Section 3: Measures and Measure Spaces

Intuitively, in $\mathbb{R}^2$, we expect the area of a disjoint union of sets $A \mathbin{\dot\cup} B$ to be the sum of area of A and area of B (i.e. total area = sum of smaller areas).

What if $(A_n)_{n=1}^{\infty}$ is a sequence of disjoint subsets of $\mathbb{R}^2$? We would hope that the area of $\bigcup_{n=1}^{\infty} A_n$ would equal $\sum_{n=1}^{\infty}$ (area of A_n).

In $\mathbb{R}$ the equivalent notion is that of length. We want to define a function $\lambda\colon \mathcal{P}(\mathbb{R}) \to [0,\infty]$ to measure the length of as many sets as possible, such that

$$\lambda((a,b]) = b-a \quad \text{and} \quad \lambda(A \mathbin{\dot\cup} B) = \lambda(A)+\lambda(B), \quad \lambda\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \lambda(A_n) \quad \text{etc.}$$

Unfortunately this cannot be done for all subsets of $\mathbb{R}$.

Area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$ have the same problems. But we will succeed in defining our measurements of size on at least all the Borel sets.

Definition 3.1

Let X be a set, let $\mathcal{C} \subseteq \mathcal{P}(X)$ s.t. $\varnothing \in \mathcal{C}$, and let $\mu\colon \mathcal{C} \to [0,\infty]$.

Then μ is a *measure* on $\mathcal{C}$ if

(i) $\mu(\varnothing) = 0$,

(ii) whenever $A_1, A_2, \ldots,$ is a sequence of pairwise disjoint sets in $\mathcal{C}$ s.t. $\dot{\bigcup_{n=1}^{\infty}} A_n$ is in $\mathcal{C}$, then $\mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Examples

(i) $X = \mathbb{R}$, $\mathcal{C} = \mathcal{P}(\mathbb{R})$,

define

$$\mu(E) = \begin{cases} \infty & \text{if } E \text{ has infinitely many elements} \\ n & \text{if } E \text{ has exactly } n \text{ elements} \end{cases}$$

Easy exercise: check μ is a measure. This measure μ is called *counting measure* on $\mathbb{R}$.

[Counting measure is usually used on $\mathbb{N}$ rather than on an uncountable set.]

(ii) 'point mass' measures. Let X be a set, $\mathcal{C} = \mathcal{P}(X)$. Let x be any fixed point in X. Define

$$\mu(E) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Certainly $\mu(\emptyset) = 0$.

If $A_1, A_2, \ldots,$ are disjoint subsets of X, then either $x \in \dot{\bigcup}_{n=1}^{\infty} A_n$, in which case x is in exactly one set A_n or $x \notin \bigcup_{n=1}^{\infty} A_n$, in which case x is in none of the A_n. In both cases $\mu\left(\dot{\bigcup}_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$. This measure μ is called the *point mass at x*, and is often denoted by δ_x.

If a and $b \geqslant 0$, μ, ν are measures on $\mathscr{C}$, then so is $a\mu + b\nu$ defined by

$$(a\mu + b\nu)(E) = a\mu(E) + b\nu(E).$$

In the examples above (i) and (ii) $\mathscr{C}$ was a σ-field.

Definition 3.2

A *measurable space* is a pair $(X, \mathscr{F})$ where X is a set and $\mathscr{F}$ is a σ-field of subsets of X.

A *measure space* is a triple $(X, \mathscr{F}, \mu)$ where $\mathscr{F}$ is a σ-field on X, and

$$\mu \colon \mathscr{F} \to [0, \infty] \quad \text{is a measure.}$$

By abuse of terminology, X is a measurable space and μ is a 'measure on X', provided we know which σ-field we are working with.

Our aim: with $\mathscr{B}$ = Borel subsets of $\mathbb{R}$, we wish to find a measure $\lambda \colon \mathscr{B} \to [0, \infty]$ s.t.

$$\lambda((a, b]) = b - a \quad \forall\ a \leqslant b \text{ in } \mathbb{R}.$$

Is this possible?

The first problem. Suppose $(a, b] = \dot{\bigcup}_{n=1}^{\infty} (a_n, b_n]$. We would need $\lambda((a, b]) = \sum_{n=1}^{\infty} \lambda((a_n, b_n])$, i.e. we need $b - a = \sum_{n=1}^{\infty} (b_n - a_n)$.

Is this last equality true? Yes! (See later.)

General Results about Measures on Rings

Proposition 3.3

Let X be a set, R be a ring of subsets of X, and let $\mu \colon R \to [0, \infty]$ be a measure.

(i) If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets in R then

$$\mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

(ii) If $A, B \in R$ then

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B).$$

Proof

(i) To see this, set $A_{n+1} = A_{n+2} = \dots = \emptyset$. Then

$$\dot{\bigcup_{k=1}^{\infty}} A_k = \dot{\bigcup_{k=1}^{n}} A_k \in R.$$

Thus

$$\mu\left(\dot{\bigcup_{k=1}^{n}} A_k\right) = \mu\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$
$$= \sum_{k=1}^{n} \mu(A_k),$$

since $\mu(\emptyset) = 0$.

(ii) $$\mu(A) = \mu(A \cap B) + \mu(A \setminus B)$$

because $A = (A \cap B) \dot{\cup} (A \setminus B)$.

Is it true that $\mu(A \cap B) = \mu(A) - \mu(A \setminus B)$? Not necessarily! (May have $\infty - \infty$.) [Remember $\infty - \infty$ is not defined.]

e.g. work with counting measure on $\mathbb{N}$.

Set

$$A = \{2, 4, 6, \dots\}$$

$$B = \{\text{primes}\}$$

$A \cap B = \{2\}$, $\mu(A \cap B) = 1$, $\mu(A) = \mu(B) = \mu(A \setminus B) = \infty$ so $\mu(A) - \mu(A \setminus B)$ is not defined.

Proposition 3.4

Let μ be a measure on a ring R of subsets of a set X.

(i) If $A, B \in R$ with $A \subseteq B$, then

$$\mu(A) \leqslant \mu(B). \quad \text{(Monotonicity)}$$

(ii) If $A \in R$, $B_1, B_2, \dots \in R$ and $A \subseteq \bigcup_{n=1}^{\infty} B_n$, then

$$\mu(A) \leqslant \sum_{n=1}^{\infty} \mu(B_n). \quad \text{(Countable subadditivity)}$$

Proof

(i) $\mu(B) = \mu(A) + \mu(B \setminus A) \geqslant \mu(A)$.

(ii) (N.B. The B_n are NOT assumed disjoint, and we do not assume $\bigcup_{n=1}^{\infty} B_n \in R$.)

Set $C_n = B_n \cap A$. Then

$$A = A \cap \left(\bigcup_{n=1}^{\infty} B_n \right) = \bigcup_{n=1}^{\infty} C_n .$$

Set $D_1 = C_1$ and $D_n = C_n \setminus \bigcup_{k=1}^{n-1} C_k$ $(n > 1)$. We then have

D_n are in R,
$D_n \subseteq C_n \subseteq B_n \quad \forall n$,
D_n are pairwise disjoint.

Also, for each n,

$$\bigcup_{k=1}^{n} D_k = \bigcup_{k=1}^{n} C_k .$$

We then have

$$A = \dot{\bigcup_{n=1}^{\infty}} D_n ,$$

and so

$$\mu(A) = \sum_{n=1}^{\infty} \mu(D_n) \leqslant \sum_{n=1}^{\infty} \mu(B_n). \qquad \square$$

The property that

$$\mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n)$$

is 'μ is *countably additive*'.

('μ is *finitely additive*' means $\mu\left(\dot{\bigcup_{n=1}^{N}} A_n \right) = \sum_{n=1}^{N} \mu(A_n)$.)

The property that $A \subseteq B \Rightarrow \mu(A) \leqslant \mu(B)$ is called *monotonicity* (μ is monotone).

$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$ means 'μ is *countably subadditive*'.

If $\mathcal{F}$ is σ-field on X, then $(X, \mathcal{F})$ is a measurable space. If $\mu : \mathcal{F} \to [0, \infty]$ is a measure, then $(X, \mathcal{F}, \mu)$ is a measure space.

Definition 3.5.

If $\mu(X) < \infty$ then μ is a *finite measure*.

If $\mu(X) = 1$ then μ is a *probability measure* (informally, for $A \in \mathcal{F}$, $\mu(A)$ represents the probability that a random point chosen from X will be in A).

We say that a measure is **σ-finite** if there are countably many sets $E_n \in \mathcal{F}$ with $\mu(E_n) < \infty$ all n, and s.t.

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Examples.

The point mass measures are all probability measures (and hence finite measures).

Counting measure μ on a set X

$$\mu(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

is a finite measure if and only if X is finite.

If μ is a counting measure on $\mathbb{N}$, then $\mu(\mathbb{N}) = \infty$, but

$$\mathbb{N} = \bigcup_{n=1}^{\infty} \{1, 2, 3, \ldots, n\}$$

so that μ is σ-finite.

But counting measure on $\mathbb{R}$ (or on any uncountable set) is not σ-finite.

More Standard Properties of Measures

Proposition 3.6

Let R be a ring of subsets of a set X. Suppose $\mu : R \to [0, \infty]$ is a measure and let $A_1, A_2, A_3, \ldots \in R$.

(i) If $\bigcup_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu\left(\bigcup_{k=1}^{n} A_k\right).$$

(ii) If $\mu(A_1) < \infty$ and $\bigcap_{n=1}^{\infty} A_n \in R$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu\left(\bigcap_{k=1}^{n} A_k\right).$$

Proof

(i) If $\bigcup_{n=1}^{\infty} A_n \in R$, set $A = \bigcup_{n=1}^{\infty} A_n$, set $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for $n > 1$. Then each $B_n \in R$, the sets B_n are pairwise disjoint,

$$\bigcup_{k=1}^{n} A_k = \dot{\bigcup_{k=1}^{n}} B_k \quad \forall n,$$

and $A = \dot{\bigcup_{k=1}^{\infty}} B_k$. Thus

$$\begin{aligned}\mu(A) &= \sum_{k=1}^{\infty} \mu(B_k)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \mu(B_k)\right)\\ &= \lim_{n\to\infty}\left(\mu\left(\dot{\bigcup_{k=1}^{n}} B_k\right)\right)\\ &= \lim_{n\to\infty}\left(\mu\left(\bigcup_{k=1}^{n} A_k\right)\right).\end{aligned}$$

(ii) Now suppose that $\mu(A_1) < \infty$.

If $\bigcap_{n=1}^{\infty} A_n$ is in R, then set

$$C_n = A_1 \setminus A_n \quad \forall n.$$

Then $C_n \in R$ and

$$\begin{aligned}\bigcap_{n=1}^{\infty} A_n &= A_1 \setminus \bigcup_{n=1}^{\infty} (A_1 \setminus A_n)\\ &= A_1 \setminus \bigcup_{n=1}^{\infty} C_n.\end{aligned}$$

$\bigcup_{n=1}^{\infty} C_n \subseteq A_1$ by definition of C_n and so $\bigcup_{n=1}^{\infty} C_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n \in R$. Thus

$$\mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} C_k\right)$$

(by the first part).

Now note

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu(A_1) - \mu\left(\bigcup_{n=1}^{\infty} C_n\right) \qquad \text{[this holds because } \mu(A_1) < \infty \text{ and } \bigcup_{n=1}^{\infty} C_n \subseteq A_1.\text{]}$$

$$= \mu(A_1) - \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} C_k\right)$$

$$= \lim_{n\to\infty}\left(\mu(A_1) - \mu\left(\bigcup_{k=1}^{n} C_k\right)\right)$$

$$= \lim_{n\to\infty}\left(\mu\left(A_1 \setminus \bigcup_{k=1}^{n} C_k\right)\right)$$

$$= \lim_{n\to\infty}\left(\mu\left(\bigcap_{k=1}^{n} A_k\right)\right) \quad \text{as required.}$$

□

Properties which hold almost everywhere

Definition 3.7

Let $(X, \mathcal{F}, \mu)$ be a measure space. To say that a property holds almost everywhere (with respect to μ) (a.e. (μ)) means that there is a set $E \in \mathcal{F}$ with $\mu(E) = 0$ such that the property holds $\forall\, x \in X \setminus E$.

For example:

Using Lebesgue measure (see Chapter 5 for the construction) on $\mathbb{R}$ we can say

$\chi_{\mathbb{Q}}(x) = 0$ almost everywhere (λ).

OR alternatively

$\chi_{\mathbb{Q}}(x) = 0$ for almost all x (λ).

["(λ)" means "with respect to λ".]

This is because $\lambda(\mathbb{Q}) = 0$ (see question sheet 5).

Definition 3.8

Given two functions $f, g\colon X \to Y$ where Y is some set, we say f and g are *equivalent* if

$$f(x) = g(x) \quad \text{a.e. } (\mu)$$

(this depends on the measure μ).

Check: this really is an equivalence relation (make sure your sets are really in $\mathcal{F}$).

Note that if you use *counting measure*, a.e. means *everywhere*! (Because $\mu(E) = 0 \Rightarrow E = \emptyset$ when μ is counting measure.)