G1CMIN MEASURE AND INTEGRATION: QUESTION SHEET 3

Answers to questions 1 and 4 to be handed in by the end of the lecture on Friday 7th March.

Always justify your answers!

Recall that if f is a function from X to Y, E is a subset of X and F is a subset of Y then $f(E)=\{f(x): x \in E\}$ while $f^{-1}(F) = \{x \in X : f(x) \in F\}$.

1. Let X, Y be sets, and let $f: X \longrightarrow Y$ be a function. Suppose that $\mathcal{F}_1$ is a σ-field of subsets of X, and that $\mathcal{F}_2$ is a σ-field of subsets of Y. Prove that

$$\{f^{-1}(E): E \in \mathcal{F}_2\}$$

is a σ-field on X, and that

$$\{E \subseteq Y: f^{-1}(E) \in \mathcal{F}_1\}$$

is a σ-field on Y.

Is it necessarily true that

$$\{f(E): E \in \mathcal{F}_1\}$$

is a σ-field on Y?

2. Let $(X,\mathcal{F})$ be a measurable space, let $Y=\mathbb{R}$, and let Z be either $\mathbb{R}$ or $\overline{\mathbb{R}}$ (in fact Y and Z could be any metric spaces here). Suppose that $f: X \longrightarrow Y$ is measurable, and that $g: Y \longrightarrow Z$ is measurable (note: *by default* we use the Borel σ-fields on Y and Z). Prove that $g \circ f: X \longrightarrow Z$ is measurable.

A measure space $(X,\mathcal{F},\mu)$ is said to be *complete* if, whenever A, E, B are subsets of X such that A, B are in $\mathcal{F}$, $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$, then E is also in $\mathcal{F}$. In Chapter 5 we shall see that Lebesgue measure on the σ-field of Lebesgue measurable subsets of $\mathbb{R}$ is a complete measure. In question 3 we see another way to make complete measures.

3. Suppose that $(X,\mathcal{F},\mu)$ is a measure space. Set

$$\overline{\mathcal{F}} = \{E \subseteq X: \textit{there are } A,B \in \mathcal{F}, \textit{ with } A \subseteq E \subseteq B \textit{ such that } \mu(B \setminus A) = 0\}.$$

Prove that $\overline{\mathcal{F}}$ is a σ-field on X containing $\mathcal{F}$, that there is a unique measure $\overline{\mu}$ on $\overline{\mathcal{F}}$ extending μ (i.e. such that $\overline{\mu}(E) = \mu(E)$ for all E in $\mathcal{F}$), and that this measure $\overline{\mu}$ is complete.

In the next question you may find it helpful to note that if $(X,\mathcal{F},\mu)$ is a complete measure space (see above) and E is in $\mathcal{F}$ with $\mu(E) = 0$ then *every* subset of E is also in $\mathcal{F}$.

4. Let $(X,\mathcal{F},\mu)$ be a complete measure space (see above), let $f: X \longrightarrow \overline{\mathbb{R}}$ be a measurable function, and let $g: X \longrightarrow \overline{\mathbb{R}}$ be a function which satisfies $g(x) = f(x)$ almost everywhere (i.e. there is a set $E \in \mathcal{F}$ with $\mu(E) = 0$ and such that $f(x) = g(x)$ for all $x \in X \setminus E$). Prove that g is also measurable. (This result is false without the assumption that μ is a complete measure).

REMARK: thus, for a complete measure space, we can change a function however we like on a set of measure zero without affecting its measurability. In particular, this applies when we use Lebesgue measure on the σ-field of Lebesgue measurable subsets of $\mathbb{R}$. (See Chapter 5 for more details.)