G1CMIN MEASURE AND INTEGRATION: QUESTION SHEET 1

Answers to questions 1 and 2 to be handed at the end of the lecture on Friday February 7th.

Although you are not required to hand in the questions on Riemann integration, I *strongly recommend* that you try them! They give you a chance to revise Riemann integration, and to test your understanding of fields and σ-fields.

Always justify your answers.

1.(i) Let $E \subseteq \overline{\mathbb{R}}$. Prove that

$$\inf E = -\sup\{-x : x \in E\}.$$

(ii) Use (i) to show that, as claimed in the notes, for every sequence $(x_n) \subseteq \overline{\mathbb{R}}$,

$$\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty}(-x_n)$$

2.(i) Prove carefully the following statement: "Let (x_n), (y_n) be *nondecreasing* sequences in $[0,\infty]$, and suppose that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then $x_n y_n \to xy$ as $n \to \infty$." (Throughout convergence is as defined for sequences in $\overline{\mathbb{R}}$.)

(ii) Does the statement in (i) remain true if the word "nondecreasing" is omitted?

(iii) Show that the statement in (i) is false if the interval $[0,\infty]$ where x_n and y_n are assumed to be is replaced by the interval $(-\infty,\infty]$.

In the remaining questions, χ_E denotes the characteristic function of the set E:

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{otherwise.} \end{cases}$$

3. (Revision) Show that for every *finite* set $E \subseteq [0,1]$, χ_E is Riemann integrable on $[0,1]$.

4. Let $\mathcal{F}$ be the family of all sets $E \subseteq [0,1]$ such that χ_E is Riemann integrable on $[0,1]$. Prove that $\mathcal{F}$ is a field of subsets of $[0,1]$, but not a σ-field.

5. Give an example of a sequence of functions $f_n : [0,1] \longrightarrow [0,1]$ and a function $f : [0,1] \longrightarrow [0,1]$ such that each function f_n is Riemann integrable on $[0,1]$, $f_n \to f$ pointwise on $[0,1]$ but f is *not* Riemann integrable on $[0,1]$.

6. (Harder)

(i) Let C be the Cantor middle-thirds set. Is χ_C Riemann integrable on $[0,1]$? If so, what is its integral?

(ii) Find a *closed* set $E \subseteq [0,1]$ such that χ_E is *not* Riemann integrable on $[0,1]$.