G13MTS: Metric and Topological Spaces
Blow-by-blow account of the module

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Other staff involved: None

Lecture 1: Module information, including details of assessment. General description of the nature and content of the module.
Chapter 1. Review of the theory of mathematical analysis from earlier modules: handout, to be read before Lecture 3's question and answer session.

Lecture 2: **Chapter 2. Introduction to metric spaces**. Definitions of metric and pseudometric. Metric spaces and pseudometric spaces. The zero (indiscrete) pseudometric on any set. The **usual metrics** on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$ (Euclidean metric on $\mathbb{R}^n$, denoted by d_2) and on subsets of these. Convergence and divergence of sequences in (pseudo)metric spaces: usual limit notation available. In pseudometric spaces, limits need not be unique. Stated: limits in metric spaces are unique when they exist.

Lecture 3: Question and answer session on material from G1BMAN.
Proof that limits in metric spaces are unique when they exist. Standard metrics on $\mathbb{R}^n$: d_∞, d_1, d_2 (Euclidean metric) and, more generally, d_p for $p \in [1, \infty)$. Proof that d_1 is a metric. For d_∞ see the first question sheet. The metric d_2 is just the usual Euclidean distance. For general p, the proof of the triangle law for d_p is harder and needs Minkowski's Inequality: see books if interested.

Lecture 4: The discrete metric (on any set). Convergence/divergence of a given sequence depends on which set you are working in and which metric you are using on it. Standard metrics on $C[0, 1]$ (distance between two functions): d_∞, d_1, d_2 and (more generally) d_p for $p \in [1, \infty)$. The metric d_1 is sometimes called the **area** metric; d_∞ is sometimes called the **uniform** or **sup norm** metric (norms are discussed below) and the corresponding type of convergence is **uniform convergence** of a sequence of functions (as in G1BMAN). The jungle river metric d_J on $\mathbb{R}^2$. Norms on vector spaces. Metrics induced by norms.

Lecture 5: Standard norms on $\mathbb{R}^n$ and on $C[0, 1]$ (corresponding to the standard metrics above). Open balls in metric spaces: notation $B_X(x, r)$ for clarity, or $B(x, r)$ if there is no ambiguity. Examples of open balls in various metric spaces.

Lecture 6: More examples of open balls. Convergence of sequences in terms of open balls. Functions between sets: image $f(E)$, pre-image $f^{-1}(F)$, injections, surjections, bijections, inverse functions. Good behaviour of pre-image with respect to intersections, unions, complements and set differences. Image only behaves well with respect to unions.

Lecture 7: Definitions (for metric spaces, in terms of open balls, ε-δ): continuity at a point; discontinuity at a point; continuous functions; discontinuous functions. Uniform continuity and Lipschitz continuity for functions between metric spaces: definitions and examples. (Instead of 'Lipschitz continuous', some authors say that such a function f 'satisfies a Lipschitz condition of order 1').

Lecture 8: For fixed x_0 in a metric space (X, d), the function $x \mapsto d(x, x_0)$ is Lipschitz continuous from X to $\mathbb{R}^+$. Equivalent definitions of continuity and uniform continuity in terms of sequences in metric spaces. The set of continuous, real-valued functions on a metric space X, (denoted by $C(X)$). Pointwise addition, multiplication etc. for functions in $C(X)$.

Lecture 9: Composition of continuous functions. New metrics from old: metrics on subsets (the subspace metric $\tilde{d}$ induced by a metric d); sums and scalar multiples of metrics etc. Open sets in metric spaces: definition. Examples of open subsets of $[0,1]$.

Lecture 10: Examples of open sets in metric spaces. In particular, $\emptyset$ and X are always open in X. Every open ball is an open set but usually not every open set is an open ball. Finite intersections of open sets are open. Arbitrary unions of open sets are still open. Summary of properties of open sets. Characterization of open balls and open sets with respect to the discrete metric and the indiscrete pseudometric.

Lecture 11: **Chapter 3. Topological Spaces**. Topologies and topological spaces. The discrete topology and the indiscrete topology. The topology induced by a metric: every metric space is a topological space. Pseudometrics also induce topologies. In particular, the discrete metric induces the discrete topology, where every subset is open and the indiscrete pseudometric induces the indiscrete topology where only $\emptyset$ and X are open in X. Metrizable and pseudometrizable topological spaces. Other examples: Hausdorff topological spaces. Closed sets in topological spaces. Examples of closed sets, clopen (open and closed) sets and sets which are neither open nor closed. The term **half-open** is misleading, even when it makes sense. In metric spaces, and more generally in Hausdorff spaces, single-point sets are closed. Finite unions and arbitrary intersections of closed sets are closed (see question sheet for details).

Lecture 12: The interior, closure and boundary of a subset Y of a topological space X, with notation $\operatorname{int} Y$, $\overline{Y}$ (or $\operatorname{clos} Y$) and ∂Y. Examples. Connections between interior, closure, complement and boundary. Neighbourhoods of points. Dense subsets of a topological space: several equivalent definitions.

Lecture 13: Characterisations of neighbourhood, interior, closure and boundary in metric spaces, in terms of open balls. Characterisations of closure and closed sets in metric spaces in terms of sequences. Characterisation of continuous functions between metric spaces in terms of pre-images of open sets or of closed sets. Continuous functions between topological spaces: defined in terms of pre-images of open sets. Conditions equivalent to continuity in terms of closed sets.

Lecture 14: Composition of continuous functions between topological spaces. Continuity in terms of neighbourhoods. Homeomorphisms and homeomorphic topological spaces. Isometries between metric spaces. Isometric metric spaces. Examples.

Lecture 15: More examples. Topological properties as opposed to metric space properties. Equivalent metrics and uniformly equivalent metrics. Examples.

Lecture 16: Separation conditions: Hausdorffness (revised), regularity and normality. Note that some authors include Hausdorffness in the definitions of regular and normal, but we do not. Distance from a non-empty subset of a metric space to a point of the metric space. Characterisation of closure in terms of distance from a point to a set. Discussion of the problem associated with Urysohn's Lemma: necessary conditions for the separation of two subsets E and F of a topological space X using a function $f \in C(X)$.

Lecture 17: Urysohn's Lemma for normal topological spaces (statement but not proof). Urysohn's Lemma for metric spaces. Metric spaces are normal (exercise). Bases for topologies (also called analytic bases, and note that some authors use the term basis where we use the term base). Examples. Sub-bases.

Lecture 18: Every collection of subsets of X is a (synthetic) sub-base for some topology on X (using next result). Necessary and sufficient conditions on a collection of subsets of X to ensure that it is a (synthetic) base for some topology on X. (Proof completed next lecture.)

Lecture 19: Conclusion of proof of characterization of synthetic bases. Neighbourhood bases: definition and examples. Continuity in terms of bases and neighbourhood bases.
Chapter 4. Subspaces, quotients and products. Definition of the subspace topology. Connection with the subspace metric.

Lecture 20: Given a topological space X, a set Y and a function q from X to Y, how to find topologies on Y such that q is continuous: the quotient topology on Y induced by q is the strongest such topology. Universal property of quotient spaces: see Question Sheet 4. Quotient topologies and equivalence relations: 'gluing topological spaces together'. In particular, $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle $\mathbb{T}$. (Details of the homeomorphism claims are best checked later using compactness methods.)

Lecture 21: The product topology (also called the weak topology) on $X \times Y$, (for topological spaces X and Y) defined using the standard synthetic base. The product topology is the weakest topology on $X \times Y$ such that the coordinate projections are continuous. Products of finitely many topological spaces. Brief discussion of infinite products: see books for details.

Lecture 22: Standard product metrics d_1, d_2, d_∞ on a product of two metric spaces (X, d_X) and (Y, d_Y). These three standard metrics are uniformly equivalent to each other, and, with respect to these metrics on $X \times Y$, $(x_n, y_n) \to (x, y)$ in $X \times Y$ as $n \to \infty$ if and only if $x_n \to x$ in X as $n \to \infty$ and $y_n \to y$ in Y as $n \to \infty$. If X and Y are metric spaces, then each of these standard metrics induces the product topology on $X \times Y$. These results extend to finite products. (Infinite products are beyond the scope of this module.) The special cases of $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}^n$.
Chapter 5. Compactness. Extension of definition of sequential compactness (met in G1BMAN for $\mathbb{R}^n$) to metric spaces and their subsets. Revision of Heine-Borel theorem for sequentially compact subsets of $\mathbb{R}^n$. Special case of $[a, b]$ (Bolzano-Weierstrass). $\mathbb{R}$ is NOT sequentially compact.

Lecture 23: The boundedness theorem for non-empty, sequentially compact metric spaces. Tychonoff's Theorem for a product of two sequentially compact metric spaces (true also for finite products, by induction). A useful lemma concerning the existence of convergent subsequences.

Lecture 24: Covers (coverings) of sets. Finite subcovers. Open covers of topological spaces. Examples of open covers: some open covers of $\mathbb{R}$ do have finite subcovers, while others do not. Compactness of topological spaces and their subsets. Examples. We will show later that metric spaces are compact if and only if they are sequentially compact. From this (and Heine-Borel: sequential compactness version) the compact subsets of $\mathbb{R}^n$ are the subsets which are both closed and bounded. Generally, finite sets are always compact. Stated easy classification of compact subsets in the case of the indiscrete or the discrete topology.

Lecture 25: Collections of sets with the Finite Intersection Property. Compactness in terms of closed sets. Every compact metric space is sequentially compact. Total boundedness. Every sequentially compact metric space is totally bounded.

Lecture 26: Lebesgue number for a cover. Every open cover of a sequentially compact metric space has a positive Lebesgue number. Every sequentially compact metric space is compact. Corollaries: all results for sequentially compact metric spaces can now be restated for compact metric spaces.

In particular we have the Heine-Borel theorem classifying the compact subsets of $\mathbb{R}^n$ as those which are closed and bounded (this is also valid in $\mathbb{C}^n$), and Tychonoff's theorem for a product of two compact metric spaces. (Tychonoff's theorem is valid for arbitrary products of compact topological spaces, but this is beyond the scope of this module.) The continuous image of a compact set is compact. This is one way to see that compactness is a topological property (preserved by homeomorphisms). The boundedness theorem for compact topological spaces.

Lecture 27: Every compact subset of a Hausdorff space is closed (in particular compact subsets of metric spaces are always closed). Every closed subset of a compact topological space is compact. Every continuous bijection from a compact topological space to a Hausdorff space is automatically a homeomorphism. No compact topology can be strictly stronger than a Hausdorff topology.

Lecture 28: **Chapter 6. Connectedness**. Clopen (open-closed) subsets of X. Connected and disconnected topological spaces and subsets: many equivalent definitions. Examples. The continuous image of a connected set is connected. Connectedness is a topological property.

Lecture 29: Disconnected sets must have at least two points. Connectedness in terms of continuous functions from X to the discrete topological space $\{0, 1\}$ (note that the discrete topology is the same as the usual topology on $\{0, 1\}$). With the usual topology, every interval in $\mathbb{R}$ is connected. Paths in topological spaces. The image of a path is connected. Classification of connected subsets: the only connected subsets of $\mathbb{R}$ are the intervals (bounded, unbounded or empty!). The only connected subsets of $\mathbb{Q}$ are the empty set and single-point sets.

Lecture 30: Unions and closures of connected sets. Connected components and their properties. Examples. A product of two connected topological spaces is connected. Totally disconnected topological spaces (e.g. $\mathbb{Q}$). Path-connectedness. Every path-connected subset of a topological space is connected.

Lecture 31: For open subsets of $\mathbb{R}^n$, connectedness, path-connectedness and stepwise connectedness are equivalent.
Chapter 7. Completeness. Cauchy sequences in metric spaces. Complete and incomplete metric spaces. $(0, 1)$ is incomplete with the usual metric. Elementary facts (proof is an exercise): every Cauchy sequence is bounded, and if a Cauchy sequence has a convergent subsequence, then the Cauchy sequence must itself converge. Every (sequentially) compact metric space is complete. With the usual metric $\mathbb{R}$ is complete, and so are $\mathbb{R}^n$, $\mathbb{C}$ (with any of their usual metrics). The metric space $(C[0, 1], d_1)$ is incomplete.

Lecture 32: The metric space $(C[0, 1], d_\infty)$ is complete. Completeness is not a topological property. Classification of complete subsets (with the subspace metric): (a) every closed subset of a complete metric space is complete when given the subspace metric; (b) given a metric space (X, d) and $E \subseteq X$ then if E is complete with respect to the subspace metric on E induced by d, then E is closed in X. Thus, if (X, d) is a complete metric space, then a subset of X, with the subspace metric, is complete if and only if it is closed in X. In particular, with the usual metric, a subset of $\mathbb{R}$ is complete if and only if it is closed in $\mathbb{R}$. The same applies for our other standard complete metric spaces, e.g. $\mathbb{R}^n$, $\mathbb{C}$ and $(C[0, 1], d_\infty)$.

Lecture 33: Tutorial/revision/question and answer session. Student Evaluation of Teaching forms.