Section 2: Classes of Sets

Notation:

If A, B are subsets of X, then $A \setminus B$ denotes the set difference,

$$A \setminus B = \{x \in A : x \notin B\}.$$

$A \triangle B$ denotes the *symmetric difference* .

$$\begin{aligned} A \triangle B &= (A \setminus B) \cup (B \setminus A) \\ &= (A \cup B) \setminus (A \cap B). \end{aligned}$$

Remarks:

(i) $A \triangle A = \varnothing$.

(ii) If $A \cap B = \varnothing$, then $A \triangle B = A \cup B$.

(iii) If $B \subseteq A$ then $A \triangle B = A \setminus B$.

(iv) In fact

$$A \setminus B = A \triangle (A \cap B),$$

$$A \cup B = (A \cap B) \triangle (A \triangle B).$$

Let X be a set. Then $\mathscr{P}(X)$ denotes the set of all subsets of X. If we write

$$A = \dot{\bigcup_{i=1}^{n}} A_i$$

we mean that $A_1, \ldots, A_n$ are pairwise disjoint and

$$A = \bigcup_{i=1}^{n} A_i.$$

Definition 2.1. Let X be a set. Then a collection of sets $\mathscr{S} \subseteq \mathscr{P}(X)$ is a semi-ring if

(i) $\varnothing \in \mathscr{S}$,

(ii) if $A, B \in \mathscr{S}$ then $A \cap B \in \mathscr{S}$,

(iii) if $A, B \in \mathscr{S}$ then there is an $n \in \mathbb{N}$ and there are sets $A_1, A_2, \ldots, A_n \in \mathscr{S}$ such that A_i are pairwise disjoint and $A \setminus B = \bigcup_{i=1}^{n} A_i$.

Example 2.2. Set $P = \{(a, b] : a, b \in \mathbb{R}, a \leqslant b\}$. Then P is a semi-ring of subsets of $\mathbb{R}$.

[It is not hard to check this. For example, if $a < c < d < b$, then $(a,b]\setminus(c,d] = (a,c]\cup(d,b]$.]

Similarly in $\mathbb{R}^2$ or $\mathbb{R}^n$ we can consider P^n, the collection of all subsets of $\mathbb{R}^n$ of the form

$$(a_1,b_1]\times(a_2,b_2]\times\ldots\times(a_n,b_n],$$

e.g. $P^2 = \{(a,b]\times(c,d] : a,b,c,d\in\mathbb{R},\ a \leqslant b \text{ and } c \leqslant d\}$.

Then P^n is a semi-ring of subsets of $\mathbb{R}^n$. This is not as obvious as in the case $n = 1$. For example, for P^2 note that the set difference of two half-open rectangles is the disjoint union of (at most) four half-open rectangles.

Other examples of semi-rings of subsets of X:

(i) $\mathcal{P}(X)$ is a semi-ring;

(ii) $\{\varnothing\}$ is a semi-ring;

(iii) $\{\varnothing\}\cup\{\{x\}: x\in X\}$ = collection of all subsets of X containing $\leqslant 1$ point.

Definition 2.3. Let X be a set, let $R\subseteq\mathcal{P}(X)$. Then R is a *ring* of subsets of X if

(i) $\varnothing\in R$;

(ii) if $A,B\in R$ then $A\cap B$, $A\cup B$ and $A\setminus B$ are all in R.

Remarks

(i) Every ring is a semi-ring.

(ii) Rings are closed under finite intersection and union: if $A_1, A_2, \ldots, A_n\in R$, then

$$\bigcap_{i=1}^{n} A_i \in R \quad\text{and}\quad \bigcup_{i=1}^{n} A_i\in R.$$

Examples

(i) $\mathcal{P}(X)$, $\{\varnothing\}$ are both rings of subsets of X.

(ii) $R = \{A\subseteq X: A \text{ is finite}\}$.

(iii) $R = \{A\subseteq\mathbb{R}: A \text{ is bounded}\}$.

In this course, our main example of a ring will be the following.

Example 2.4. Set

$$\mathcal{E} = \{A\subseteq\mathbb{R}: A \text{ is a finite union of half open intervals in } \mathbb{R}\text{, each of the form } (a,b]\}.$$

$\mathcal{E}$ is called the collection of *elementary figures* in $\mathbb{R}$.

$(_{a_1}{-}{-}{-}{-}{-}{-}]_{b_1}\quad (_{a_2}{-}{-}{-}{-}{-}{-}]_{b_2}\quad (_{a_3}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}{-}]_{b_3}\quad ({-}{-}{-}{-}{-}{-}]\ \cdots\quad (_{a_n}{-}{-}{-}{-}]_{b_n}$

$\mathcal{E}$ contains all sets of form $\bigcup_{i=1}^{n}(a_i,b_i]$.

The fact that $\mathcal{E}$ is a ring follows from Lemma 2.6 below. First we give a definition.

Definition 2.5. Let $\mathcal{S}$ be a semi-ring of subsets of a set X. Then $R(\mathcal{S})$ is defined to be the collection of all finite *disjoint* unions of sets in $\mathcal{S}$.

Lemma 2.6. Let $\mathcal{S}$ be a semi-ring of subsets of a set X. Then $R(\mathcal{S})$ is a ring, and for any ring R' satisfying $\mathcal{S} \subseteq R'$, we have $R(\mathcal{S}) \subseteq R'$ also.

Remarks. It will follow that $R(\mathcal{S})$ is also the collection of all finite unions of sets in $\mathcal{S}$. $R(\mathcal{S})$ is the smallest ring containing $\mathcal{S}$.

Proof. First note $\mathcal{S} \subseteq R(\mathcal{S})$.

Next note that if $A, B \in \mathcal{S}$, then $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$ (by property (iii) of semi-rings). Thus $A \setminus B \in R(\mathcal{S})$. Suppose now that $A, B \in R(\mathcal{S})$ and $A \cap B = \emptyset$.

Then $A = \dot{\bigcup}_{i=1}^{n} A_i$, $B = \dot{\bigcup}_{j=1}^{m} B_j$ with all A_i, B_j in $\mathcal{S}$.

Then $A \cup B = \dot{\bigcup}_{i=1}^{n} A_i \,\dot{\cup}\, \dot{\bigcup}_{j=1}^{m} B_j$, a finite disjoint union of sets in $\mathcal{S}$. Thus $A \cup B \in R(\mathcal{S})$. Hence if $A_1, A_2, \ldots, A_n \in R(\mathcal{S})$ and A_i are pairwise disjoint then $\bigcup_{i=1}^{n} A_i \in R(\mathcal{S})$.

Now suppose that $A, B \in R(\mathcal{S})$ with

$$A = \dot{\bigcup}_{i=1}^{n} A_i, \quad B = \dot{\bigcup}_{j=1}^{m} B_j, \quad A_i, A_j \in \mathcal{S}.$$

Set $C_{ij} = A_i \cap B_j$. Then $C_{ij} \in \mathcal{S}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$, and

$$A \cap B = \left(\bigcup_{i=1}^{n} A_i \right) \cap \left(\bigcup_{j=1}^{m} B_j \right) = \bigcup_{\substack{i,j \\ 1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (A_i \cap B_j) = \bigcup_{i,j} C_{ij}.$$

The sets C_{ij} are pairwise disjoint, so $A \cap B \in R(\mathcal{S})$. Hence $R(\mathcal{S})$ is closed under finite intersections.

Suppose A, B are as above in $R(\mathcal{S})$. Then

$$\begin{aligned} A \setminus B &= \left(\dot{\bigcup}_{i=1}^{n} A_i \right) \setminus \left(\dot{\bigcup}_{j=1}^{m} B_j \right) \\ &= \dot{\bigcup}_{i=1}^{n} \left(A_i \setminus \bigcup_{j=1}^{m} B_j \right) \\ &= \dot{\bigcup}_{i=1}^{n} \left(\bigcap_{j=1}^{m} (A_i \setminus B_j) \right) \\ &\in R(\mathcal{S}). \end{aligned}$$

Finally, if $A, B \in R(\mathscr{S})$, then

$$A \cup B = (A \setminus B) \dot{\bigcup} B$$

$$\in R(\mathscr{S}).$$

Hence $R(\mathscr{S})$ is a ring.

The rest of the result is obvious, since any ring containing $\mathscr{S}$ must also contain all finite unions of sets in $\mathscr{S}$. □

In particular, with P as in Example 2.2, we see that $R(P) = \mathscr{E}$, and so $\mathscr{E}$ is a ring. Similarly in $\mathbb{R}^n$, the ring generated by P^n is the set of elementary figures in $\mathbb{R}^n$, $\mathscr{E}_n$ consisting of all finite (disjoint) unions of sets in P^n.

There is an alternative definition of ring, equivalent to ours, R is a ring if

(i) $\emptyset \in R$,

(ii) for $A, B \in R$, $A \cap B$ is in R, and $A \triangle B \in R$.

With Operations $\cap$ as multiplication, $\triangle$ as addition, R really is a ring in the algebraic sense.

This is NOT true for *fields*: fields of sets are not usually really fields in the algebraic sense.

Definition 2.7. Let X be a set. A collection of sets $\mathscr{F} \subseteq \mathscr{P}(X)$ is a *field* of subsets of X if $\mathscr{F}$ is a ring and $X \in \mathscr{F}$.

Examples

(i) $\{\emptyset, X\}$ is the smallest possible field of subsets of X.

(ii) $\mathscr{P}(X)$ is a field of subsets of X.

(iii) Let $A \subseteq X$. Then $\{\emptyset, A, X \setminus A, X\}$ is a field of subsets of X.

(iv) Set $\mathscr{F} = \{A \subseteq X$: either A or $X \setminus A$ are finite (or both)$\}$.

Exercise: Check this is a field.

Fields are also called *algebras* of sets.

The next type of collection of sets is the σ-field (also known as σ-algebra or Borel algebra or Borel family).

σ-fields

A σ-*field* of subsets of X is a field of subsets of X which is closed under countable unions.

In full, the definition of a σ-field is:

Definition 2.8. Let X be a set and let $\mathscr{F} \subseteq \mathscr{P}(X)$. Then $\mathscr{F}$ is a σ-*field* of subsets of X if $\mathscr{F}$ satisfies

(i) $\varnothing, X \in \mathscr{F}$,

(ii) for all $A, B \in \mathscr{F}$, $A \setminus B \in \mathscr{F}$,

(iii) whenever $A_1, A_2, A_3, \ldots \in \mathscr{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$.

Since

$$\bigcap_{n=1}^{\infty} A_n = X \setminus \left(\bigcup_{n=1}^{\infty} (X \setminus A_n) \right),$$

$\mathscr{F}$ is closed under infinite intersections. Finite unions and intersections then follow, since ϕ and X are in $\mathscr{F}$.

Examples

(i) $\{\varnothing, X\}$, $\mathscr{P}(X)$ are both σ-fields of subsets of X.

(ii) If $\mathscr{F}$ is a field and $\mathscr{F}$ has only finitely many elements, then $\mathscr{F}$ is always a σ-field. This is because only finitely many sets are involved in the countable union.

(iii) Set $X = \mathbb{R}$. Set $\mathscr{F} = \{A \subseteq \mathbb{R}: A$ or $\mathbb{R} \setminus A$ is countable$\}$ (here countable means finite or countably infinite).

Exercise. Show that $\mathscr{F}$ is a σ-field.

The following lemma remains true if 'σ-field' is replaced throughout by 'field', 'ring' but **NOT** 'semi-ring'.

Lemma 2.9. Let $\{\mathscr{F}_\gamma : \gamma \in \Gamma\}$ be a set of σ-fields on a set X, where Γ is a non-empty indexing set. Then $\bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$ is also a σ-field on X.

Proof

(i) For each γ, $\varnothing$ and X are in $\mathscr{F}_\gamma$. Thus $\varnothing \in \bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$ and $X \in \bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$.

(ii) Let $A, B \in \bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$. Then, for all $\gamma \in \Gamma$, A and B are in $\mathscr{F}_\gamma$.

Since $\mathscr{F}_\gamma$ is a σ-field, $A \setminus B \in \mathscr{F}_\gamma$ for all γ, and so $A \setminus B \in \bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$.

(iii) Let $A_1, A_2, A_3, \ldots$ be a sequence of sets in $\bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$. Then, just as before, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}_\gamma$ for every γ and so $\bigcup_{n=1}^{\infty} A_n \in \bigcap_{\gamma \in \Gamma} \mathscr{F}_\gamma$. □

Definition 2.10. Suppose that X is a set and $\mathcal{C}$ is any set of subsets of X. There is at least one σ-field on X containing $\mathcal{C}$, namely, $\mathcal{P}(X)$. Now define

$$\mathcal{F}(\mathcal{C}) = \cap\{\mathcal{B} \text{ a } \sigma\text{-field on } X\colon \mathcal{C} \subseteq \mathcal{B}\}$$

the intersection of all the σ-fields on X containing $\mathcal{C}$. By Lemma 2.9, $\mathcal{F}(\mathcal{C})$ is a σ-field on X. Any σ-field on X which contains $\mathcal{C}$ must contain $\mathcal{F}(\mathcal{C})$ also.

$\mathcal{F}(\mathcal{C})$ is called the *σ-field generated by* $\mathcal{C}$.

Definition 2.11. Let (X, d) be a metric space. Let $\mathcal{C}$ be the collection of all open subsets of X. Then the *Borel subsets* of X are the sets in $\mathcal{F}(\mathcal{C})$.

Thus the collection of Borel sets on X is the σ-field generated by the set of open subsets of X.

We are interested mainly in $\mathbb{R}$ and $\overline{\mathbb{R}}$.

Let $\mathcal{B}$ denote the collection of Borel subsets of $\mathbb{R}$. So $\mathcal{B}$ is a σ-field which includes all open sets. Since fields are closed under complement, all closed subsets of $\mathbb{R}$ are also in $\mathcal{B}$. Since $\mathcal{F}$ is closed under countable unions and intersections, we see that every countable subset of $\mathbb{R}$ is in $\mathcal{B}$, in particular $\mathbb{Q} \in \mathcal{B}$. There are very many sets in $\mathcal{B}$ but we shall see later that $\mathcal{B} \neq \mathcal{P}(\mathbb{R})$.

We have, for example, the Cantor middle thirds set is in $\mathcal{B}$.

Example 2.12 (the Cantor Middle Thirds Set). Start with $X_0 = [0, 1]$. Delete the middle third $(\frac{1}{3}, \frac{2}{3})$ to form $X_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. X_1 consists of two closed intervals. Form X_2 by deleting the middle third of both intervals to leave four closed intervals.

e.g.

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X0   |-----------------|
X1   |-------|   |-------|
X2   |--|  |--|   |--|  |--|
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X_n consists of 2^n closed intervals, each with length $\dfrac{1}{3^n}$ obtained by deleting the middle third of all the intervals forming X_{n-1}.

Set

$$C = \bigcap_{n=0}^{\infty} X_n$$

$$= \text{those points in none of the deleted open intervals, but in } [0, 1].$$

Then C is a closed subset of $[0, 1]$, called the Cantor middle thirds set.

In fact C consists of all x in $[0, 1]$ which have a base 3 expansion of the form

$$0 \cdot a_1 a_2 a_3 \ldots$$

where all a_i are 0 or 2.

C is an example of a metric space with no isolated points but such that the only connected subsets are single points. Although C contains no intervals of positive length, C has the same cardinality as $\mathbb{R}$.

Every half-open interval $(a, b]$ is a Borel set. This is because

$$(a, b] = \bigcup_{n=1}^{\infty} \left[a + \frac{(b-a)}{n}, b\right].$$

Thus $(a, b]$ is a countable union of closed sets and hence $(a, b] \in \mathcal{B}$.

We have $P \subseteq \mathcal{B}$. Since $\mathcal{B}$ is a ring we have $\mathcal{E} \subseteq \mathcal{B}$, i.e. every elementary figure is a Borel set. Also, since $\mathcal{B}$ is a σ-field containing P, we have $\mathcal{F}(P) \subseteq \mathcal{B}$.

However, every open interval (a, b) is a countable union of sets in P.

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{(b-a)}{n}\right].$$

Thus $(a, b) \in \mathcal{F}(P)$ for all $a < b$ in $\mathbb{R}$. But *any* open set $U \subseteq \mathbb{R}$ is a countable union of open intervals, e.g. $U = \cup\{(p, q): p, q \in \mathbb{Q}$ and $(p, q) \subseteq U\}$. Thus $\mathcal{F}(P)$ is a σ-field containing all open subsets of $\mathbb{R}$. Since $\mathcal{B}$ is the *smallest* σ-field containing all the open sets, it follows that $\mathcal{B} \subseteq \mathcal{F}(P)$. We already had $\mathcal{F}(P) \subseteq \mathcal{B}$. Thus $\mathcal{F}(P) = \mathcal{B}$. We have thus proven the following.

Proposition 2.13. The σ-field generated by P is precisely the set of Borel subsets of $\mathbb{R}$. □