A non-Borel set

Using transfinite induction (which is beyond the scope of this module) one can show that the cardinality of the collection $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ is the same as the cardinality of $\mathbb{R}$.

Since this is strictly less than the cardinality of $\mathcal{P}(\mathbb{R})$, it follows that there are very many non-Borel sets.

See books for more details.

Here we show how to find a non-Borel set by choosing one point from each of an appropriate set of equivalence classes of elements of $[0, 1]$.

The word 'choose' here indicates that we are making use of the Axiom of Choice here: see books for more on this.

Notation Let A be a subset of $\mathbb{R}$ and let $c \in \mathbb{R}$. Then we denote by $A + c$ (or $c + A$) the **translate of** A **by** c, i.e.

$$A + c = \{x + c : x \in A\} .$$

The next lemma shows how we can prove facts about Borel sets without having a precise description these sets.

It says that every translate of a Borel set is still a Borel set.

Recall that our standard semi-ring of subsets of $\mathbb{R}$ is $P = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$, and that we know that $\mathcal{B} = \mathcal{F}_{\mathbb{R}}(P)$.

Lemma 1 For every $E \in \mathcal{B}$ and every $c \in \mathbb{R}$, $E + c$ is also in $\mathcal{B}$.

We will need properties of Lebesgue outer measure on $\mathbb{R}$.

These properties will be discussed in more detail in Section 5.

First, define $\mu : P \to [0, \infty)$ by $\mu((a, b]) = b - a$.

Thus μ measures the length of intervals in P.

Definition 2 The **Lebesgue outer measure**, λ^*, on $\mathbb{R}$ is the function from $\mathcal{P}(\mathbb{R})$ to $[0, \infty]$ defined as follows. For $E \subseteq \mathbb{R}$, set

$$S_E = \left\{ \sum_{n=1}^{\infty} \mu(I_n) : I_1, I_2, \cdots \in P, E \subseteq \bigcup_{n=1}^{\infty} I_n \right\} .$$

Thus S_E is the set of all possible sums of lengths of sequences of intervals $I_n \in P$ which cover E.
Then

$$\lambda^*(E) = \inf S_E ,$$

the infimum of all these possible sums.

For more details on the following proposition, see Section 5.

Proposition 3 Lebesgue outer measure on $\mathbb{R}$, λ^*, has the following properties.

(a) (Monotonicity) Let A and B be subsets of $\mathbb{R}$ with $A \subseteq B$. Then $\lambda^*(A) \leq \lambda^*(B)$.

(b) (Translation invariance) For all $A \subseteq \mathbb{R}$ and all $c \in \mathbb{R}$ we have $\lambda^*(A + c) = \lambda^*(A)$.

(c) We have $\lambda^*(\emptyset) = 0$ (more generally, $\lambda^*(S) = 0$ for every countable subset S of $\mathbb{R}$).

(d) (Countable additivity on the Borel sets) For every sequence $A_1, A_2, A_3, \ldots$ of pairwise disjoint Borel subsets of $\mathbb{R}$, we have

$$\lambda^* \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \lambda^*(A_n) \, .$$

(e) (Correct length for closed intervals) For a and b in $\mathbb{R}$ with $a \leq b$ we have $\lambda^*([a, b]) = b - a$.

You may assume these standard properties of λ^* throughout Sections 3 and 4, but NOT in Section 5, where these properties will finally be established.

You should convince yourself at the end of the module that our arguments are not circular!

We are now able to prove the existence of a non-Borel set.

The set we describe below is often called a **non-measurable** set, for reasons that will become clear in Section 5.

Example 4 (For more details, see Section 5.)

Define an equivalence relation on $[0, 1]$ by $x \sim y$ if (and only if) $x - y \in \mathbb{Q}$.

It is clear that this is an equivalence relation.

This equivalence relation partitions $[0, 1]$ into equivalence classes.

We may form a set E by choosing exactly one point from each of these equivalence classes (remember that these equivalence classes are pairwise disjoint).

From the choice of E it follows that the sets $E + q$ $(q \in \mathbb{Q})$ are pairwise disjoint and, for all $y \in [0, 1]$ there is a (unique) $q \in \mathbb{Q} \cap [-1, 1]$ such that $y \in E + q$.

Since $\mathbb{Q} \cap [-1, 1]$ is countable, we may choose a sequence (q_k) such that every element of $\mathbb{Q} \cap [-1, 1]$ appears exactly once in this sequence.

Consider the set

$$A = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (E + q) = \bigcup_{k=1}^{\infty} (E + q_k) \, .$$

From above, we see that $[0, 1] \subseteq A$. It is also clear that $A \subseteq [-1, 2]$.

Thus, by monotonicity of λ^*, we have $1 \leq \lambda^*(A) \leq 3$.

Note also that $\lambda^*(E + q_k) = \lambda^*(E)$ for all $k \in \mathbb{N}$.

Suppose, for contradiction, that $E \in \mathcal{B}$.

Then, $E + q_k \in \mathcal{B}$ for all k, and **as these sets are pairwise disjoint**, we would have

$$\lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(E + q_k)\,.$$

However, as noted above, all of the non-negative numbers $\lambda^*(E + q_k)$ are equal, and so this sum must either be 0 or $+\infty$.

This contradicts the fact that $1 \leq \lambda^*(A) \leq 3$.

Thus E is not a Borel set.

It is now clear that there is no satisfactory way to assign a 'total length' to this set E: assuming desirable properties such as those possessed by λ^* results in the loss of countable additivity on the translates $E + q_k$.

It is easy to see that the set $E \times [0, 1]$ leads to similar problems for area in $\mathbb{R}^2$ (etc.)