G1CMIN MEASURE AND INTEGRATION: QUESTION SHEET 4

Answers to questions 1 and 2 to be handed in by the end of the lecture on Friday April 25th

Always justify your answers!

1. Let $(X,\mathcal{F},\mu)$ be a measure space, and let $E \in \mathcal{F}$. As in the course we define $\mathcal{F}_E$ by

$$\mathcal{F}_E = \{F \cap E : F \in \mathcal{F}\}.$$

(i) Show that $\mathcal{F}_E$ is a σ-field on E.

(ii) (Easy!) Let ν be the restricton of μ to $\mathcal{F}_E$. Show that ν is a measure on E.

(iii) Let $s: X \longrightarrow [0,\infty)$ be a simple measurable function. Let t be the restriction of s to E. Prove that, with respect to the σ-field $\mathcal{F}_E$, t is a simple measurable function from E to $[0,\infty]$, and that

$$\int_E s\,d\mu = \int_E t\,d\nu.$$

(iv) Now let $f: X \longrightarrow [0,\infty]$ be a measurable function, and let g be the restriction of f to E. Using (iii) and the Monotone Convergence Theorem, or otherwise, prove that g is $\mathcal{F}_E$-measurable, and that

$$\int_E f\,d\mu = \int_E g\,d\nu.$$

[Hint: one way to do this is to consider a sequence of simple functions approximating $f\chi_E$ on X and apply the Monotone Convergence Theorem twice].

2. Let $(X,\mathcal{F},\mu)$ be a measure space, and let f, g be measurable functions from X to $[0,\infty]$. Suppose that, with respect to μ, the functions f and g are equivalent (i.e. $f(x)=g(x)$ almost everywhere). Prove that, for every set $E \in \mathcal{F}$,

$$\int_E f\,d\mu = \int_E g\,d\mu.$$

(Thus functions which agree almost everywhere are pretty much indistinguishable from the point of view of integration).

3. Let μ be counting measure on $\mathbb{N}$, and let $f(n) = 3^{-n}$. Calculate

$$\int_{\mathbb{N}} f\,d\mu\,.$$

Either by direct calculation, or quoting an appropriate theorem, prove that as n tends to ∞,

$$\int_{\mathbb{N}} f^n\,d\mu$$

tends to zero.

4. (From 1994-5 G13AN4 exam)

(a) Using the dominated convergence theorem, or otherwise, prove carefully that

$$\lim_{n \to \infty} \left(\int_0^\infty \frac{\sin(n^2 x)}{e^x + nx^3} \, dx \right) = 0.$$

(b) Let (f_n) be a sequence of Borel measurable functions from $[0,\infty)$ to $\mathbb{R}$ such that $\int_0^\infty |f_n(x)| \, dx \leqslant 1$ for all n, and such that $f_n \to 0$ uniformly on $[0,\infty)$. Is it necessarily true that

$$\lim_{n \to \infty} \left(\int_0^\infty f_n(x) \, dx \right) = 0?$$

5. Show that the following limit exists, and find its value

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2} \right)^{1+\frac{1}{n}} \right)$$