# G13MIN Measure and Integration: introductory material

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## 1 Review of the theory of mathematical analysis from earlier modules

The material in this chapter is mostly taken from the syllabus of **G1BMAN: Mathematical Analysis** and the first year Core. It is **essential** that you read all sections of this chapter except for those which are marked as optional. It will be assumed that you are familiar with this material, and so you should revise carefully any parts which you do not feel fully confident about. **There will be a Question and Answer Session on this material in the third lecture.**

### 1.1 Standard notation and terminology

#### 1.1.1 Familiar sets

You should by now be very familiar with the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$. However, as different authors disagree over whether $0$ is in $\mathbb{N}$, note that **throughout this module** it is not, and so

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

You should be familiar with all the different types of intervals in $\mathbb{R}$: these intervals are bounded or unbounded, and are commonly described as open, closed or half-open, though we shall see that this last term can be a little misleading. As there is some variation in the notation and terminology used for intervals, we clarify this now.

The empty set counts as an interval, and is denoted by $\emptyset$. Subsets of $\mathbb{R}$ with exactly one element (**single-point sets**) are also intervals. The intervals mentioned so far may be described as **degenerate** intervals.

Intervals which have more than one point are **non-degenerate** intervals or **intervals of positive length**.

Non-degenerate, bounded intervals always have two end-points $a < b$ where $a$ and $b$ are in $\mathbb{R}$. The four types of non-degenerate, bounded interval are then the **closed interval** $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, the open interval $]a, b[= (a, b) = \{x \in \mathbb{R} : a < x < b\}$ and the **half-open intervals** $[a, b[= [a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $]a, b] = (a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. Note that the degenerate intervals may also be denoted in this form by allowing $a = b$: for example $[a, a]$ is the single-point set $\{a\}$.

The remaining intervals are unbounded, and the notation involves using the symbols $-\infty$ and $\infty$ (or $+\infty$, which is regarded as the same as $\infty$ in the context of $\mathbb{R}$). For example, $(0, \infty) = ]0, \infty[$ is the set of strictly positive real numbers, and we have $\mathbb{R} = ]-\infty, \infty[= (-\infty, \infty)$: this is the biggest possible interval in $\mathbb{R}$.

**Please note** that there are many subsets of $\mathbb{R}$ which are not intervals and which can not be described as open, closed or half-open (e.g $\mathbb{Q}$).

### 1.1.2 Bounded subsets of $\mathbb{R}$

You should be familiar with the following fact about $\mathbb{R}$.

**Proposition 1.1** Let $E$ be a non-empty subset of $\mathbb{R}$ which is bounded above, then $E$ has a **least upper bound** (also called the **supremum** of $E$). Similarly, if $E$ is non-empty and bounded below, then $E$ has a **greatest lower bound** (or **infimum**).

We will use the terms infimum and supremum throughout this module, and use the following notation.

**Notation.** Let $E$ be a non-empty subset of $\mathbb{R}$. If $E$ is bounded above, then we denote the supremum of $E$ (the least upper bound of $E$) by $\sup(E)$. If $E$ is bounded below then we denote the infimum of $E$ (the greatest lower bound of $E$) by $\inf(E)$.

### 1.1.3 Set operations

#### Set difference and complements

We denote the set difference of two sets $A$ and $B$ by $A \setminus B$. So, for example, the set of irrational real numbers is $\mathbb{R} \setminus \mathbb{Q}$. We can also use notation for complements: if we are working with subsets of some particular set $X$, then for $A \subseteq X$ we may denote $X \setminus A$ by $A^c$, the **complement of $A$ in $X$**. Of course complementation depends on the set $X$ you are working with. With this notation, as a subset of $\mathbb{R}$, the set of irrational numbers is $\mathbb{Q}^c$.

#### Unions, intersections and de Morgan's laws

You should be familiar with the notions of union and intersection. This includes:
the union and intersection of two sets $A$ and $B$

$$A \cup B \quad \text{and} \quad A \cap B;$$

finite unions and intersections

$$\bigcup_{j=1}^{n} B_j \quad \text{and} \quad \bigcap_{j=1}^{n} B_j$$

for sets $B_1, B_2, \ldots, B_n$ and arbitrary unions and intersections (often involving infinitely many sets)

$$\bigcup_{i \in I} A_i \quad \text{and} \quad \bigcap_{i \in I} A_i$$

where the $A_i$ are sets $(i \in I)$ and $I$ is some indexing set (which is usually non-empty).

In the case where $I = \mathbb{N}$, we often use the notation

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i \in \mathbb{N}} A_i \quad \text{and} \quad \bigcap_{i=1}^{\infty} A_i = \bigcap_{i \in \mathbb{N}} A_i \,.$$

If $I = \emptyset$ the definition of $\bigcap_{i \in I} A_i$ is problematic: in the context where you are looking at subsets of some particular set $X$, many authors use the convention that $\bigcap_{i \in \emptyset} A_i = X$.

An important connection between intersections and unions is given by **de Morgan's laws**, which we summarise in the following proposition.

**Proposition 1.2** With notation as above, suppose that all of the sets concerned are subsets of some particular set $X$ in which we wish to take complements. For finite intersections and unions we have

$$\left(\bigcup_{j=1}^{n} B_j\right)^c = \bigcap_{j=1}^{n} B_j^c \quad \text{and} \quad \left(\bigcap_{j=1}^{n} B_j\right)^c = \bigcup_{j=1}^{n} B_j^c$$

and for arbitrary unions and intersections we have

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c\,.$$

The following important result tells us a lot about the special nature of closed intervals in $\mathbb{R}$. (It is not valid for the other types of interval.)

**Proposition 1.3 Nested Intervals Theorem** Let $(a_n)$ and $(b_n)$ be sequences of real numbers with $a_n \le b_n$ and such that

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots.$$

(so the sequence $(a_n)$ is non-decreasing and the sequence $(b_n)$ is non-increasing). Then there is at least one real number $c$ with the property that, for all $n \in \mathbb{N}$, $c \in [a_n, b_n]$. We then have $c \in \bigcap_{n\in\mathbb{N}}[a_n, b_n]$, and so

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] \ne \emptyset.$$

If $\lim_{n\to\infty}(b_n - a_n) = 0$ then this point $c$ is unique,

$$\bigcap_{n\in\mathbb{N}} [a_n, b_n] = \{c\},$$

and $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

### Cartesian products and Cartesian powers

Given two sets $A$ and $B$ we may form the **Cartesian product**

$$A \times B = \{(x, y) : x \in A, y \in B\}.$$

More generally, for $n \in \mathbb{N}$ and sets $A_1$, $A_2$, $\ldots A_n$, we may form the Cartesian product

$$\prod_{j=1}^{n} A_j = \{(x_1, x_2, \ldots, x_n) : x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n\}\,.$$

and this may be regarded as the same thing as $A_1 \times A_2 \times \cdots \times A_n$ (by suppressing any extra brackets in whichever version of the latter you take).

For a Cartesian product to be empty, it is necessary and sufficient for **at least** one of the sets involved in the product to be empty.

In the special case where all of the sets in the product are the same, we form **Cartesian powers**: if $A_1 = A_2 = \ldots = A_n = X$, then

$$X^n = \prod_{j=1}^{n} A_j = \{(x_1, x_2, \ldots, x_n) : x_j \in X \text{ for } j = 1, 2, \ldots, n\}.$$

In particular we have $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, etc.., while we usually regard $\mathbb{R}^1$ as being the same as $\mathbb{R}$.

For $n \in \mathbb{N}$, you may regard elements of $\mathbb{R}^n$ as either vectors or points. We will not make a distinction in this module. Typically we may work with elements $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, etc.. In this case we can perform the usual vector operations, including forming $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$.

### 1.1.4 Sequences and subsequences

Let $X$ be a set. We use the notation $(x_n)_{n=1}^{\infty} \subseteq X$ to mean that $x_1, x_2, x_3, \ldots$ is a sequence of elements of $X$. If there is no danger of ambiguity we will often shorten this to

$$(x_n) \subseteq X.$$

A **subsequence** of $(x_n)$ is a sequence of the form $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ (or $(x_{n_k})_{k=1}^{\infty}$) with $n_1 < n_2 < n_3 < \ldots$ (and where all the $n_k$ are in $\mathbb{N}$). If we say that $(y_n)$ is a subsequence of $(x_n)$, it means that there is a strictly increasing sequence of positive integers $(n_k)_{k=1}^{\infty}$ as above with $y_k = x_{n_k}$ for $k = 1, 2, 3, \ldots$.

## 1.2 Topology of $\mathbb{R}^n$ ($n$-dimensional Euclidean space)

### 1.2.1 The Euclidean norm and distance in $\mathbb{R}^d$

Recall that the Euclidean distance in $\mathbb{R}^n$ is defined in terms of the **Euclidean norm on** $\mathbb{R}^n$ which we denote by $\|\cdot\|_2$: for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^d$,

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{j=1}^{n} |x_j|^2}$$

(the non-negative square root). For $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ it makes sense to subtract one from the other and then the **Euclidean distance** between $\mathbf{x}$ and $\mathbf{y}$ is simply

$$\|\mathbf{x} - \mathbf{y}\|_2 = \|\mathbf{y} - \mathbf{x}\|_2 = \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2}.$$

You may think of this as the distance obtained using Pythagoras's theorem.

We will meet several other norms and the distances they induce in this module.

### 1.2.2 Sequences and series in $\mathbb{R}$ and in $\mathbb{R}^n$

You should be familiar with the **usual** notions of convergence of sequences and series in $\mathbb{R}$ and in $\mathbb{R}^n$.

**Definition 1.4** Let $(x_n) \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Then we say $(x_n)$ **converges** to $x$ in $\mathbb{R}$ if, for every positive real number $\varepsilon$, from some term onwards the sequence $x_n$ stays within the interval $(x-\varepsilon, x+\varepsilon)$. Equivalently, we may state this definition (more formally) in terms of $\varepsilon$ and $N$: $(x_n)$ converges to $x$ if, for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that, for all $n \geq N(\varepsilon)$, $|x_n - x| < \varepsilon$. If $(x_n)$ converges to $x$, we write $x_n \to x$ **as** $n \to \infty$, or $\lim_{n\to\infty} x_n = x$. We also say that the **limit as** $n \to \infty$ of the sequence $(x_n)$ is $x$. If the sequence $(x_n)$ does not converge, then we say that $(x_n)$ **diverges**. In this case the notation $\lim_{n\to\infty} x_n$ does not mean anything. If we ever write

$$\lim_{n\to\infty} x_n = x$$

we always mean that the sequence $(x_n)$ converges and the limit of the sequence is $x$.

The following is one of the most basic results concerning sequences in $\mathbb{R}$.

**Proposition 1.5 (Monotone Sequence Theorem)** Let $(x_n)$ be a bounded sequence of real numbers. Suppose that $(x_n)$ is monotone (i.e. $x_1 \leq x_2 \leq x_3 \leq \cdots$ or $x_1 \geq x_2 \geq x_3 \geq \cdots$). Then $(x_n)$ is a convergent sequence in $\mathbb{R}$.

**Definition 1.6** A sequence $(\mathbf{x}_k) \subseteq \mathbb{R}^n$ **converges** to $\mathbf{a} \in \mathbb{R}^n$ if and only if , with the usual notion of convergence in $\mathbb{R}$ (as above), $\lim_{k\to\infty} \|\mathbf{x}_k - \mathbf{a}\|_2 = 0$. In this case, we may write $\mathbf{x}_k \to \mathbf{a}$ as $k \to \infty$ or $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$. As above, if a sequence does not converge then it **diverges**.

We have the following standard result.

**Proposition 1.7** Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and let $(\mathbf{x}_k) \subseteq \mathbb{R}^n$, where $\mathbf{x}_k = (x_{k1}, x_{k2}, \ldots, x_{kn})$ for $k \in \mathbb{N}$. With the above definition of convergence in $\mathbb{R}^n$, $(\mathbf{x}_k)$ converges to $\mathbf{a}$ in $\mathbb{R}^n$ if and only if, for all $j \in \{1, 2, \ldots, n\}$, $x_{kj} \to a_j$ as $k \to \infty$.

When we look at other notions of distance, we will see that they give rise to different notions of convergence. We shall always describe the type of convergence in $\mathbb{R}$ and in $\mathbb{R}^n$ given above as the **usual** type of convergence, and will be the type of convergence we use **unless otherwise specified**.

There are various forms of the Algebra of Limits available. Here is one of the most useful.

**Proposition 1.8 (The algebra of limits in $\mathbb{R}$)** Let $(x_n)$ and $(y_n)$ be convergent sequences of real numbers, with $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$, then

(i) $x_n + y_n \to x + y$ as $n \to \infty$,

(ii) $x_n y_n \to xy$ as $n \to \infty$,

(iii) if $y \neq 0$ then $x_n/y_n \to x/y$ as $n \to \infty$.

**Remark:** strictly speaking, in (iii) we should be worried about division by zero. Just because $y \neq 0$ does not mean that none of the $y_n$ are zero. However, since $\lim_{n\to\infty}(y_n)$ is not zero, it follows that from some term onwards we will have $y_n \neq 0$. From this term onwards, $x_n/y_n$ makes sense, and gives a sequence in $\mathbb{R}$ which converges to $x/y$.

The following well-known false statement and proof illustrates the danger of assuming that sequences are convergent.

**False theorem** Let $(x_n) \subseteq \mathbb{R}$ and suppose that $\lim_{n\to\infty}(x_n^2) = 1$. Then

$$\lim_{n\to\infty}(x_n) = 1 \text{ or } -1.$$

**False proof** By the algebra of limits,

$$\lim_{n\to\infty}(x_n^2) = \left(\lim_{n\to\infty}(x_n)\right)^2,$$

i.e. $(\lim_{n\to\infty}(x_n))^2 = 1$. The result follows. □

The fact that this statement is false is shown by using, for example, the example $x_n = (-1)^n$. This sequence satisfies the conditions of the false theorem, but it does not converge at all, and this is where the false proof breaks down.

the following result helps get around this type of problem.

**Proposition 1.9 (Squeeze rule, or Sandwich Theorem)** Let $(a_n)$, $(b_n)$, $(c_n)$ be sequences of real numbers such that, for all $n \in \mathbb{N}$,

$$a_n \leq b_n \leq c_n.$$

Suppose that the sequences $(a_n)$ and $(c_n)$ are both convergent, and that they have the same limit. Then the sequence $(b_n)$ also converges, and

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(b_n) = \lim_{n\to\infty}(c_n).$$

When we look at bounded sequences, one key result is the Bolzano-Weierstrass Theorem.

**Proposition 1.10 (The Bolzano-Weierstrass Theorem)** Let $(x_n)$ be a bounded sequence of real numbers. Then $(x_n)$ has at least one convergent subsequence.

**Remark.** Of course, the sequence $(x_n)$ need not itself converge, as the sequence $x_n = (-1)^n$ shows. The same example shows that a sequence may have different convergent subsequences with different limits. For this example, some subsequences converge to $1$, some converge to $-1$, and the remaining subsequences diverge.

The same result is valid in $\mathbb{R}^n$.

**Proposition 1.11** Every bounded sequence in $\mathbb{R}^n$ has at least one convergent subsequence.

### 1.2.3 Open sets and closed sets in $\mathbb{R}$ and $\mathbb{R}^n$

You should be familiar with the definitions and properties of open balls, closed balls, open sets and closed sets in $\mathbb{R}$ and in $\mathbb{R}^n$. In particular, you should be familiar with the following results about open sets and closed sets. **Remember: 'closed' is not the same thing as 'not open'.**

**Proposition 1.12** (i) The sets $\emptyset$ and $\mathbb{R}$ are open in $\mathbb{R}$ and they are also closed in $\mathbb{R}$.

(ii) The intersection of any two (or finitely many) open subsets of $\mathbb{R}$ is again an open subset of $\mathbb{R}$.

(iii) Taking the union of **any** collection of open subsets of $\mathbb{R}$ always gives another open subset of $\mathbb{R}$.

(iv) The union of any two (or finitely many) closed subsets of $\mathbb{R}$ is again a closed subset of $\mathbb{R}$.

(v) Taking the intersection of **any** (non-empty) collection of closed subsets of $\mathbb{R}$ always gives another closed subset of $\mathbb{R}$.

As usual, exactly the same result is true if you replace $\mathbb{R}$ with $\mathbb{R}^n$ throughout this result. We will revisit this material again in a more general setting in this module.

### 1.2.4 Sequential compactness

There are several different equivalent definitions of compactness for subsets of $\mathbb{R}$ and of $\mathbb{R}^n$ which you have met in earlier modules. Some caution is needed, as not all of these definitions generalize in a satisfactory way. The main definition of compactness that you met in the module G1BMAN was given in terms of convergent subsequences. We shall call that type of compactness **sequential compactness**. We give the definition for $\mathbb{R}^n$: of course this is also valid for $\mathbb{R}$, since $\mathbb{R}$ may be regarded as the same as $\mathbb{R}^1$.

**Definition 1.13** Let $A$ be a subset of $\mathbb{R}^n$. Then $A$ is **sequentially compact** if, for every sequence $(a_k)$ of points of $A$, $(a_k)$ has at least one subsequence which converges to a point of $A$.

The most important result here is a characterization of sequential compactness for subsets of $\mathbb{R}^n$ (this is closely related to the Bolzano-Weierstrass Theorem).

**Proposition 1.14 (Heine-Borel theorem, sequential compactness version)** Let $A$ be a subset of $\mathbb{R}^n$. Then $A$ is sequentially compact if and only if it is both closed and bounded in $\mathbb{R}^n$.

## 1.3 Functions

You should be familiar with the standard definitions and the theory of limits, continuity, differentiation and integration for functions defined on subsets of $\mathbb{R}^n$. In particular you should be familiar with the following results.

**Proposition 1.15 (Boundedness theorem, sequential compactness version)** Let $A$ be a non-empty, sequentially compact subset of $\mathbb{R}^n$ and let $f$ be a continuous, real-valued function defined on $A$. Then Then $f$ is bounded on $A$ and moreover $f$ attains both a minimum and a maximum value on $A$, i.e. there are $a_1$ and $a_2$ in $A$ such that, for all $x \in A$,

$$f(a_1) \leq f(x) \leq f(a_2).$$

**Proposition 1.16 (Intermediate Value Theorem)** Let $a$ and $b$ be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for every $c$ between $f(a)$ and $f(b)$ (inclusive) there exists at least one point $d$ in $[a, b]$ such that $f(d) = c$.

**Proposition 1.17 (Mean Value Theorem)** Let $a$ and $b$ be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that $f$ is differentiable on $(a, b)$. Then there exists at least one point $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Geometrical interpretation:** at some point between $a$ and $b$ the tangent to the curve must be parallel to the straight line joining the point $(a, f(a))$ to the point $(b, f(b))$.

**Proposition 1.18** (Fundamental Theorem of Calculus) Let $a$ and $b$ be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Then $F$ is differentiable on $[a, b]$ and, for all $x \in [a, b]$, $F'(x) = f(x)$.

You should also be familiar with various forms of convergence for sequences of functions: in particular we shall revisit the notion of **uniform convergence** in this module.