G1CMIN MEASURE AND INTEGRATION
TUTORIAL PROBLEMS

The following are the main theorems on integration in Chapter 4

MONOTONE CONVERGENCE THEOREM

If $(X, \mathcal{F}, \mu)$ is a measure space, and $f, f_n : X \to [0, \infty]$ are measurable functions, with

$$f_n(x) \uparrow f(x) \quad \text{as} \quad n \to \infty$$

for all x in X, then

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

FATOU'S LEMMA

If $(X, \mathcal{F}, \mu)$ is a measure space, and $f_n : X \to [0, \infty]$ are measurable functions, then

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

DOMINATED CONVERGENCE THEOREM

If $(X, \mathcal{F}, \mu)$ is a measure space, $h : X \to [0, \infty]$ a measurable function with $\int_X h \, d\mu < \infty$, $f, f_n : X \to \mathbb{R}$ measurable functions such that $|f_n(x)| \leq h(x)$ for all $x \in X$ and all n, and

$$f_n(x) \to f(x) \quad \text{as} \quad n \to \infty \quad (x \in X).$$

Then

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Bearing in mind the facts that, if μ is counting measure on $\mathbb{N}$, and $f : \mathbb{N} \to [0, \infty]$

$$\int_{\mathbb{N}} f \, d\mu \quad \text{means} \quad \sum_{n=1}^{\infty} f(n)$$

and that if f is Riemann integrable on $[a, b]$, then (taking λ to be Lebesgue measure on the Borel subsets of $[a, b]$)

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x) \, dx,$$

discuss with each other the following questions (and try to answer them!).

(1) Find a sequence of non-negative measurable functions f_n on a measure space $(X, \mathcal{F}, \mu)$ such that $f_n \to 0$ pointwise, but

$$\int_X f_n \, d\mu \not\to 0$$

[so monotonicity cannot be omitted in the MCT].

(2) Find a sequence of non-negative measurable functions f_n on a measure space $(X, \mathcal{F}, \mu)$ such that

$$\int_X (\liminf_{n\to\infty} f_n) \, d\mu < \liminf_{n\to\infty} \int_X f_n \, d\mu$$

(so the inequality in Fatou's Lemma may be strict).

(3) Find a sequence of real-valued measurable functions f_n on a measure space $(X, \mathcal{F}, \mu)$ such that $f_n \to 0$ *uniformly*, but

$$\int_X f_n \, d\mu \not\to 0.$$

Why does your example not violate the Dominated Convergence Theorem?

(4) What can go wrong with Fatou's Lemma if the functions are allowed to be

$$f_n : X \to \overline{\mathbb{R}}$$

rather than $f_n : X \to [0, \infty]$?