OPTIONAL REVISION SESSION
THUR JAN 11^{th}, 3PM.

5 Sequences of Functions

5.1 Pointwise and uniform convergence

Let $D \subseteq \mathbb{R}^d$, and let $f_n : D \to \mathbb{R}$ be functions $(n \in \mathbb{N})$.

We may think of the functions $f_1, f_2, f_3, \ldots$ as forming a **sequence of functions**.

$(f_n)_{n=1}^{\infty}$ or (f_n)

This is very different from a sequence of numbers but it is still possible to define the concept of a limit of such a sequence.

Such a notion is quite important.

For instance when solving differential equations or other problems one is often able to produce a sequence of approximate solutions and then needs to know in which sense the approximate solutions converges to the exact one.

We shall discuss two of the main notions of convergence of sequences of functions $f_n : D \to \mathbb{R}$.

To help us, recall the following **non-standard terminology**, introduced on Question Sheet 4.

Definition 5.1.1 Let $(\boldsymbol{x}_n) \subseteq \mathbb{R}^d$ and let $A \subseteq \mathbb{R}^d$.
We say that the set A **absorbs** the sequence $(\boldsymbol{x}_n)$ if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } \boldsymbol{x}_n \in A, \qquad (*)$$

i.e., all terms of the sequence from $\boldsymbol{x}_N$ onwards lie in the set A.

A absorbs (x_n)

$\Leftrightarrow$ there are at most finitely many n with $\underline{x_n} \notin A$, i.e.,

$$\{n \in \mathbb{N} \mid \underline{x_n} \notin A\}$$ is finite $\underbrace{\text{(or empty)}}_{\text{redundant, as } \phi \text{ is finite.}}$

redundant, as ϕ $\underline{\text{is}}$ finite.

The following is a slight variation of an exercise on Question Sheet 4.

The proof is an **exercise**.

Proposition 5.1.2 Let $a \in \mathbb{R}$ and let $(x_n) \subseteq \mathbb{R}$. Then the following statements are equivalent:

(a) the sequence (x_n) converges to a;

(b) $\boxed{\text{for all } \varepsilon > 0,}$ the closed interval $[a - \varepsilon, a + \varepsilon]$ absorbs the sequence (x_n).

Definition 5.1.3 The sequence (f_n) **converges pointwise (on D)** to the function f if, for every $\boldsymbol{x} \in D$, the sequence $(f_n(\boldsymbol{x}))_{n=1}^{\infty}$ converges to $f(\boldsymbol{x})$, i.e., $f_n(\boldsymbol{x}) \to f(\boldsymbol{x})$ as $n \to \infty$.

The notion of uniform convergence is more subtle.

To explain this, we first extend our notions of closed ball and of sets absorbing sequences.

We need to consider sets and sequences of **functions**.

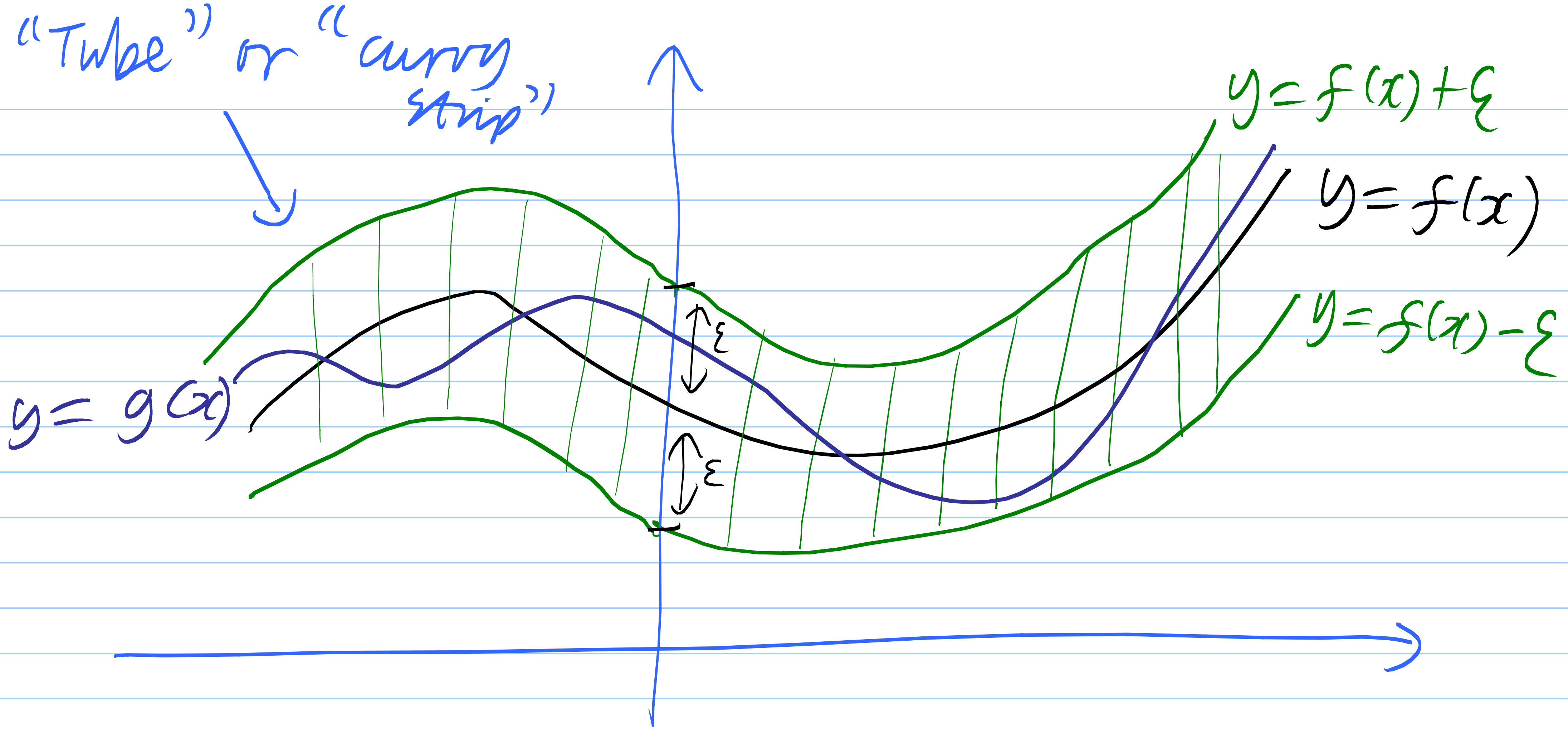
Definition 5.1.4 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \mapsto \mathbb{R}$, and let (f_n) be a sequence of functions from D to $\mathbb{R}$.

For $\varepsilon > 0$, we define the **closed ball** centred on f and with radius ε, $\bar{\boldsymbol{B}}_\varepsilon(f)$, by

$$\bar{\boldsymbol{B}}_\varepsilon(f) = \{g : D \to \mathbb{R} \mid \overbrace{g(\boldsymbol{x}) \in [f(\boldsymbol{x}) - \varepsilon, f(\boldsymbol{x}) + \varepsilon] \text{ for all } \boldsymbol{x} \in D}^{|g(x)-f(x)| \leq \varepsilon \quad (\underline{x} \in D).}\}\,.$$

Note that this closed ball is a set of **functions** from D to $\mathbb{R}$.

Gap to fill in



Here $g \in \bar{B}_\varepsilon(f)$ because (domain)

$|g(x) - f(x)| \leq \varepsilon$ for all $x \in D$

As in $(*)$ above, we say that the closed ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ **absorbs** the sequence of functions (f_n) if there exists $N \in \mathbb{N}$ such that the following condition holds:

$$\text{for all } n \geq N, \text{ we have } f_n \in \bar{\boldsymbol{B}}_\varepsilon(f), \qquad (**)$$

i.e., all terms of the sequence of functions from f_N onwards lie in $\bar{\boldsymbol{B}}_\varepsilon(f)$.

The sequence (f_n) **converges uniformly (on D)** to the function f if, for every $\varepsilon > 0$, the closed ball $\bar{\boldsymbol{B}}_\varepsilon(f)$ absorbs the sequence (f_n).

In full, this means the following:

For all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $\boldsymbol{x} \in D$, we have

$$|f_n(x) - f(x)| \leq \varepsilon\,.$$

The N in the full definition of uniform convergence depends only on ε; the same N works for all $\boldsymbol{x} \in D$.

Roughly this means that the sequences of numbers $(f_n(\boldsymbol{x}))$ converge to $f(\boldsymbol{x})$ at the same rate.

Pointwise convergence on the other hand means simply that the sequence of numbers $(f_n(\boldsymbol{x}))$ converges to $f(\boldsymbol{x})$ for each $\boldsymbol{x} \in D$.

At different points the speed of convergence could be very different.

The next result follows directly from the definitions.

(**Exercise.** Convince yourself that this is correct.)

Lemma 5.1.5 If (f_n) converges uniformly to f (on D) then it converges pointwise to f on D.

The converse of this lemma is NOT true.

It is time for some examples to illustrate this.

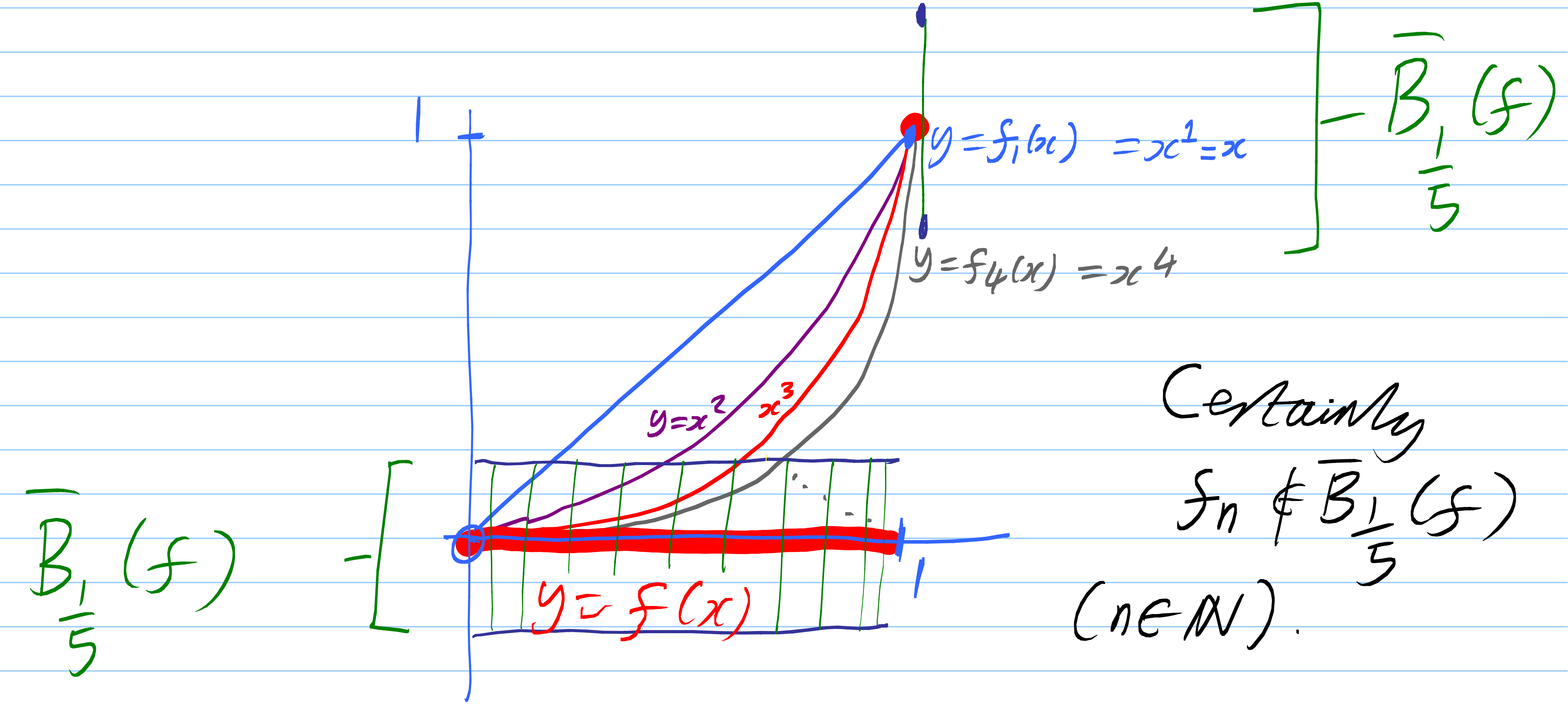
These examples will be discussed in detail in an examples class.

Examples 1) Let $D = [0, 1]$ and $f_n(x) = x^n$. Then (f_n) converges pointwise, but NOT uniformly, to the function

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Gap to fill in

There are no continuous functions at all in $\overline{B}_{\frac{1}{5}}(f)$.
[Details: exercise.]



Certainly $f_n \notin \overline{B}_{\frac{1}{5}}(f)$ $(n \in \mathbb{N})$.

Proof that $f_n \to f$ pointwise on $[0,1]$.

Case by case analysis:

Suppose $x=1$. Then $f_n(x)=1$ all n, so $f_n(x) \to 1 = f(x)$ as $n \to \infty$.

Suppose $0 \le x < 1$. Then $|x|<1$, so $f_n(x) = x^n \to 0$ as $n \to \infty$, i.e. $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$ $(=0)$.

We have checked all x in $[0,1]$, so $f_n \to f$ pointwise on $[0,1]$.

(f_n) does <u>NOT</u> converge to f <u>uniformly</u> on $[0,1]$, because e.g. $B_{\frac{1}{5}}(f)$ does not absorb (f_n).

$\boxed{\text{One bad } \varepsilon \text{ is enough!}}$

Any $\varepsilon < \frac{1}{2}$ is easily seen to be "bad". In fact you can show here that any $\varepsilon < 1$ will do.

2) Suppose $D = [0, 1/2]$ and again $f_n(x) = x^n$. Now (f_n) converges uniformly to the function which is identically 0 i.e. the function given by $f(x) = 0$ for all $x \in [0, 1/2]$ (we say it converges uniformly to 0).

Gap to fill in

$$|f_n(x)| \leq \frac{1}{2^n}, \text{ and so}$$

$$f_n \in \overline{B}_{\frac{1}{2^n}}(f).$$

Given $\varepsilon > 0$, choose N such that $2^{-N} < \varepsilon$. Then $n \geq N \Rightarrow f_n \in \overline{B}_\varepsilon(f)$.

3) Suppose $D = \mathbb{R}_+$ and $f_n : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$f_n(x) = \begin{cases} x/n & \text{if } 0 \le x \le n \\ 1 & \text{if } x > n. \end{cases}$$

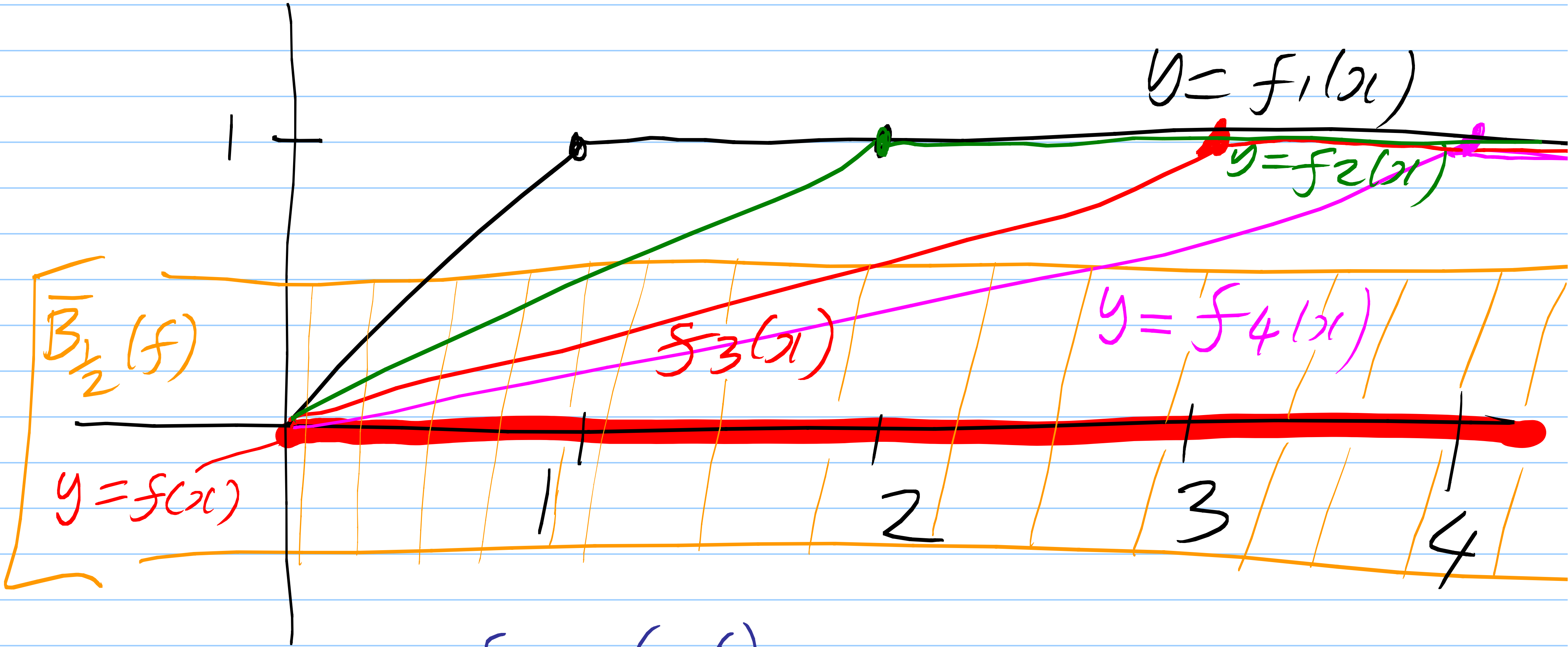
Then the sequence of functions (f_n) converges to 0 pointwise, but not uniformly, on $\mathbb{R}_+$.

Gap to fill in

What is the sequence $\left(f_n(1000000)\right)_{n=1}^{\infty}$?

Sketch functions $f_1(x), f_2(x), f_3(x), \ldots$

$f(x) = 0 \quad \text{for } x \in \mathbb{R}^+$



$$f_{10^6}(10^6) = 1$$

Here $f_n \to f = 0$ pointwise, but not uniformly.

<u>Ex.</u> Check pointwise convergence (see hints above).

Uniform convergence fails: only need one bad ε. Here e.g. $\varepsilon = \frac{1}{2}$ (or any $\varepsilon \in \,]0,1[$) has the property that

$\overline{B_\varepsilon}(f)$ does not absorb (f_n).

<u>Indeed</u>, NONE of the functions f_n are in $\overline{B_{1/2}}(f)$. (Here we only <u>need</u> infinitely many outside.)

A variety of methods to investigate the convergence of sequences of functions will be discussed in an examples class.

For an alternative approach, involving the **uniform norm**, see Question Sheet 5.

The above examples indicate that uniform convergence is much stronger than pointwise convergence and more difficult to establish.

One of the reasons that uniform convergence is important is that it has much better properties.

In the above examples, all of the functions f_n are continuous.

Unfortunately, as Example 1), shows the limit of a pointwise convergent sequence of continuous functions need not be continuous.

For uniformly convergent sequences the situation is much better.

Theorem 5.1.6 Let D be a non-empty subset of $\mathbb{R}^d$, let $f : D \to \mathbb{R}$, and let (f_n) be a sequence of continuous functions from D to $\mathbb{R}$. Suppose that the functions f_n converge uniformly on D to f. Then f must also be continuous.

Proof. Omitted. See Wade, Theorem 7.9, if you are interested. $\square$

From now on **you may quote this theorem as standard**.

It may be summarized as follows:

Uniform limits of sequences of continuous functions are always continuous.

This theorem is one of the main reasons why uniform convergence is so important.

It sometimes gives you a quick way to see that certain pointwise convergent sequences do not converge uniformly.

If all the f_n are continuous but f isn't then your sequence cannot converge uniformly! (e.g. Example 1)

However, this trick does not always work.

Example 3) shows that a pointwise but non-uniform limit can sometimes be continuous.

We conclude this section with some additional standard facts which can sometimes help to establish that uniform convergence fails.

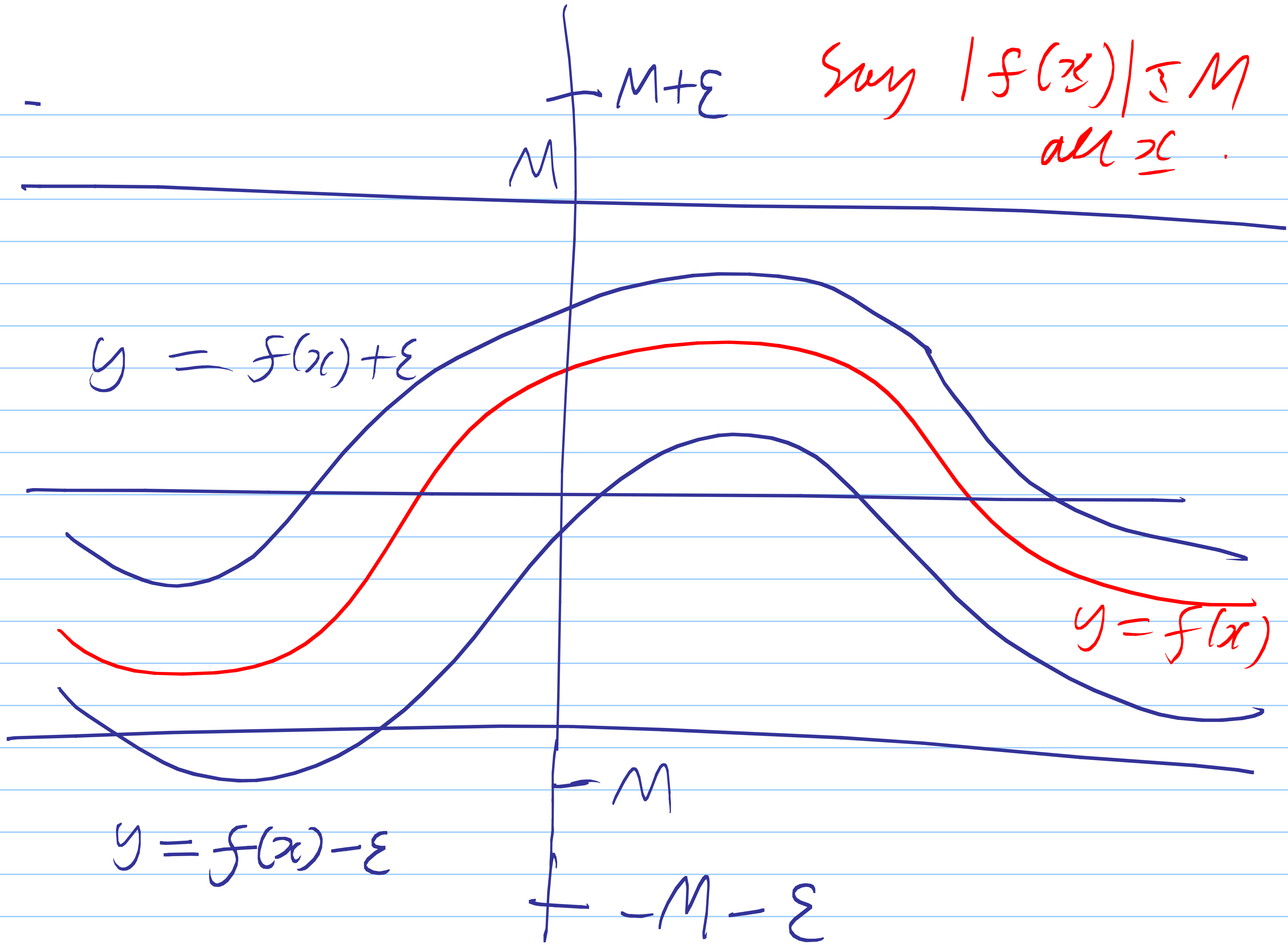
The proofs of these are an **exercise**.

Proposition 5.1.7 Let D be a non-empty subset of $\mathbb{R}^d$ and suppose that f is a **bounded** function from D to $\mathbb{R}$, i.e., $f(D)$ is a bounded subset of $\mathbb{R}$.

(i) Let $\varepsilon > 0$ and suppose that $g \in \bar{\boldsymbol{B}}_\varepsilon(f)$. Then g is also bounded on D.

(ii) Let (f_n) be a sequence of functions from D to $\mathbb{R}$. Suppose that all of the functions f_n are unbounded on D. Then (f_n) can not converge uniformly on D to f.

Say $|f(x)| \le M$ all x.



Another way to state this last result is:

It is impossible for a sequence of UNBOUNDED, real-valued functions from D to $\mathbb{R}$ to converge uniformly on D to a BOUNDED real-valued function.

Since $g \in \bar{\boldsymbol{B}}_\varepsilon(f) \Leftrightarrow f \in \bar{\boldsymbol{B}}_\varepsilon(g)$, a similar proof shows the following:

It is impossible for a sequence of BOUNDED functions from D to $\mathbb{R}$ to converge uniformly on D to an UNBOUNDED real-valued function.

Chapter 5 Summary

- Definitions and examples of pointwise and uniform convergence of sequences of functions.
- A uniform limit of continuous functions is continuous.
- Elementary results which help when investigating uniform convergence.