G1CMIN MEASURE AND INTEGATION

TUTORIAL PROBLEMS 2

Let $\mathcal{F}$ be a σ-field on a set X. **Recall** that a function $\mu : \mathcal{F} \to [0, \infty]$ is FINITELY ADDITIVE if, for all pairwise disjoint $A_1, A_2, \ldots, A_n$ in $\mathcal{F}$

$$\mu\left(\dot{\bigcup_{k=1}^{n}} A_k\right) = \sum_{k=1}^{n} \mu(A_k),$$

and μ is a **measure** on $\mathcal{F}$ if

(i) $\mu(\phi) = 0$

(ii) whenever $A_1, A_2, A_3 \ldots$ are pairwise disjoint sets in $\mathcal{F}$, then

$$\mu\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Recall also that a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an OUTER MEASURE if

(i) $\mu^*(\phi) = 0$

(ii) whenever $A \subseteq B \subseteq X$ then $\mu^*(A) \leq \mu^*(B)$

(iii) if $A, A_1, A_2, \ldots$ are subsets of X, and $A \subseteq \bigcup_{k=1}^{\infty} A_k$, then

$$\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(A_k).$$

Question For each of the following functions $\mu_i : \mathcal{P}(\mathbb{R}) \to [0, \infty]$, determine whether or not μ_i is (a) finitely additive,(b) a measure, (c) an outer measure, (d) monotone, (e) countably subadditive.

For $E \subseteq \mathbb{R}$, set

$$\mu_1(E) = \begin{cases} 0 & \text{if } E \text{ is a bounded subset of } \mathbb{R}, \\ 1 & \text{otherwise.} \end{cases}$$

$$\mu_2(E) = \begin{cases} 0 & \text{if } E \text{ is a bounded subset of } \mathbb{R}, \\ \infty & \text{otherwise.} \end{cases}$$

$$\mu_3(E) = \begin{cases} \text{The number of points in } E, & \text{if } E \text{ is a finite set,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_4(E) = \begin{cases} 0 & \text{if } E \subseteq \mathbb{Q}, \\ \infty & \text{otherwise.} \end{cases}$$