Modified extract from Chapter 5: Lebesgue outer measure, Lebesgue measurable sets and Lebesgue measure

Definition 5.9

Let X be a set, and let

$$\mu^*\colon \mathscr{P}(X) \to [0,\infty].$$

Then μ^* is an *outer measure* on X if

(i) $\mu^*(\phi) = 0$,

(ii) if $A, B \in \mathscr{P}(X)$ and $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$ (μ^* is *monotone*),

(iii) if $A \subseteq \bigcup_{n=1}^{\infty} A_n$, where A, A_n are subsets of X, then $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n)$. ($\mu^*$ is *countably subadditive*.)

Note

An outer measure on X is defined on *all* subsets of X.

Examples

(i) Any measure on $\mathscr{P}(X)$ is also an outer measure on X.

(ii) Defining

$$\mu^*(E) = \begin{cases} 0 & (E = \varnothing), \\ 1 & (E \neq \varnothing), \end{cases}$$

defines an outer measure which is not a measure (provided that X has at least two points!).

Definition 5.10

Let X be a set and let μ^* be an outer measure on X. A set $A \subseteq X$ is said to be *measurable with respect to* μ^* (or μ^**-measurable*) if, for *every* set $E \subseteq X$, the equality

$$\mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$$

holds.

Theorem 5.13

Let μ^* be an outer measure on a non-empty set X. Let $\mathscr{F}$ be the set of μ^*-measurable subsets of X. Then

(i) $\mathscr{F}$ is a σ-field on X,

(ii) for any set $E \subseteq X$ and pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathscr{F}$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} E \cap A_n\right) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n),$$

(iii) the restriction of μ^* to $\mathscr{F}$ is a measure.

In fact the measure on $\mathscr{F}$ obtained this way is always *complete* in the sense discussed on question sheet 3.

The construction of Lebesgue measure is now based on the following facts about Lebesgue outer measure, λ^*. Recall we are using the following definition. Let E be a subset of $\mathbb{R}$. Define S_E to be the following set of extended real numbers

$$S_E = \left\{ \sum_{n=1}^{\infty} (b_n - a_n): \ a_n \leqslant b_n \text{ in } \mathbb{R}, E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}$$

Then $\lambda^*(E) = \inf(S_E)$.

We need the following facts:

(i) λ^* really *is* an outer measure on $\mathbb{R}$.

(ii) Every half-open interval $A=(a,b]$ is λ^*-measurable, and for such A, $\lambda^*(A) = b-a$ (the length of A).

λ^*-measurable sets are also called *Lebesgue measurable* sets.

Given these facts, let $\mathscr{F}$ be the set of all λ^*-measurable subsets of $\mathbb{R}$. We know that $\mathscr{F}$ is a σ-field and that it contains our semi-ring P of all half-open intervals $(a,b]$. Since P generates the Borel sets we see that every Borel set is Lebesgue measurable.

We now have that the restriction of λ^* to $\mathscr{F}$ is a measure. We call this measure *Lebesgue measure on* $\mathbb{R}$ and denote it by λ. In fact λ is a complete measure on $\mathscr{F}$, and is the completion of the measure you get by restricting λ to the Borel sets. Note that for every half-open interval $A=(a,b]$ we have $\lambda(A)=\lambda^*(A) = b-a$. It follows fairly easily that λ is the length of A for all sets which are finite unions of any kind of intervals. (See question sheet 5 for the various types of intervals.) This makes it reasonable to take Lebesgue measure, λ, as our notion of length.

Unfortunately, not all subsets of $\mathbb{R}$ are Lebesgue measurable. We proved this earlier in the lectures when we constructed a non-measurable set. Full details may also be found in Chapter 5 of the printed notes.