G1CMIN MEASURE AND INTEGRATION: QUESTION SHEET 2

Answers to questions 1 and 4 to be handed in by the end of the lecture on Friday February 21st.

1. Let X be a set, and let R be a collection of subsets of X. Prove that R is a ring if and only if

(i) $\emptyset \in R$

(ii) for all A, B in R, both $A \cap B$ and $A \Delta B$ are in R.

Assuming that R is a ring, show that, with $\cap$ as multiplication and Δ as addition, the associative and distributive laws are satisfied.

2. Show that proposition 3.6 (ii) may fail if the condition $\mu(A_1) < \infty$ is removed, i.e. give an example where the measure of a countable intersection is not the limit of the measures of the finite intersections.

3. (i) Without using the Riemann integral, prove from first principles that if $(a_k, b_k]$ are finitely many pairwise disjoint half open intervals contained in $\mathbb{R}$ $(1 \leqslant k \leqslant n)$ and

$$\bigcup_{k=1}^{n} (a_k, b_k] = (a, b]$$

then

$$b - a = \sum_{k=1}^{n} (b_k - a_k).$$

(you must not use standard properties of Lebesgue measure here, since this result is used in the construction of Lebesgue measure).

(ii) Now consider the two dimensional case: suppose that $(a_k, b_k] \times (c_k, d_k]$ are finitely many pairwise disjoint half open rectangles $(1 \leqslant k \leqslant n)$ and that

$$\bigcup_{k=1}^{n} (a_k, b_k] \times (c_k, d_k] = (a, b] \times (c, d].$$

Prove (by any method, as long as you do not use properties of Lebesgue area measure) that

$$(b - a)(d - c) = \sum_{k=1}^{n} (b_k - a_k)(d_k - c_k).$$

Does your method generalise to the n-dimensional case?

4. Prove that, for each of the following families of subsets of $\mathbb{R}$ the σ-field they generate on $\mathbb{R}$ is the set of all Borel subsets of $\mathbb{R}$.

(i) $\{(a, b) : a, b \in \mathbb{R}\}$

(ii) $\{[a, b] : a, b \in \mathbb{R}\}$

(iii) $\{(-\infty, a] : a \in \mathbb{R}\}$.

5. Let $\mathscr{B}$ denote the collection of all Borel subsets of $\mathbb{R}$. Set

$\mathscr{F} = \{A \cup B : A \in \mathscr{B}, B \subseteq \{-\infty, \infty\}\}$. Show that $\mathscr{F}$ is a σ-field of subsets of $[-\infty, \infty]$.

Prove that this σ-field is generated by the following family of sets: $\{[-\infty, a] : a \in \mathbb{R}\}$.

6*. The σ-field $\mathscr{F}$ constructed in question 5 is known as the collection of *Borel* subsets of the extended real line. However, there is an alternative definition for the collection of Borel subsets of $[-\infty,\infty]$, namely the σ-field generated by the open subsets of $[-\infty,\infty]$ (as defined in the notes, or using the metric given in the notes). Prove that these two definitions agree. [Hint: every open subset of $\mathbb{R}$ is also open in $[-\infty,\infty]$. What other open subsets of $[-\infty,\infty]$ are there?].