Module code: G1CMIN **Module name: Measure and Integration**
Session: 2002-3 **Credits: 15**
Module Lecturer: Dr J.F. Feinstein **Other staff involved: None**

Detailed blow-by-blow account

Lecture 1: *Chapter 0. Introduction* General description of content and motivation for the module (length and area, connections with integrals, modes of convergence of functions, convergence theorems for integrals, formal manipulation of ∞).

Lecture 2: *Chapter 1. The extended real line* The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, a totally ordered set. Intervals, upper bounds, lower bounds. Most details left as exercises (some on question sheet 1). EVERY subset E of $\overline{\mathbb{R}}$ has an infimum and a supremum in $\overline{\mathbb{R}}$, denoted by $\inf(E)$ and $\sup(E)$ respectively. The minus operator $x \mapsto -x$. Sequences in $\overline{\mathbb{R}}$: the limit infimum and limit supremum of a sequence ($\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$). Warning over notation. Convergent sequences in $\overline{\mathbb{R}}$ defined in terms of lim inf and lim sup: this extends the usual definition of convergence in $\mathbb{R}$, but sequences which previous diverged to $+\infty$ now *converge* to $+\infty$ in $\overline{\mathbb{R}}$. Equivalent definitions of convergence to $\pm\infty$ in $\overline{\mathbb{R}}$. A standard homeomorphism between $\mathbb{R}$ and $(-1,1)$ extends to a homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$.

Lecture 3: The monotone sequence theorem in $\overline{\mathbb{R}}$. The algebra of limits in $[0,\infty]$ (with limitations: see question sheet 1). Arithmetic in $\overline{\mathbb{R}}$: addition and subtraction (where possible) and multiplication in $\overline{\mathbb{R}}$. Series in $\overline{\mathbb{R}}$. Series with terms in $[0,\infty]$. Fact: series with non-negative terms can be rearranged arbitrarily and still give the same sum (finite or infinite). (Some special cases are proved in the printed notes. These results also follow from results on integration in Chapter 4.)

Lecture 4: Sums over countable sets. Open sets in $\mathbb{R}$, defined as countable unions of open intervals. Properties and examples of open sets. Closed sets in $\mathbb{R}$. Examples. Countable intersections and unions, set difference, complements and De Morgan's laws (complements of countable unions and intersections). Implications for properties of open/closed sets. Open subsets of $\overline{\mathbb{R}}$.

Lecture 5: *Chapter 2. Classes of sets* Motivation recalled: aim to measure the size (total length) of as many subsets of $\mathbb{R}$ as possible. This will be possible (using Lebesgue measure) for a large class of sets (the Borel sets, to be defined shortly). Symmetric difference introduced, various equivalent definitions including addition of characteristic functions modulo 2. The power set of X, $\mathcal{P}(X)$ (or 2^X). Levels of abstraction, notation e.g $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$, $A \in \mathcal{P}(R)$, $\mathcal{C} \subseteq \mathcal{P}(X)$, $\mathcal{C} \in \mathcal{P}(\mathcal{P}(X))$. Semi-rings of sets: Intervals in $\mathbb{R}$. Half-open intervals $P = \{(a,b] : a, b \in \mathbb{R}, a \leq b\}$. Half-open rectangles. Rings of sets.

Lecture 6: Elementary figures in $\mathbb{R}$: finite (disjoint) unions of half-open intervals from P. Elementary figures in $\mathbb{R}^2$. The ring generated by a semi-ring. Fields (or algebras) of subsets of a set X (also called 'fields on X'). Alternative definitions, deductions from axioms.

Lecture 7: The field generated by a ring. Examples. Definition and examples of σ-fields of subsets of a set X.

Lecture 8: What can you say about a σ-field on $\mathbb{R}$ if it includes all of the half-open intervals $(a,b]$? Indexing sets, intersections of indexed families of collections of subsets of X, $\displaystyle\bigcap_{\gamma\in\Gamma} \mathcal{S}_\gamma$. Whenever you have some σ-fields of subsets of X, say $\mathcal{F}_\gamma$ ($\gamma \in \Gamma$), then $\bigcap_{\gamma\in\Gamma} \mathcal{F}_\gamma$ is also a σ-field of subsets of X. The σ-field on X generated by a collection $\mathcal{C}$ of subsets of X, denoted in this module by $\mathcal{F}(\mathcal{C})$ or, to avoid ambiguity, $\mathcal{F}_X(\mathcal{C})$. This is the smallest possible σ-field on X which includes all of the sets in $\mathcal{C}$. More formally, $\mathcal{F}_X(\mathcal{C})$ is a σ-field on X, $\mathcal{C} \subseteq \mathcal{F}_X(\mathcal{C})$ and, whenever $\mathcal{G}$ is a σ-field on X such that $\mathcal{C} \subseteq \mathcal{G}$ then we have also $\mathcal{F}_X(\mathcal{C}) \subseteq \mathcal{G}$. Proof of the existence and properties of $\mathcal{F}_X(\mathcal{C})$.

Lecture 9: Comparison of $\mathcal{F}_X(\mathcal{C})$ and $\mathcal{F}_Y(\mathcal{C})$ when $X \subseteq Y$ (exercise: see also question sheet 2). The σ-field, $\mathcal{B}$, of all Borel sets in $\mathbb{R}$ (also called Borel subsets of $\mathbb{R}$ or Borel measurable subsets of $\mathbb{R}$): $\mathcal{B}$ is the σ-field generated by the collection of all open subsets of $\mathbb{R}$. Borel subsets of $\overline{\mathbb{R}}$ (and other metric spaces). Examples of Borel subsets of $\mathbb{R}$, including open sets, closed sets, $\mathbb{Q}$, countable interesections of open sets, countable unions of closed sets. There are many other Borel sets. Brief comments on transfinite induction (beyond the scope of this module, but see books if interested). The Cantor set and the Cantor function. Proof that $\mathcal{F}_{\mathbb{R}}(P) = \mathcal{B}$ (with P and $\mathcal{B}$ as above). Related facts are on question sheet 2. All these proofs are based on the fact that whenever $\mathcal{C} \subseteq \mathcal{P}(\mathbb{R})$ and a σ-field $\mathcal{G}$ on $\mathbb{R}$ is such that $\mathcal{C} \subseteq \mathcal{G}$ then the σ-field on $\mathbb{R}$ generated by $\mathcal{C}$, $\mathcal{F}_{\mathbb{R}}(\mathcal{C})$, must also be $\subseteq \mathcal{G}$.

Lecture 10: Short cuts for proving $\mathcal{F}(\mathcal{C}_1) = \mathcal{F}(\mathcal{C}_2)$: $\mathcal{F}(\mathcal{C}_1) \subseteq \mathcal{F}(\mathcal{C}_2)$ if and only if $\mathcal{C}_1 \subseteq \mathcal{F}(\mathcal{C}_2)$.

Chapter 3. Measures and measure spaces Brief mention of notions of size: counting measure, length, area, volume. Definition of (positive) measure on a collection $\mathcal{C}$ of subsets of X (with $\emptyset \in \mathcal{C}$). Examples: length of half-open intervals (see Chapter 5), counting measure (on any set). Measurable spaces and measure spaces. Properties of measures on rings: countable additivity (part of definition), finite additivity.

Lecture 11: Monotonicity and countable subadditivity of measures on rings. Finite measures, probability measures, σ-finite measures. Positive measures (the measures used in this module) and other kinds of measures: complex measures, real measures, signed measures. Hahn decomposition for complex/signed measures stated in the form $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ (where μ_i are positive measures, $1 \leq i \leq 4$). More examples: the zero measure, point mass measures, sums and multiples of measures.

Lecture 12: Continuity properties of measures on rings: measures (when defined) of countable unions and countable intersections of sets in rings. Properties that hold almost everywhere e.g. almost every real number is irrational (because $\lambda(\mathbb{Q}) = 0$, where λ is Lebesgue measure on $\mathbb{R}$).

Lecture 13: Equivalence (almost everywhere equality) of functions on measure spaces. What this means for some specific measures. Completeness of measures (see question sheet 3 for more details). An example of a 'non-measurable' (in particular, non-Borel) subset of $[0, 1]$ (using equivalence classes modulo the rationals). This uses facts that will be proven in Chapter 5 (see below). Translates of subsets of $\mathbb{R}$ ($E + x$ or $x + E$). Translates of Borel sets are also Borel sets (exercise, or see later). Some standard properties of λ^* (Lebesgue outer measure on $\mathbb{R}$) stated (see Chapter 5 for details): monotonicity, translation invariance, countable additivity on Borel sets, correct length for intervals. We can show that λ^* does not add up correctly for translates of the set we constructed above, so it is 'non-measurable'.

Lecture 14: Final details concerning the non-measurable set constructed in the previous lecture.

Chapter 4: The Integral Revision of Riemann integration: approximation of functions from below and above using step functions (staircase functions). The idea behind the Lebesgue integral: start with finite linear combinations of characteristic functions of sets more general than intervals (can use any measurable sets). These will be easy to integrate (in particular we will have no problem integrating $\chi_{\mathbb{Q}}$, which gave problems with the Riemann integral). Simple functions on X: definition (n.b. simple functions are real-valued), examples, standard form (using the distinct values, partition the set X and so form a finite linear combination of characteristic functions). Sums, products and linear combinations of simple functions are still simple functions. Every finite linear combination of characteristic functions is a simple function (even if the sets do not form a partition of X or the coefficients are not distinct).

Lecture 15: Over-use of the word measurable: measurable spaces, measurable sets and measurable functions. Continuous functions, images and pre-images revised. Topological definition of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (in terms of pre-images of open sets). Measurable functions from one measurable space to another. Using the Borel sets on the codomain it is enough to check the pre-images of open sets.

Lecture 16: Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is (Borel) measurable. (By default we use the Borel sets as our σ-field on $\mathbb{R}$. See Chapter 5, however, for discussion of the Lebesgue measurable sets.) Measurability of functions taking values in $\overline{\mathbb{R}}$. Four conditions equivalent to measurability (from X to $\mathbb{R}$ or $\overline{\mathbb{R}}$), including that, for all $a \in \mathbb{R}$, the set $\{x \in X : f(x) \leq a\}$ be a measurable set. The function $-f$ is measurable if and only if f is measurable.

Lecture 17: The pointwise sup, inf, lim sup and lim inf of any sequence of measurable functions is measurable. Hence every pointwise limit of a sequence of measurable functions is a measurable function.

Lecture 18: Characteristic functions of measurable sets are measurable functions, while those of non-measurable sets are non-measurable functions. Most sensible functions are measurable, but the characteristic function of the non-Borel set we constructed earlier is a non-measurable function on $\mathbb{R}$. Simple measurable functions (measurable simple functions): finite linear combinations of characteristic functions of measurable sets. A sum or product of two simple measurable functions is again a simple measurable function. Sketch shown of approximation of the function $f(x) = x$ by simple functions on $[0, \infty)$. Monotone approximation from below of non-negative measurable functions using non-negative simple measurable functions. Deduction (from the corresponding result for simple measurable functions) of the fact that the sum and product of two non-negative measurable functions is measurable. Many results for general measurable functions can be deduced in the same way using the results for simple measurable functions and this method of approximation.

Lecture 19: The pointwise maximum/minimum of two measurable functions is measurable. Recall: the pointwise maximum of two $\overline{\mathbb{R}}$-valued measurable functions is a measurable function. Decomposition of $\overline{\mathbb{R}}$-valued functions into positive and negative parts: $f = f^+ - f^-$. The function f is measurable if and only if both f^+ and f^- are. Definition of the (Lebesgue) integral of non-negative, simple measurable functions (notation: $I_E(s, \mu)$ [non-standard]). Connection with Riemann integrals of staircase functions. Brief discussion of some standard facts (mostly intuitively obvious, proofs in printed notes or on question sheet 4, some proofs discussed in lectures).

Lecture 20: When s is non-negative, simple measurable, then the function $\phi(E) = I_E(s, \mu)$ is a measure on $\mathcal{F}$. Definition of the Lebesgue integral of a non-negative measurable function, $\int_E f \, \mathrm{d}\mu$. In particular this gives the same value as before for simple measurable functions: $I_E(s, \mu) = \int_E s \, \mathrm{d}\mu$. Thus we may safely switch to the new notation, but maintain our old results. For example, when s is non-negative, simple measurable, then the function $\psi(E) = \int_E s \, \mathrm{d}\mu$ is a measure on $\mathcal{F}$. Brief discussion of standard facts about integrals of non-negative, measurable functions (proofs in printed notes, most follow directly from the definitions and the results for non-negative, simple measurable functions).

Lecture 21: Further elementary problems of the integral. Revision of continuity properties of measures. Statement and proof of the Monotone Convergence Theorem.

Lecture 22: Typical application of MCT: deduction of less elementary facts about integrals of non-negative, measurable functions using facts about simple measurable functions and monotone approximation. Integral of a sum of two non-negative, measurable functions:

$$\int_X (f+g)\mathrm{d}\mu = \int_X f \, \mathrm{d}\mu + \int_X g \, \mathrm{d}\mu.$$

For a non-negative measurable function f and $\alpha \in [0, \infty)$ we have

$$\int_X \alpha f \mathrm{d}\mu = \alpha \int_X f \, \mathrm{d}\mu$$

(this may also be proved by elementary means). For non-negative measurable functions f_n,

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \, \mathrm{d}\mu \right).$$

For any non-negative measurable function f, the function $\Phi(E) = \int_E f \, \mathrm{d}\mu$ is a measure on $\mathcal{F}$.

Lecture 23: Counting measure on $\mathbb{N}$: connection between integrals and series. In particular, another proof of the fact that for non-negative extended real numbers $a_{n,k}$,

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{n,k} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{n,k} \right).$$

Statement and proof of Fatou's Lemma.

Lecture 24: Defined (where possible) the integral over E of a measurable $\overline{\mathbb{R}}$-valued function f to be $\int_E f \mathrm{d}\mu = \int_E f^+ \mathrm{d}\mu - \int_E f^- \mathrm{d}\mu$. The (measurable) function f is integrable on E if both f^+ and f^- have finite integral on E. The set of integrable functions f such that f take values in $\mathbb{R}$ (non-standard) is denoted by $L^1(\mu)$ (or $L^1(X, \mu)$ or $L^1(X, \mathrm{d}\mu)$). Most authors allow f to be $\overline{\mathbb{R}}$-valued, but this makes no real difference to the theory. For integrable functions

$$\left| \int_X f \, \mathrm{d}\mu \right| \le \int_X |f| \, \mathrm{d}\mu.$$

Lecture 25: $L^1(\mu)$ is a vector space of functions on X, and, for f, g in $L^1(\mu)$ and α, β in $\mathbb{R}$, we have

$$\int_X (\alpha f + \beta g)\ \mathrm{d}\mu = \alpha \int_X f\ \mathrm{d}\mu + \beta \int_X g\ \mathrm{d}\mu.$$

Statement and proof of the Dominated Convergence Theorem (DCT).

Lecture 26: **Problem class/Tutorial session** on the main three theorems of Chapter 4: the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem (discussed in the context of Riemann integrals of functions and sums of series). Students worked in groups to find counterexamples when conditions of the theorems are weakened, and an example where the inequality in Fatou's Lemma is strict. Answers were discussed.

Lecture 27: Sets of measure zero have no effect on integration. Countable unions of sets of measure 0 still have measure 0. As a result, conditions in convergence theorems are only required to hold almost everywhere. Recalled the definition of Lebesgue outer measure λ^* in terms of possible sums of lengths of sequences of half-open intervals which cover the set, and stated some properties of Lebesgue measure (more details in Chapter 5). The connection between the Riemann integral and the Lebesgue integral w.r.t. Lebesgue measure λ (see below). The two agree for all Riemann integrable functions (idea of proof sketched, based on approximation of a Riemann integrable function by staircase functions). This allows us to use the notation $\int_a^b f(x)\ \mathrm{d}x$ for the Lebesgue integral $\int_{[a,b]} f\ \mathrm{d}\lambda$ of a Lebesgue integrable function (even if it is not Riemann integrable).

Chapter 5. Outer measures and the construction of Lebesgue measure Definition and examples of outer measures. Definition of μ^*-measurable sets for an outer measure μ^*. Statements of some standard results (see printed notes for full details): the set of μ^*-measurable sets is a σ-field and the restriction of μ^* to this σ-field is a complete measure. Lebesgue outer measure λ^* is an outer measure on $\mathbb{R}$ and the half-open intervals $(a,b]$ are λ^*-measurable with $\lambda^*((a,b]) = b - a$.

Lecture 28: Tutorial/problem class session on measures and outer measures.

Lecture 29: Fubini's theorem for double sums (another proof). Using a measure on a ring to define an outer measure.

Lecture 30: Revision of concept of measurability with respect to an outer measure μ^*. Lemma: The collection of μ^* measurable sets is a field on which μ^* is finitely additive.

Lect. 31-32: Proofs of some further details of the construction of Lebesgue measure, including the fact that length gives a measure on our semi-ring P of half-open intervals. Statement and discussion of the extension theorem: every measure on a semi-ring may be extended to a complete measure on a σ-field containing the semi-ring, using the standard outer measure construction. The particular case of Lebesgue outer measure and Lebesgue measure. Question and answer session. Student opinion forms.