G1CMIN MEASURE AND INTEGRATION: QUESTION SHEET 5

Answers to questions 2 and 5 to be handed in by the end of the lecture on Friday May 2nd 2003.

1. If you want to obtain area measure for subsets of $\mathbb{R}^2$, you begin with half-open rectangles by defining $\mu((a,b]\times(c,d]) = (b-a)(d-c)$. On question sheet 2, you showed this was finitely additive. Now prove that μ is a *measure* on the semi-ring of all half-open rectangles $(a,b]\times(c,d]$.

You may assume the following compactness result: if

$$[a,b]\times[c,d] \subseteq \bigcup_{k=1}^{\infty} (a_k,b_k)\times(c_k,d_k)$$

then there exists $n \in \mathbb{N}$ such that

$$[a,b]\times[c,d] \subseteq \bigcup_{k=1}^{n} (a_k,b_k)\times(c_k,d_k).$$

Once you have this measure, our powerful extension machinery immediately provides a measure on a σ-field containing all the elementary figures in $\mathbb{R}^2$, and in particular we can measure the area of all Borel sets. Of course, all these results also work for dimensions higher than 2. This allows us to construct Lebesgue measure on the Borel subsets of $\mathbb{R}^n$, for each n in $\mathbb{N}$.

In the remaining questions, λ^* denotes Lebesgue outer measure on $\mathbb{R}$, and λ denotes Lebesgue measure on the set of Lebesgue measurable sets. You may assume that λ is a measure, and that $\lambda((a,b]) = b-a$ whenever a, b are in $\mathbb{R}$ with $a<b$.

2. Prove that $\lambda((a,b)) = \lambda([a,b)) = \lambda([a,b]) = b-a$ for all a, b in $\mathbb{R}$ with $a<b$. (One method is to start by looking at the Lebesgue measure of single-point sets.)

3. Let S be any countable subset of $\mathbb{R}$. Prove that $\lambda(S) = 0$. (Note, in particular, that $\lambda(\mathbb{Q}) = 0$).

4. (i) Show that the Lebesgue measure of the Cantor middle-thirds set is 0.

(ii) Give an example of a closed subset of [0,1] which has positive Lebesgue measure, but which contains no non-empty open intervals. [Hint: either modify the construction of the cantor set, or simply enumerate the rationals in [0,1] and then delete a suitable collection of open intervals from [0,1].]

5. Starting from the definition

$$\lambda^*(E) = \inf\{\sum_{n=1}^{\infty} (b_n-a_n): a_n \leqslant b_n \in \mathbb{R},\, E \subseteq \bigcup_{n=1}^{\infty} (a_n,b_n]\}$$

given in the notes for Lebesgue outer measure λ^*, prove the result claimed in the notes that, for any $E \subseteq \mathbb{R}$ **(note that E need not be measurable!)**,

$$\lambda^*(E) = \inf\{\sum_{n=1}^{\infty} (b_n-a_n): a_n \leqslant b_n \in \mathbb{R},\, E \subseteq \bigcup_{n=1}^{\infty} (a_n,b_n)\}$$

and

$$\lambda^*(E) = \inf\{\sum_{n=1}^{\infty} (b_n-a_n): a_n \leqslant b_n \in \mathbb{R},\, E \subseteq \bigcup_{n=1}^{\infty} [a_n,b_n]\}.$$