F}$. For $x \in X$,

$$\begin{aligned} x \in f^{-1}([-\infty, a]) \quad &\text{iff} \quad f(x) \leqslant a, \\ &\text{iff} \quad f_n(x) \leqslant a \quad \forall\, n, \\ &\text{iff} \quad x \in \bigcap_{n \in \mathbb{N}} f_n^{-1}([-\infty, a]). \end{aligned}$$

Thus

$$f^{-1}([-\infty, a]) = \bigcap_{n \in \mathbb{N}} f_n^{-1}([-\infty, a]) \in \mathcal{F}.$$

Proposition 4.13. Suppose $f_1, f_2, f_3, \ldots \; X \to \overline{\mathbb{R}}$ are all measurable. Then so are the functions $\inf f_n$, $\liminf f_n$, $\limsup f_n$.

Remark. Here the relevant functions are defined pointwise, looking at the sequence $f_n(x)$.

Proof Let

$$g(x) = \inf\{f_n(x) : n \in \mathbb{N}\}.$$

Then

$$g(x) = -\sup\{-f_n(x) : n \in \mathbb{N}\}$$

and so g a measurable function by 4.11 and 4.12.

Set

$$h(x) = \limsup_{n \to \infty} (f_n(x))$$

$$= \inf_{n \in \mathbb{N}} (\sup_{k \geqslant n} f_k(x)).$$

Then h is a measurable function, using the above and Proposition 4.12. Finally,

$$\liminf_{n \to \infty} (f_n(x)) = -\limsup_{n \to \infty} (-f_n(x))$$

which is measurable by the above and 4.11.

Corollary 4.14 If f_n is a sequence of measurable functions from X to $\overline{\mathbb{R}}$, and if $f_n(x) \to f(x) \;\; \forall\, x \in X$, then f is also measurable.

Proof. $\limsup_{n \to \infty} f_n(x) = f(x)$, and so f is measurable.

In other words, the collection of measurable functions is closed under the operation of taking pointwise limits.

Theorem 4.15

Let $(X, \mathcal{F})$ be a measurable space, and let $f, g \colon X \to \overline{\mathbb{R}}$ be measurable functions. Suppose that $f(x) + g(x)$ is defined for all $x \in X$. Then the function $f + g$ is measurable.

Proof. It is enough to show that, $\forall\, a \in \mathbb{R}$,

$$\{x \in X : f(x) + g(x) < a\} \quad \text{is in} \quad \mathcal{F}.$$

But

$$\{x \in X\colon\ f(x)+g(x) < a\} = \bigcup_{\substack{p,q \in \mathbb{Q} \\ p+q<a}} \{x \in X\colon\ f(x) \leqslant p \text{ and } g(x) \leqslant q\}$$

$$= \bigcup_{\substack{p,q \in \mathbb{Q} \\ p+q<a}} f^{-1}([-\infty, p]) \cap g^{-1}([-\infty, q]),$$

a countable union of measurable sets. □

Returning to simple functions, suppose $(X, \mathscr{F})$ is measurable space, and $s\colon X \to \mathbb{R}$ is a simple function. We have

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$$

for some sets A_k with $X = \dot{\bigcup}_{k=1}^{n} A_k$, where the α_k are the distinct values taken by s.

When is s measurable? With this notation it is easily shown that s is measurable if and only if each set A_k is measurable.

Note, however, that if $A_1, A_2, \ldots, A_n \in \mathscr{F}$, not necessarily disjoint, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ then $\sum_{k=1}^{n} \alpha_k \chi_{A_k}$ is a sum of measurable functions, and so is measurable. It is also simple.

Integration theory begins with simple measurable functions (measurable simple functions).

Proposition 4.16 Let $(X, \mathscr{F})$ be a measurable space, and let s, t be simple measurable functions on X. Then $s+t$ and st are also simple measurable functions.

Proof. This is immediate from Proposition 4.2 and Theorem 4.15, except for the measurability of st. Write

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k} \qquad A_k \text{ all measurable,}$$

$$t = \sum_{j=1}^{m} \beta_j \chi_{B_j} \qquad B_j \text{ all measurable.}$$

Then

$$st = \sum_{\substack{k,j \\ 1 \leqslant k \leqslant n \\ 1 \leqslant j \leqslant m}} (\alpha_k \beta_j) \chi_{A_j \cap B_k}$$

which is a measurable simple function, as required.

So the collection of simple measurable functions is closed under multiplication and addition.

Lemma 4.17

Let $(X, \mathcal{F})$ be a measurable space, and let

$$f\colon X \to [0,\infty]$$

be a function. Then there is a sequence of simple functions

$$s_n\colon X \to [0,\infty) \quad \text{with} \quad 0 \leqslant s_1(x) \leqslant s_2(x) \leqslant \ldots \leqslant f(x)$$

and

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \forall\, x \in X.$$

If f is measurable, the s_n may be chosen to be measurable simple functions. If f is bounded then we can choose s_n to converge to f uniformly.

Proof. Define $s_n\colon X \to \mathbb{R}$ as follows.

$$s_n(x) = \begin{cases} n & \text{if} \quad f(x) \geqslant n \\ \dfrac{j}{2^n} & \text{if} \quad f(x) < n \quad \text{and} \quad j \in \mathbb{Z}^+ \quad \text{satisfies} \quad \dfrac{j}{2^n} \leqslant f(x) < \dfrac{j+1}{2^n}. \end{cases}$$

NB: $f(x) < n \Rightarrow s_n(x) = \dfrac{j}{2^n}$ for some integer $0 \leqslant j \leqslant n2^n - 1$, and in this case

$$s_n(x) \leqslant f(x) < s_n(x) + \frac{1}{2^n}.$$

Certainly s_n is simple, and $0 \leqslant s_n(x) \leqslant f(x)$ all x.

If $k \in \mathbb{N}$, and $f(x) \geqslant k$, then certainly

$$s_k(x) \geqslant k \qquad (\text{because } \ s_k(x) = k).$$

In fact, $\forall\, n \geqslant k$, $s_n(x) \geqslant k$ (you should check this).

For all $x \in X$, we can see that $s_n(x) \to f(x)$ as $n \to \infty$ because, if $f(x) < \infty$, then $\forall\, n > f(x)$,

$$|s_n(x) - f(x)| < \frac{1}{2^n},$$

while if $f(x) = \infty$ then $s_n(x) = n \ \ \forall\, n$ and so $s_n(x) \to f(x)$.

To see that $s_n(x) \leqslant s_{n+1}(x)$ there are two cases:

(i) $f(x) \geqslant n$

In this case $s_n(x) = n$ and $s_{n+1}(x) \geqslant n$.

(ii) $f(x) < n$

Then there is $j < n2^n$ with $\dfrac{j}{2^n} \leqslant f(x) < \dfrac{j+1}{2^n}$.

Then $s_n(x) = \frac{j}{2^n}$. But also

$$\frac{2j}{2^{n+1}} \leqslant f(x) < \frac{2j+2}{2^{n+1}}$$

and so $s_{n+1}(x) = \frac{2j}{2^{n+1}}$ or $\frac{2j+1}{2^{n+1}}$.

In either case, $s_{n+1}(x) \geqslant s_n(x)$.

In all cases $s_n(x) \leqslant s_{n+1}(x)$.

If f is bounded then there is $N \in \mathbb{N}$ with

$$0 \leqslant f(x) \leqslant N \qquad \forall\ x \in X.$$

But then, $\forall\ n \geqslant N$,

$$|s_n(x) - f(x)| < \frac{1}{2^n} \quad \text{all } x.$$

So in this case $s_n \to f$ uniformly.

Note:

$$s_n = n\chi\{x \in X:\ f(x) \geqslant n\} + \sum_{j=0}^{n2^n-1} \frac{j}{2^n}\chi\left(\left\{x \in X:\ \frac{j}{2^n} \leqslant f(x) < \frac{j+1}{2^n}\right\}\right).$$

If f is measurable, each of these subsets is measurable, and so s_n is a measurable function. □

Corollary 4.18

Let $f, g\colon X \to [0,\infty]$ be measurable functions, where $(X,\mathcal{F})$ is a measurable space. Then fg is also measurable.

Proof

We can choose simple functions s_n, t_n such that s_n, t_n are measurable,

$$0 \leqslant s_n(x) \leqslant s_{n+1}(x)$$

$$0 \leqslant t_n(x) \leqslant t_{n+1}(x) \quad \text{all } n$$

and all $x \in X$,

$$s_n(x) \to f(x),$$

$$t_n(x) \to g(x).$$

Then $\forall\ n$, $s_n t_n$ is a simple measurable function

$$\forall\ x \in X, \qquad (s_n t_n)(x) = s_n(x)t_n(x) \to f(x)g(x) \ \text{ as } \ n \to \infty,$$

because the sequences $s_n(x)$ and $t_n(x)$ are nondecreasing. Thus fg is a pointwise limit of measurable functions and so fg is measurable. □

Recall:

If (f_n) is a sequence of measurable functions, then the function

$$x \mapsto \sup_n f_n(x)$$

is also measurable. It follows that if f, g are measurable then

$$x \mapsto \max\{f(x), g(x)\}$$

is also measurable.

Definition 4.19. Let X be a set and let $f\colon X \to \overline{\mathbb{R}}$. We define

$$f^+(x) = \max\{f(x), 0\},$$

$$f^-(x) = \max\{-f(x), 0\}.$$

f^+ is the *positive part* of f, f^- is the *negative part*.

Note that if X is a measurable space and f is measurable, then $f^+, f^-\colon X \to [0,\infty]$ are measurable. We always have $f(x) = f^+(x) - f^-(x)$ all $x \in X$.

The Integral

We begin by defining the integral of a non-negative, simple measurable function.

Definition 4.20

Let $(X, \mathcal{F}, \mu)$ be a measure space, let $s\colon X \to [0,\infty)$ be a simple measurable function. Then, for every $E \in \mathcal{F}$ we define the *integral of s over E with respect to* μ, $I_E(s,\mu)$,
as follows.

Let $\alpha_1, \ldots, \alpha_n$ be the distinct values taken by s. Let $A_k = \{x \in X\colon s(x) = \alpha_k\}$. Then

$$I_E(s,\mu) = \sum_{k=1}^{n} \alpha_k \mu(E \cap A_k).$$

NB: α_k are all real numbers, but $\mu(E \cap A_k)$ may be ∞. $I_E(s,\mu)$ is a well defined element of $[0,\infty]$.

Proposition 4.21. (a) If $s(x) = \alpha \quad \forall\, x \in X$, then

$$I_E(s,\mu) = \alpha \cdot \mu(E) \quad \forall\, E \in \mathcal{F}.$$

(b)

$$I_\emptyset(s,\mu) = 0$$

for any simple measurable s.

(c) If $E \in \mathscr{F}$ and s, t are simple measurable functions with

$$s(x) \leqslant t(x) \quad \text{all } x \in E$$

then

$$I_E(s,\mu) \leqslant I_E(t,\mu).$$

Proof. Parts (a) and (b) are trivial. To prove (c),

let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the values taken by s.

Let $\beta_1, \beta_2, \ldots, \beta_n$ be the values taken by t

and set

$$A_j = \{x \in X\colon s(x) = \alpha_i\}$$

$$B_k = \{x \in X\colon t(x) = \beta_k\}.$$

Since $s(x) \leqslant t(x) \;\; \forall\, x \in E$, it follows that if $A_j \cap B_k \cap E \neq \emptyset$, then $\alpha_j \leqslant \beta_k$. Also

$$X = \dot{\bigcup_{j=1}^{m}} A_j = \dot{\bigcup_{k=1}^{n}} B_k.$$

$$\begin{aligned} I_E(s,\mu) &= \sum_{j=1}^{m} \alpha_j \mu(A_j \cap E) \\ &= \sum_{j=1}^{m} \alpha_j \sum_{k=1}^{n} \mu(A_j \cap B_k \cap E) \\ &= \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_j \mu(A_j \cap B_k \cap E) \\ &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_k \mu(A_j \cap B_k \cap E) \\ &= \sum_{k=1}^{n} \beta_k \mu(\beta_k \cap E) \quad \text{(reversing order)} \\ &= I_E(t,\mu). \end{aligned}$$

□

Further Properties of the Integral

Proposition 4.22 $(X, \mathscr{F}, \mu)$ is a measure space. $s\colon X \to [0,\infty)$ is simple measurable.

(a) For any $E \in \mathscr{F}$ such that $\mu(E) = 0$,

$$I_E(s,\mu) = 0.$$

(b) If $E \in \mathscr{F}$ and c is such that $s(x) = c \;\; \forall\, x$ in E, then

$$I_E(s,\mu) = c\mu(E).$$

(c) Let $E \in \mathcal{F}$. Then recall $\mathcal{F}_E$ is the σ-field $\{A \cap E \colon A \in \mathcal{F}\}$ on E. Let ν be $\mu|_{\mathcal{F}_E}$, (the restriction of μ to $\mathcal{F}_E$), so that $(E, \mathcal{F}_E, \nu)$ is a measure space. Then $s|_E$ is a simple measurable function $E \to [0, \infty)$, and

$$I_E(s,\mu) = I_E(s|_E, \nu).$$

Proof. Easy exercise! (See question sheet 4).

Lemma 4.23.

Let $(X, \mathcal{F}, \mu)$ be a measure space.

(i) Let $s \colon X \to [0, \infty)$ be a simple measurable function. Define

$$\phi(E) = I_E(s,\mu) \quad (E \in \mathcal{F}).$$

Then ϕ is a measure on $\mathcal{F}$.

(ii) Let $s, t \colon X \to [0, \infty)$ be simple measurable functions and let $E \in \mathcal{F}$. Then

$$I_E((s+t),\mu) = I_E(s,\mu) + I_E(t,\mu).$$

Proof

(i) To show ϕ is a measure, note that $\phi(E) \in [0, \infty] \ \forall \ E \in \mathcal{F}$ and that $\phi(\emptyset) = 0$ because $I_\emptyset(s,\mu) = 0$.

It remains to show that ϕ is countably additive.

Let $E \in \mathcal{F}$, and suppose that

$$E = \dot{\bigcup_{n=1}^{\infty}} E_n$$

where E_n is in $\mathcal{F} \ \ \forall \ n$. We show that $\phi(E) = \sum_{n=1}^{\infty} \phi(E_n)$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the distinct values taken by s, and set

$$A_k = \{x \in X \colon s(x) = \alpha_k\}.$$

As usual $X = \dot{\bigcup_{k=1}^{m}} A_k$.

By definition

$$\phi(E) = I_E(s,\mu) = \sum_{k=1}^{m} \alpha_k \mu(E \cap A_k)$$

$$\phi(E_n) = I_{E_n}(s,\mu) = \sum_{k=1}^{m} \alpha_k \mu(E_n \cap A_k)$$

since

$$E \cap A_k = \dot{\bigcup_{n=1}^{\infty}} (E_n \cap A_k).$$

We have

$$\mu(E \cap A_k) = \sum_{n=1}^{\infty} \mu(E_n \cap A_k)$$

and so

$$\begin{aligned}\phi(E) &= \sum_{k=1}^{m} \alpha_k \sum_{n=1}^{\infty} \mu(E_n \cap A_k) \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{m} \alpha_k \mu(E_n \cap A_k) \\ &= \sum_{n=1}^{\infty} \phi(E_n).\end{aligned}$$

Thus ϕ is a measure.

(ii) Let $s, t\colon X \to [0, \infty)$ be simple measurable functions and let $E \in \mathscr{F}$. Then $s+t$ is also simple measurable.

To show that

$$I_E((s+t),\mu) = I_E(s,\mu) + I_E(t,\mu)$$

define

$$\begin{aligned}\phi_1(A) &= I_A(s,\mu) \qquad (A \in \mathscr{F}) \\ \phi_2(A) &= I_A(t,\mu) \qquad (A \in \mathscr{F}) \\ \phi_3(A) &= I_A((s+t),\mu) \quad (A \in \mathscr{F}).\end{aligned}$$

We must show

$$\phi_1(E) + \phi_2(E) = \phi_3(E).$$

We know ϕ_1, ϕ_2, ϕ_3 are measures.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the distinct values taken by s, $\beta_1, \beta_2, \ldots, \beta_n$ be the values taken by t.

Set

$$\begin{aligned}A_j &= \{x \in X\colon s(x) = \alpha_j\}, \\ B_k &= \{x \in X\colon t(x) = \beta_k\}.\end{aligned}$$

Set $E_{jk} = E \cap A_j \cap B_k$. Then

$$E = \dot{\bigcup_{j=1}^{m}} \dot{\bigcup_{k=1}^{n}} E_{jk}.$$

On E_{jk} s is constantly α_j, t is constantly equal to β_k and $(s+t)$ is constantly equal to $\alpha_j+\beta_k$. By 4.22(b),

$$I_{E_{jk}}((s+t),\mu) = (\alpha_j+\beta_k)\,\mu(E_{jk}),$$

$$I_{E_{jk}}(s,\mu) = \alpha_j\,\mu(E_{jk}),$$

$$I_{E_{jk}}(t,\mu) = \beta_k\,\mu(E_{jk}).$$

Hence $\phi_3(E_{jk}) = \phi_1(E_{jk})+\phi_2(E_{jk})$. But ϕ_1,ϕ_2,ϕ_3 are measures, and

$$E = \dot{\bigcup_{j,k}} E_{jk},$$

so

$$\begin{aligned}\phi_3(E) &= \sum_{j,k} \phi_3(E_{j,k}),\\ &= \sum_{j,k} (\phi_1(E_{jk})+\phi_2(E_{jk})),\\ &= \sum_{j,k} \phi_1(E_{jk}) + \sum_{j,k} \phi_2(E_{jk}),\\ &= \phi_1(E)+\phi_2(E).\end{aligned}$$

□

Note in particular that if $\alpha_1,\alpha_2,\ldots,\alpha_n \in \mathbb{R}^+$ and $A_1,A_2,\ldots,A_n \in \mathcal{F}$, then

$$I_X\Big(\sum_{k=1}^{n} \alpha_k\chi_{A_k},\mu\Big) = \sum_{k=1}^{n} \alpha_k\,\mu(A_k)$$

even if the α_k are not distinct and the A_k are not be disjoint.

Recall:

$$s \leqslant t \Rightarrow I_E(s,\mu) \leqslant I_E(t,\mu).$$

The following result follows immediately.

Proposition 4.24 For $s\colon X \to [0,\infty)$, measurable simple.

$$I_E(s,\mu) = \sup\left\{ I_E(t,\mu) \;\middle|\; \begin{array}{l} t\colon X \to [0,\infty) \text{ simple, measurable}\\ \text{and } 0 \leqslant t(x) \leqslant s(x) \text{ all } x \in X\end{array}\right\}.$$

Definition 4.25 We now define, for any $f\colon X \to [0,\infty]$ measurable, and $E \in \mathcal{F}$

$$\int_E f\,\mathrm{d}\mu = \sup\left\{ I_E(s,\mu) \;\middle|\; \begin{array}{l} s\colon X \to [0,\infty) \text{ simple measurable and}\\ 0 \leqslant s(x) \leqslant f(x) \quad \forall\, x \in X\end{array}\right\}.$$

In view of proposition 4.24, we can safely call $\int_E f\,\mathrm{d}\mu$ the (Lebesgue) integral of f over E with respect to μ.

All our results about the integrals of simple measurable functions remain true (for simple measurable functions) if we change to our new version of the integral (which has the same value for such functions). From now on, this is the version of the integral which we shall use.

Properties

Proposition 4.26.

(a) If $f(x) \leqslant g(x) \;\; \forall\, x \in X$ then

$$\int_E f \,\mathrm{d}\mu \leqslant \int_E g \,\mathrm{d}\mu$$

(f, g non-negative measurable functions).

(b) If $E \in \mathcal{F}$ and $\mu(E) = 0$ then

$$\int_E f \,\mathrm{d}\mu = 0$$

(even if $f(x) = \infty$ all $x \in X$) for any measurable function $f\colon X \to [0, \infty]$.

(c) Let $f\colon X \to [0, \infty)$ be measurable, $E \in \mathcal{F}$ and suppose that $f(x) = 0 \;\; \forall\, x$ in E. Then

$$\int_E f \,\mathrm{d}\mu = 0.$$

(d)

$$\int_E f \,\mathrm{d}\mu = \int_E (f\chi_E) \,\mathrm{d}\mu = \int_X (f\chi_E) \,\mathrm{d}\mu$$

for $f\colon X \to [0, \infty]$ measurable and $E \in \mathcal{F}$.

(e) Let $f, g\colon X \to [0, \infty]$ be measurable, let $E \in \mathcal{F}$, and suppose $f(x) \leqslant g(x) \;\; \forall\, x \in E$. Then

$$\int_E f \,\mathrm{d}\mu \leqslant \int_E g \,\mathrm{d}\mu.$$

Proof

(a) This is because we take the sup of a larger set (for g).

(b) This is because $\int_E s \,\mathrm{d}\mu = 0$ for all simple functions which are measurable and satisfy $0 \leqslant s \leqslant f$.

(c)

$$\int_E f \,\mathrm{d}\mu = \sup\left\{ \int_E s \,\mathrm{d}\mu \colon s \text{ measurable simple, } 0 \leqslant s \leqslant f \right\}$$

since $f(x) = 0 \;\; \forall\, x$ in E, then whenever $0 \leqslant s \leqslant f$ we have $s(x) = 0$ all x in E, and so

$$\int_E s \, d\mu = 0$$

for all such measurable simple s. Hence

$$\int_E f \, d\mu = 0.$$

(d) Certainly $f\chi_E$ is measurable. Since $f\chi_E \leqslant f$, we have

$$\int_E (f\chi_E) \, d\mu \leqslant \int_E f \, d\mu.$$

Now suppose s is a simple function with s measurable and $0 \leqslant s \leqslant f$. We shall show

$$\int_E s \, d\mu \leqslant \int_E f\chi_E \, d\mu.$$

(Taking sup over s will then give equality.)

$s = s\chi_E + s\chi_{X \setminus E}$ (the sum of two simple measurable functions).

$$\begin{aligned} \int_E s \, d\mu &= \int_E (s\chi_E) \, d\mu + \int_E (s\chi_{X \setminus E}) \, d\mu, \\ &= \int_E s\chi_E \, d\mu, \\ &\leqslant \int_E f\chi_E \, d\mu. \end{aligned}$$

Taking sup over s,

$$\int_E f \, d\mu \leqslant \int_E f\chi_E \, d\mu,$$

hence equality.

For the rest: if $0 \leqslant s \leqslant f\chi_E$ then $s \equiv 0$ on $X \setminus E$, so

$$\begin{aligned} \int_E s \, d\mu &= \int_E s \, d\mu + \int_{X \setminus E} s \, d\mu \\ &= \int_X s \, d\mu \end{aligned}$$

so taking sup over s,

$$\int_E f\chi_E \, d\mu = \int_X f\chi_E \, d\mu.$$

(e)

$$\int_E f\chi_E \, d\mu = \int_E f \, d\mu,$$

$$\int_E g\chi_E \, d\mu = \int_E g \, d\mu.$$

But $f(x)\chi_E(x) \leqslant g(x)\chi_E(x) \ \ \forall\, x$ in $\boldsymbol{X}$, therefore, by property (a),

$$\int_E g\chi_E \, d\mu \geqslant \int_E f\chi_E \, d\mu.$$

Corollary 4.27. Let $(X,\mathcal{F},\mu)$ be a measure space, let $f: X \to [0,\infty]$ be measurable, and let $A, B \in \mathcal{F}$ with A contained in B. Then

$$\int_A f \, d\mu \leqslant \int_B f \, d\mu.$$

Proof This is because $f\chi_A \leqslant f\chi_B$.

Proposition 4.28

Let $f, g\colon X \to [0,\infty]$ be measurable. Then $\{x \in X\colon f(x) \leqslant g(x)\}$ is measurable.

Proof

Easy exercise (using $\mathbb{Q}$ as usual).

The following trivial result is used in the proof of the Monotone Convergence Theorem.

Lemma 4.29. If $(X, \mathcal{F}, \mu)$ is a measure space, $s\colon X \to [0,\infty)$ is simple measurable and $\alpha \in \mathbb{R}^+$, then αs is also a simple measurable function, and $\forall\ E \in \mathcal{F}$,

$$\int_E (\alpha s) \, d\mu = \alpha\left(\int_E s \, d\mu\right).$$

This is because $s = \sum_{k=1}^{n} \beta_k \chi_{A_k}$ for some $\beta_1, \beta_2, \ldots, \beta_k \in [0,\infty)$ and measurable sets $A_1, \ldots, A_n$. But then

$$\alpha s = \sum_{k=1}^{n} (\alpha\beta_k)\chi_{Ak},$$

which is simple, measurable, and

$$\begin{aligned}\int_E (\alpha s) \, d\mu &= \sum_{k=1}^{n} (\alpha\beta_k)\mu(E \cap A_k) \\ &= \alpha \sum_{k=1}^{n} \beta_k \mu(E \cap A_k) \\ &= \alpha \int_E s \, d\mu.\end{aligned}$$

Theorem 4.30 (Monotone Convergence Theorem)

Let (X, f, μ) be a measure space, let

$$f_n \colon X \to [0, \infty]$$

be a sequence of measurable functions with

$$0 \leqslant f_1(x) \leqslant f_2(x) \ldots \quad \forall\, x \in X.$$

Suppose

$$f(x) = \lim_{n \to \infty} f_n(x) \quad \forall\, x \in X.$$

Then f is measurable and

$$\int_X f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu.$$

Remark

Without the assumption that $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots$ the result is false: there are many examples of functions which converge pointwise, but whose integrals do not converge.

Proof Since $f(x) = \lim_{n \to \infty} f_n(x)$, f is a pointwise limit of measurable functions, and hence f is measurable, and $f \colon X \to [0, \infty]$.

We have

$$0 \leqslant f_1 \leqslant f_2 \leqslant \ldots \leqslant f$$

so, $\forall n$,

$$0 \leqslant \int_X f_n \, \mathrm{d}\mu \leqslant \int_X f_{n+1} \, \mathrm{d}\mu \leqslant \int_X f \, \mathrm{d}\mu.$$

Certainly there is an α in $[0, \infty]$ such that

$$\alpha = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$$

and note

$$\alpha \leqslant \int_X f \, \mathrm{d}\mu.$$

It remains to prove $\int_X f \, \mathrm{d}\mu \leqslant \alpha$.

From the definition of the integral, it is enough to show that, if s is simple measurable and $0 \leqslant s \leqslant f$, then

$$\int_X s \, \mathrm{d}\mu \leqslant \alpha.$$

Let s be such a function. Note that s does not take the value ∞. Then it is enough to show that $\forall c$ with $0 < c < 1$,

$$c \int_X s \, \mathrm{d}\mu \leqslant \alpha,$$

since then

$$\int_X s \, \mathrm{d}\mu = \lim_{n \to \infty} \left(\left(1 - \frac{1}{2n} \right) \int_X s \, \mathrm{d}\mu \right) \leqslant \alpha.$$

But, for such c,

$$c \int_X s \, \mathrm{d}\mu = \int_X (cs) \, \mathrm{d}\mu.$$

We show this is $\leqslant \alpha$. Set $A_n = \{x \in X \colon (cs)(x) \leqslant f_n(x)\}$. Then each A_n is measurable, and the sets A_n are nested. Also

$$X = \bigcup_{n=1}^{\infty} A_n$$

because (two cases):

(i) if $s(x) = 0$, then $x \in A_n \quad \forall n$;

(ii) if $s(x) > 0$, then, since $s(x) \neq \infty$, $cs(x) < s(x) \leqslant f(x)$.

Since $f(x) = \lim_{n \to \infty} f_n(x)$ there is an n with

$$f_n(x) \geqslant cs(x), \qquad \text{i.e.} \quad x \in A_n.$$

But now, for all n,

$$\int_{A_n} (cs) \, \mathrm{d}\mu \leqslant \int_{A_n} f_n \, \mathrm{d}\mu \leqslant \int_X f_n \, \mathrm{d}\mu.$$

But, recall,

$$E \mapsto \int_E (cs) \, \mathrm{d}\mu$$

is a measure on $\mathcal{F}$, so

$$\int_X (cs) \, \mathrm{d}\mu = \lim_{n \to \infty} \int_{A_n} (cs) \, \mathrm{d}\mu \qquad \text{by standard properties of measures}$$

$$\leqslant \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu \qquad \text{by above.} \qquad \square$$

We now give some corollaries to the monotone convergence theorem.

Corollary 4.31

Let $f, g\colon X \to [0,\infty]$ be measurable functions and let $\alpha \in [0,\infty)$. Then

(i) $$\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu,$$

(ii) αf is measurable and

$$\int_X (\alpha f)\,\mathrm{d}\mu = \alpha \int_X f\,\mathrm{d}\mu.$$

Proof

Let s_n, t_n be simple measurable functions with

$$0 \leqslant s_n \leqslant s_{n+1}, \qquad 0 \leqslant t_n \leqslant t_{n+1}$$

and $s_n \to f$ pointwise, $t_n \to g$ pointwise. Then $s_n + t_n$ is simple measurable and $s_n + t_n$ converges pointwise to $f+g$. Also $0 \leqslant s_n + t_n \leqslant s_{n+1} + t_{n+1}$, so this convergence is monotone.

By MCT we have

$$\int_X s_n\,\mathrm{d}\mu \to \int_X f\,\mathrm{d}\mu$$

$$\int_X t_n\,\mathrm{d}\mu \to \int_X g\,\mathrm{d}\mu$$

and $$\int_X (s_n+t_n)\,\mathrm{d}\mu \to \int_X (f+g)\,\mathrm{d}\mu.$$

But s_n, t_n are simple, so

$$\int_X (s_n+t_n)\,\mathrm{d}\mu = \int_X s_n\,\mathrm{d}\mu + \int_X t_n\,\mathrm{d}\mu.$$

Taking the limit as $n \to \infty$, using the above,

$$\int_X (f+g)\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu + \int_X g\,\mathrm{d}\mu.$$

This proves (i).

Also, (αs_n) is a simple measurable function with

$$\int_X \alpha s_n\,\mathrm{d}\mu = \alpha \int_X s_n\,\mathrm{d}\mu.$$

Also, αs_n tends monotonically pointwise up to αf, and so by MCT, (αf is measurable) and

$$\int_X \alpha f\,\mathrm{d}\mu = \lim_{n\to\infty} \int_X (\alpha s_n)\,\mathrm{d}\mu = \lim_{n\to\infty} \alpha \int_X s_n\,\mathrm{d}\mu$$

$$= \alpha \lim_{n\to\infty} \int_X s_n\,\mathrm{d}\mu$$

$$= \alpha \int_X f \, d\mu. \qquad \square$$

Corollary 4.32

Let f_n be a sequence of measurable functions ($f_n\colon X \to [0,\infty]$). Set $g(x) = \sum_{n=1}^{\infty} f_n(x)$. Then g is measurable, and

$$\int_X g \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Proof

Set $g_n(x) = \sum_{k=1}^{n} f_k(x) \quad (x \in X)$.
i.e. $g_n = f_1 + f_2 + \ldots + f_n$.
Then g_n is measurable,

$$0 \leqslant g_n \leqslant g_{n+1} \quad \forall n \text{ and}$$

$$g_n(x) \to g(x) \quad \text{as} \quad n \to \infty.$$

By MCT, g is measurable, and

$$\int_X g \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu.$$

But $g_n = f_1 + f_2 + \ldots + f_n$ and so by corollary 4.31,

$$\int_X g_n \, d\mu = \sum_{k=1}^{n} \left(\int_X f_k \, d\mu \right)$$

and so $\lim_{n\to\infty} \int_X g_n \, d\mu$ is just

$$\sum_{k=1}^{\infty} \left(\int_X f_k \, d\mu \right). \qquad \square$$

Corollary 4.33

Let $f\colon X \to [0,\infty]$ be measurable. Define

$$\Phi(E) = \int_E f \, d\mu.$$

Then Φ is a measure on $\mathscr{F}$.

Proof

Certainly $\Phi(\varnothing) = 0$.

Now suppose that $E \in \mathcal{F}$ and let $E = \dot{\bigcup}_{n=1}^{\infty} E_n$ for some set $E_n \in \mathcal{F}$. We show that

$$\Phi(E) = \sum_{n=1}^{\infty} \Phi(E_n).$$

To see this, note

$$\Phi(E) = \int_E f \, \mathrm{d}\mu = \int_X (f\chi_E) \, \mathrm{d}\mu$$

and
$$\Phi(E_n) = \int_X (f\chi_{E_n}) \, \mathrm{d}\mu.$$

But

$$E = \dot{\bigcup}_{n=1}^{\infty} E_n$$

and so

$$f\chi_E(x) = \sum_{n=1}^{\infty} (f\chi_{E_n})(x) \quad \text{all} \quad x \in X.$$

By Corollary 4.32,

$$\int_X (f\chi_E) \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X (f\chi_{E_n}) \, \mathrm{d}\mu,$$

i.e.
$$\Phi(E) = \sum_{n=1}^{\infty} \int \Phi(E_n).$$
□

Example

Set $X = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\mathbb{N})$, μ = counting measure on $\mathbb{N}$. All functions $f \colon \mathbb{N} \to [0, \infty]$ are now measurable. For such an f, what is $\int_{\mathbb{N}} f \, \mathrm{d}\mu$? It is $\sum_{n=1}^{\infty} f(n)$.

Proof

$$\mathbb{N} = \left(\dot{\bigcup}_{n=1}^{\infty} \{n\} \right):$$

$$\int_{\{n\}} f \, \mathrm{d}\mu = \int_{\mathbb{N}} (f\chi_{\{n\}}) \, \mathrm{d}\mu = \int_{\mathbb{N}} f(n)\chi_{\{n\}} \, \mathrm{d}\mu = f(n)\mu(\{n\})$$
$$= f(n)$$

setting $\Phi(E) = \int_E f \, \mathrm{d}\mu$, Φ is a measure so

$$\int_{\mathbb{N}} f \, \mathrm{d}\mu = \Phi(\mathbb{N}) = \sum_{n=1}^{\infty} \Phi(\{n\})$$

$$= \sum_{n=1}^{\infty} f(n).$$

Now let $a_{m,n} \in [0, \infty]$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.

Set $f_n(m) = a_{m,n}$.

This defines a sequence of (measurable) functions

$$f_n \colon \mathbb{N} \to [0, \infty].$$

Then

$$\int_{\mathbb{N}} f_n \, d\mu = \sum_{m=1}^{\infty} f_n(m) = \sum_{m=1}^{\infty} a_{m,n}.$$

By Corollary 4.32,

$$\sum_{n=1}^{\infty} \int_{\mathbb{N}} f_n \, d\mu = \int_{\mathbb{N}} \left(\sum_{n=1}^{\infty} f_n \right) d\mu$$

i.e.
$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{m,n} \right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{m,n} \right)$$

and we have recovered proposition 1.9 by other means! In fact, if you look carefully at our development of integration theory, you will find that there is no circularity in taking this as our proof of 1.9.

Recall: if $(X, \mathcal{F})$ is a measurable space,

$$(f_n)_{n=1}^{\infty} \qquad f_n \colon X \to [0, \infty]$$

f_n measurable. Then

$$x \mapsto \limsup_{n \to \infty} f_n(x)$$

$$x \mapsto \liminf_{n \to \infty} f_n(x)$$

are both measurable functions. The first is usually denoted by

$$\limsup_{n \to \infty} f_n$$

and the second by

$$\liminf_{n \to \infty} f_n.$$

Theorem 4.34 (Fatou's Lemma).

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f_n \colon X \to [0, \infty]$ be measurable. Then

$$\int_X (\liminf_{n \to \infty} f_n) \, d\mu \leqslant \liminf_{n \to \infty} \int_X f_n \, d\mu.$$

Proof

Recall:

$$\liminf_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} \inf_{k\geqslant n} f_k(x)$$
$$= \lim_{n\to\infty} (\inf_{k\geqslant n} f_k(x)).$$

Set $g_n(x) = \inf_{k\geqslant n} f_k(x)$. Then $0 \leqslant g_1(x) \leqslant g_2(x) \leqslant \ldots$ and $g_n(x) \to \liminf_{m\to\infty} f_m(x)$ as $n \to \infty$. So, by the MCT,

$$\int_X (\liminf_{m\to\infty} f_m)\, d\mu = \lim_{n\to\infty} \int_X g_n\, d\mu$$
$$= \liminf_{n\to\infty} \int_X g_n\, d\mu.$$

But

$$g_n(x) \leqslant f_n(x) \quad (\forall n \in \mathbb{N},\, x \in X)$$

so

$$\liminf_{n\to\infty} \int_X g_n\, d\mu \leqslant \liminf_{n\to\infty} \int_X f_n\, d\mu. \qquad \square$$

Officially we will not construct Lebesgue measure λ until Chapter 5, but we will assume for now the following properties of λ: λ is a *complete* measure (see question sheet 3) on a σ-field which includes all the Borel sets, and for all intervals I, $\lambda(I)$ is the length of I. The σ-field on which the complete measure λ is defined is the collection of *Lebesgue measurable* subsets of $\mathbb{R}$.

Example.

Working with the Lebesgue integral on $\mathbb{R}$, taking

$$f_n(x) = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$$

i.e. $f_n = \chi_{[n,n+1]}$. Then

$$\int_{\mathbb{R}} f_n\, d\lambda = \lambda([n, n+1]) = 1.$$

But $f_n(x) \to 0$ pointwise. So

$$\lim_{n\to\infty} \int f_n\, d\lambda = 1, \qquad \int \lim_{n\to\infty} (f_n)\, d\lambda = 0.$$

But Fatou's Lemma DOES hold,

$$\liminf_{n\to\infty} \int f_n\, d\lambda = 1, \qquad \liminf_{n\to\infty} f_n = \text{zero function}.$$

Definition 4.35. Let $(X, \mathscr{F})$ be a measurable space and let $f\colon X \to \overline{\mathbb{R}}$ be measurable, then

$$f^+(x) = \max\{0, f(x)\}$$

$$f^-(x) = \max\{0, -f(x)\}$$

$$f(x) = f^+(x) - f^-(x) \quad \text{all } x \in X,$$

f^+, f^- are measurable.

We can define $|f(x)| = f^+(x) + f^-(x)$ to coincide with the usual definition.

If $(X, \mathscr{F}, \mu)$ is a measure space, f, f^+, f^- as above.

We already know how to define

$$\int_X f^+ \, \mathrm{d}\mu, \qquad \int_X f^- \, \mathrm{d}\mu.$$

Let $E \in \mathscr{F}$. If

$$\int_E f^+ \, \mathrm{d}\mu < \infty \quad \text{or} \quad \int_E f^- \, \mathrm{d}\mu < \infty$$

then we can define

$$\int_E f \, \mathrm{d}\mu = \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu$$

so $\displaystyle\int_E f \, \mathrm{d}\mu$ is in $\overline{\mathbb{R}}$.

If further both $\displaystyle\int_E f^+ \, \mathrm{d}\mu, \int_E f^- \, \mathrm{d}\mu$ are finite we say that f is *integrable* or *summable* on E. We say f is integrable if it is integrable on X. We denote the set of all functions

$$f\colon X \to \mathbb{R}$$

which are integrable with respect to μ by $L^1(\mu)$. [We include measurable in the definition of integrable.]

For all $f \in L^1(\mu)$ and all $E \in \mathscr{F}$,

$$\int_E f \, \mathrm{d}\mu = \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu \in \mathbb{R}.$$

For example, when $X = \mathbb{N}$, $\mathscr{F} = \mathscr{P}(\mathbb{N})$, μ = counting measure, then

$$f\colon \mathbb{N} \to \mathbb{R} \in L^1(\mu) \quad \text{iff} \quad \sum_{n=1}^{\infty} f(n)$$

is *absolutely* convergent.

Since $|f(x)| = f^+(x) + f^-(x)$ we have, for all $E \in \mathcal{F}$,

$$\int_E f^+ \, d\mu \leqslant \int_E |f| \, d\mu$$

$$\int_E f^- \, d\mu \leqslant \int_E |f| \, d\mu$$

$$\int_E |f| \, d\mu = \int_E f^+ \, d\mu + \int_E f^- \, d\mu.$$

So clearly f is integrable on E iff $|f|$ is integrable on E. In particular, f is integrable iff $|f|$ is. (This statement is false if f is not assumed measurable: it is possible for $|f|$ to be measurable and f to be non-measurable). Also

$$\begin{aligned} -\int_E |f| \, d\mu \leqslant -\int_E f^- \, d\mu &\leqslant \int_E f \, d\mu \\ &\leqslant \int_E f^+ \, d\mu \\ &\leqslant \int_E |f| \, d\mu. \end{aligned}$$

Thus we have, for integrable functions f:

Proposition 4.36

$$\left| \int_E f \, d\mu \right| \leqslant \int_E |f| \, d\mu \quad \forall \ E \in \mathcal{F}.$$

Note: $(-f)^+ = f^-$ and $(-f)^- = f^+$.

So for $f \in L^1(\mu)$, we have $\forall \ E \in \mathcal{F}$,

$$\begin{aligned} \int_E (-f) \, d\mu &= \int_E (-f)^+ \, d\mu - \int_E (-f)^- \, d\mu \\ &= \int_E f^- \, d\mu - \int_E f^+ \, d\mu \\ &= -\int_E f \, d\mu. \end{aligned}$$

Now if $\alpha \geqslant 0$ then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$ so

$$\int_E (\alpha f) \, d\mu = \alpha \int_E f \, d\mu$$

from the definition because

$$\int_E (\alpha f^+) \, d\mu = \alpha \int_E f^+ \, d\mu$$

etc. Now let $\alpha < 0$. Then $\alpha f = (-\alpha)(-f)$ and $(-\alpha) \geqslant 0$, so

$$\int_E (\alpha f)\, d\mu = \int_E (-\alpha)(-f)\, d\mu$$
$$= (-\alpha)\int_E (-f)\, d\mu$$
$$= (-\alpha)\left(-\int_E f\, d\mu\right)$$
$$= \alpha \int_E f\, d\mu.$$

We have now proved the following.

Proposition 4.37.

For all $f \in L^1(\mu)$ and all $\alpha \in \mathbb{R}$ and all $E \in \mathcal{F}$,

$$\int_E (\alpha f)\, d\mu = \alpha \int_E f\, d\mu.$$

Proposition 4.38.

Let $(X, \mathcal{F}, \mu)$ be a measure space, let $f, g \in L^1(\mu)$. Then $(f+g) \in L^1(\mu)$ and

$$\int_E (f+g)\, d\mu = \int_E f\, d\mu + \int_E g\, d\mu \quad \forall\ E \in \mathcal{F}.$$

Proof

Set $h = f+g$. Then

$$h^+(x) \leqslant f^+(x) + g^+(x)$$
$$h^-(x) \leqslant f^-(x) + g^-(x)$$

$\forall\ x \in X$. (Easy exercise.)

So

$$\int_X h^+(x)\, d\mu \leqslant \int_X f^+\, d\mu + \int_X g^+\, d\mu < \infty,$$

and similarly for h^-, so certainly $h \in L^1(\mu)$.

We have

$$h(x) = h^+(x) - h^-(x)$$
$$f(x) = f^+(x) - f^-(x)$$
$$g(x) = g^+(x) - g^-(x)$$
$$h(x) = f(x) + g(x)$$

$$h^+(x) + h^-(x) = f^+(x) - f^-(x) + g^+(x) - g^-(x).$$

These are all real numbers, so

$$h^+(x) + f^-(x) + g^-(x) = h^-(x) + f^+(x) + g^+(x).$$

Thus, for $E \in \mathscr{F}$,

$$\int_E (h^+ + f^- + g^-) \, \mathrm{d}\mu = \int_E (h^- + f^+ + g^+) \, \mathrm{d}\mu$$

$$\int_E h^+ \, \mathrm{d}\mu + \int_E f^- \, \mathrm{d}\mu + \int_E g^- \, \mathrm{d}\mu = \int_E h^- \, \mathrm{d}\mu + \int_E f^+ \, \mathrm{d}\mu + \int_E g^+ \, \mathrm{d}\mu.$$

Rearranging gives

$$\int_E h \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu + \int_E g \, \mathrm{d}\mu$$

as required. □

With a little care, we can now prove the following fact: Let $h\colon X \to [0, \infty]$ measurable with $\int_X h \, \mathrm{d}\mu < \infty$, let $f\colon X \to \mathbb{R}$, $f \in L^1$, $f(x) \geqslant 0$ all x. Then

$$\int_X (h - f) \, \mathrm{d}\mu = \int_X h \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu.$$

Proof

Set $N = \{x \in X\colon h(x) = \infty\}$. Then we can see N has measure 0:

$$\infty > \int_X h \, \mathrm{d}\mu \geqslant \int_N h \, \mathrm{d}\mu$$

For all $n \in \mathbb{N}$, $h(x) \geqslant n$ on N, and so

$$\int_N h \, \mathrm{d}\mu \geqslant n\mu(N).$$

True $\forall\ n \in \mathbb{N}$. Thus $\mu(N)$ must be 0.

$$\int_X (h - f) \, \mathrm{d}\mu = \int_N (h - f) \, \mathrm{d}\mu + \int_{X \setminus N} (h - f) \, \mathrm{d}\mu \qquad \text{[check!]}$$

$$= \int_{X \setminus N} (h - f) \, \mathrm{d}\mu$$

(satisfies conditions for result proved previously)

$$= \int_{X \setminus N} h \, \mathrm{d}\mu - \int_{X \setminus N} f \, \mathrm{d}\mu$$

$$= \int_X h \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu.$$

Theorem 4.39 (Dominated Convergence Theorem)

Let $(X, \mathcal{F}, \mu)$ be a measure space, let $g\colon X \to [0, \infty]$ be a measurable function with $\int_X g \, \mathrm{d}\mu < \infty$. Let f_n, f be measurable functions from X to $\mathbb{R}$ such that

$$|f_n(x)| \leqslant g(x) \quad \forall\ x \in X \quad \text{all } n \in \mathbb{N}.$$

Suppose

$$f_n(x) \to f(x) \quad \forall\ x \in X.$$

Then

$$\text{(i)} \qquad \lim_{n \to \infty} \int_X |f_n - f| \, \mathrm{d}\mu = 0$$

$$\text{(ii)} \qquad \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu.$$

Proof

Note first that $|f(x)| \leqslant g(x)$ all $x \in X$, and so f_n, f are all in $L^1(\mu)$, with

$$\int_X |f_n| \, \mathrm{d}\mu \leqslant \int_X g \, \mathrm{d}\mu < \infty$$

$$\int_X |f| \, \mathrm{d}\mu \leqslant \int_X g \, \mathrm{d}\mu < \infty.$$

Also set

$$g_n(x) = |f - f_n(x)|.$$

Then

$$g_n(x) \leqslant 2g(x).$$

Thus

$$2g(x) - g_n(x) \geqslant 0 \quad \forall x.$$

Set

$$h_n(x) = 2g(x) - |f - f_n(x)|.$$

Then $h_n\colon X \to [0, \infty]$ and h_n is measurable.

We now apply Fatou's lemma:

$$\int_X (\liminf_{n \to \infty} h_n) \, \mathrm{d}\mu \leqslant \liminf_{n \to \infty} \int_X h_n \, \mathrm{d}\mu.$$

We have $h_n(x) \to 2g(x)$ as $n \to \infty$. So $\liminf (h_n) = 2g$,

$$\int_X (2g) \, \mathrm{d}\mu \leqslant \liminf_{n \to \infty} \left(\int_X (2g - |f - f_n|) \, \mathrm{d}\mu \right)$$

$$= \liminf_{n\to\infty}\left(\int_X (2g)\,\mathrm{d}\mu - \int_X |f - f_n|\,\mathrm{d}\mu\right)$$

$$= \int_X 2g\,\mathrm{d}\mu + \liminf_{n\to\infty}\left(-\int_X |f - f_n|\,\mathrm{d}\mu\right).$$

But $\int_X (2g)\,\mathrm{d}\mu$ is finite, so

$$0 \leqslant \liminf_{n\to\infty}\left(-\int_X |f - f_n|\,\mathrm{d}\mu\right)$$

$$= -\limsup_{n\to\infty}\int_X |f - f_n|\,\mathrm{d}\mu$$

$$\leqslant 0.$$

Thus equality holds,

$$0 = \limsup_{n\to\infty}\int_X |f - f_n|\,\mathrm{d}\mu.$$

It follows that

$$\lim_{n\to\infty}\int_X |f - f_n|\,\mathrm{d}\mu = 0$$

(proving (i)).

But now

$$\left|\int_X f\,\mathrm{d}\mu - \int_X f_n\,\mathrm{d}\mu\right| = \left|\int_X (f - f_n)\,\mathrm{d}\mu\right|$$

$$\leqslant \int_X |f - f_n|\,\mathrm{d}\mu$$

$$\to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$\lim_{n\to\infty}\int_X f_n\,\mathrm{d}\mu = \int_X f\,\mathrm{d}\mu.$$

The result is proved.

In general whenever N is a set of measure zero and $f\colon X \to \mathbb{R}$ is integrable then

$$\int_X f\,\mathrm{d}\mu = \int_{X\setminus N} f\,\mathrm{d}\mu.$$

[Write $f = f^+ - f^-$,

$$\int_X f\,\mathrm{d}\mu = \int_X f^+\,\mathrm{d}\mu - \int_X f^-\,\mathrm{d}\mu$$

$$= \int_{X \setminus N} f^+ \, d\mu - \int_{X \setminus N} f^- \, d\mu$$

$$= \int_{X \setminus \mathbb{N}} f \, d\mu. \Big]$$

Question Sheet 5: $f = g$ almost everywhere, f, g integrable

$$\Rightarrow \int_E f \, d\mu = \int_E g \, d\mu \quad \forall \text{ measurable } E.$$

All the theorems we have given have versions with the words "almost everywhere" inserted. For example, if $f_n \to f$ almost everywhere on X, f_n all measurable, f measurable, and if $|f_n(x)| \leqslant h(x)$ almost everywhere and h is integrable, then

$$\lim_{n \to \infty} \int_X |f(x) - f_n(x)| \, d\mu = 0.$$

Proof of this version

Choose set N of measure zero such that $f_n(x) \to f(x) \ \forall \ x$ in $X \setminus N$.

Choose for each $k \in \mathbb{N}$, a set N_k of measure 0 such that

$$|f_n(x)| \leqslant h(x) \quad \forall \ x \in X \setminus N_k.$$

Set

$$A = N \cup \bigcup_{k=1}^{\infty} N_k.$$

For $x \in X \setminus A$ we have $|f_n(x)| \leqslant h(x) \ \forall n$ and $f_n(x) \to f(x)$ as $n \to \infty$.

On $X \setminus A$ the conditions of the dominated convergence theorem are satisfied, so

$$\lim_{n \to \infty} \int_{X \setminus A} |f_n - f| \, d\mu = 0.$$

But A is a countable union of sets of measure zero, so $\mu(A) = 0$ also, thus

$$\int_X |f_n - f| \, d\mu = \int_{X \setminus A} |f_n - f| \, d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

Note

Working with $X = \mathbb{R}$, using Lebesgue measure λ, taking $f_n = \chi_{[n, n+1]}$. Then, with $f(x) = 0$ all x, we have

$$f_n(x) \to f(x) \quad \forall \ x \text{ in } \mathbb{R},$$

and

$$0 \leqslant f_n(x) \leqslant 1 \quad \forall \ n,$$

all x, but $\int_{\mathbb{R}} f_n \, d\mu$ does not converge to $\int_{\mathbb{R}} f \, d\mu$.

(We cannot apply the Dominated Convergence Theorem because

$$\int_{[1,\infty)} 1 d\lambda = \infty.)$$

Returning to the Riemann integral:

How does it compare with Lebesgue integral?

Let us work in the interval $[0,1]$ (any bounded interval is similar). For any interval $I \subseteq [0,1]$, χ_I is both Riemann integrable and Lebesgue integrable, with the same integral.

$$\int_{[0,1]} \chi_I \, d\lambda = \int_0^1 \chi_I(x) \, dx$$

$$= \text{length of } I = \lambda(I).$$

This is also true for finite linear combinations of characteristic functions of intervals

$$\sum_{j=1}^{n} \alpha_j \chi_{I_j},$$

i.e. the Riemann integral and the Lebesgue integral agree for all step functions on $[0,1]$. However we have $\chi_{\mathbb{Q} \cap [0,1]}$ is not Riemann integrable on $[0,1]$ but is Lebesgue integrable with integral 0.

Moreover, any (proper) Riemann integrable function f on $[0,1]$ must be bounded on $[0,1]$. However if we define

$$f(x) = \begin{cases} 0 & x = 0 \\ \dfrac{1}{\sqrt{x}} & x \in (0,1] \end{cases}$$

it is not too hard (using the next theorem, and results about measures) to prove that f is Lebesgue integrable on $[0,1]$.

Facts

1. Let $f: \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue measurable (i.e. $f^{-1}([-\infty, a])$ is a Lebesgue measurable set $\forall\, a \in \mathbb{R}$) and let $g: \mathbb{R} \to \overline{\mathbb{R}}$ be any function. If g is equivalent to f (i.e. $f(x) = g(x)$ a.e. (λ)) then g is also measurable. This is because Lebesgue measure is *complete* (see question sheet 3). This result is no longer necessarily true if we used Borel measurable functions instead.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $f: X \to [0,\infty]$ be measurable. Then

$$\int_X f \, d\mu = 0$$

if and only if $f(x) = 0$ a.e..

Proof

If $f(x) = 0$ a.e., then

$$\int_X f \, d\mu = 0$$

is trivial. Conversely, suppose that

$$\int_X f \, d\mu = 0.$$

Set

$$A_n = \left\{ x \in X: \; f(x) \geqslant \frac{1}{n} \right\}.$$

Then

$$\bigcup_{n=1}^{\infty} A_n = \{x \in X: \; f(x) > 0\}.$$

Since f is non-negative,

$$0 = \int_{A_n} f \, d\mu \geqslant \frac{1}{n} \mu(A_n)$$

and so $\mu(A_n) = 0 \;\; \forall n$ (as $\frac{1}{n} > 0$, $\mu(A_n) \geqslant 0$). Thus

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = 0.$$

Since

$$\bigcup_{n=1}^{\infty} A_n = \{x \in X: \; f(x) \neq 0\}$$

this proves $f(x) = 0$ a.e. (μ).

If f is Riemann integrable on $[0,1]$ then we can find 'step functions' s_n, t_n (finite linear combinations of characteristic functions of intervals), such that $s_n(x) \leqslant f(x) \leqslant t_n(x)$ and

$$\int_0^1 f(x) \, dx = \lim_{n \to \infty} \int_0^1 s_n(x) \, dx = \lim_{n \to \infty} \int_0^1 t_n(x) \, dx.$$

(Riemann integral)

We can arrange for $s_1 \leqslant s_2 \leqslant s_3 \leqslant \ldots$ and $t_1 \geqslant t_2 \geqslant t_3 \geqslant \ldots$. (One way to do this is to divide $[0,1]$ up into 2^n intervals and define s_n, t_n using this division of the interval.)

Theorem 4.40

Let f: $[0,1] \to \mathbb{R}$ be a Riemann-integrable function. Then f is Lebesgue integrable and

$$\int_0^1 f(x)\ \mathrm{d}x = \int_{[0,1]} f\ \mathrm{d}\lambda.$$

Proof

Choose functions s_n, t_n: $[0,1] \to \mathbb{R}$ such that

$$s_1(x) \leqslant s_2(x) \leqslant \ldots \leqslant f(x) \leqslant \ldots \leqslant t_n(x) \leqslant t_{n-1}(x)$$

and such that

$$\int_0^1 f(x)\ \mathrm{d}x = \lim_{n\to\infty} \int_0^1 s_n(x)\ \mathrm{d}x$$

$$= \lim_{n\to\infty} \int_0^1 t_n(x)\ \mathrm{d}x$$

and such that all s_n, t_n are finite linear combinations of characteristic functions of intervals. Then s_n, t_n are all simple and Lebesgue measurable. Then $s_n(x), t_n(x)$ are monotone sequences.

Set

$$f_1(x) = \lim_{n\to\infty} s_n(x), \qquad f_2(x) = \lim_{n\to\infty} t_n(x).$$

We have

$$f_1(x) \leqslant f(x) \leqslant f_2(x) \quad \forall\ x \in [0,1].$$

Then f_1, f_2 are pointwise limits of Lebesgue measurable functions and hence are Lebesgue measurable. For the functions s_n, t_n we have

$$\int_{[0,1]} s_n\ \mathrm{d}\mu = \int_0^1 s_n(x)\ \mathrm{d}x \qquad \text{and} \qquad \int_{[0,1]} t_n\ \mathrm{d}\lambda = \int_0^1 t_n(x)\ \mathrm{d}x.$$

Thus

$$\int_0^1 s_n(x)\ \mathrm{d}x \leqslant \int_{[0,1]} f_1\ \mathrm{d}\lambda \leqslant \int_{[0,1]} f_2\ \mathrm{d}\lambda \leqslant \int_0^1 t_n(x)\ \mathrm{d}x.$$

So taking the limit as $n \to \infty$ we obtain

$$\int_0^1 f(x)\ \mathrm{d}x \leqslant \int_{[0,1]} f_1\ \mathrm{d}\lambda \leqslant \int_{[0,1]} f_2\ \mathrm{d}\lambda \leqslant \int_0^1 f(x)\ \mathrm{d}x.$$

Thus

$$\int_0^1 f(x)\ \mathrm{d}x = \int_{[0,1]} f_1\ \mathrm{d}\lambda = \int_{[0,1]} f_2\ \mathrm{d}\lambda.$$

But $f_2 - f_1$ is Lebesgue measurable on $[0,1]$ and non-negative and

$$\int_{[0,1]} (f_2 - f_1)\, \mathrm{d}\lambda = 0.$$

Thus $f_2 - f_1 = 0$ a.e. on $[0,1]$. Since $f_1(x) \leqslant f(x) \leqslant f_2(x)$ on $[0,1]$, we have $f(x) = f_1(x)$ a.e. on $[0,1]$. Thus $f\colon [0,1] \to \mathbb{R}$ is also Lebesgue measurable. But then

$$\int_{[0,1]} f\, \mathrm{d}\lambda = \int_{[0,1]} f_1\, \mathrm{d}\lambda = \int_{[0,1]} f_2\, \mathrm{d}\lambda = \int_0^1 f(x)\, \mathrm{d}x. \qquad \square$$

The proof on a general interval $[a,b]$ is the same. So Riemann integrable $\Rightarrow$ Lebesgue integrable with the same value of the integral.

In view of this result, we often use Riemann-style notation for Lebesgue integrals over intervals. For example, for a Lebesgue integrable function f on $[a,b]$ we may define

$$\int_a^b f(x)\, \mathrm{d}x = \int_{[a,b]} f\, \mathrm{d}\lambda\,.$$

We conclude by using our powerful convergence theory to prove a result concerning Riemann integrable functions which is extremely hard to prove by elementary means.

Let

$$f_n\colon [0,1] \to \mathbb{R}$$

continuous or, more generally, Riemann integrable,

$$|f_n(x)| \leqslant 1 \quad \forall n,$$

and suppose that $f_n(x) \to 0$ as $n \to \infty$ for each x in $[0,1]$. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = 0.$$

Proof

Use dominated convergence. $\square$