BANACH FUNCTION ALGEBRAS WITH DENSE INVERTIBLE GROUP

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ABSTRACT. In [9], Dawson and the second author asked whether or not a Banach function algebra with dense invertible group can have a proper Shilov boundary. We give an example of a uniform algebra showing that this can happen, and investigate the properties of such algebras. We make some remarks on the topological stable rank of commutative, unital Banach algebras. In particular, we prove that $\mathrm{tsr}(A) \geq \mathrm{tsr}(C(\Phi_A))$ whenever A is approximately regular.

1. INTRODUCTION

Let A be a commutative, unital Banach algebra. The character space of A is denoted by Φ_A, as in [8]. For $a \in A$, we denote the Gel'fand transform of a by $\widehat{a}$. We say that Φ_A contains analytic structure if there is a continuous injection τ from the open unit disk $\mathbb{D}$ to Φ_A such that, for all $a \in A$, $\widehat{a} \circ \tau$ is analytic on $\mathbb{D}$. In [16], Stolzenberg gave a counter-example to the conjecture that, whenever a uniform algebra has proper Shilov boundary, its character space must contain analytic structure (see also [17, Theorem 29.19], [19] and [1]). Cole gave an even more extreme example in his thesis [3], where the Shilov boundary is proper and yet every Gleason part is trivial. It is elementary to show that the invertible group of A cannot be dense in A whenever Φ_A contains analytic structure. The converse is false, as is shown by the uniform algebra $C(\overline{\mathbb{D}})$ of all continuous, complex-valued functions on $\overline{\mathbb{D}}$. Thus the requirement that the invertible group be dense is strictly stronger than the non-existence of analytic structure in the character space of the algebra. This leads to a new conjecture: that no uniform algebra with dense invertible group can have a proper Shilov boundary. This was raised (as an open question) by Dawson and the second author in [9].

Here, we first modify the example of Stolzenberg to show, in §2, that this new conjecture is also false. Our example is of the form $P(X)$ for a compact set $X \subseteq \mathbb{C}^2$. In our example, the Shilov boundary is proper, and yet there is a dense set of functions in the algebra $P(X)$ whose spectra have empty interiors. It is clear that this latter condition is sufficient for the invertible group to be dense in the algebra.

In fact, this sufficient condition is also necessary for the invertible group to be dense, as is shown in [10] and [7, Corollary 1.10]. Note, however, that it is not true that a function in the closure of the invertible group must have a spectrum whose interior is empty. For example, the invertibles are dense in the uniform algebra

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$C(\mathbb{I})$, where $\mathbb{I} = [0, 1]$, but there are functions in $C(\mathbb{I})$ whose range (and spectrum) is the unit square.

In §3, we shall obtain a new result about the topological stable rank (to be defined below) of approximately regular commutative, unital Banach algebras; we note that a commutative, unital Banach algebra has dense invertibles if and only if it has topological stable rank equal to 1.

We recall some standard notation and results. For the basic theory of commutative, unital Banach algebras, see, for example, [8], [12], and [17].

In our terminology, a compact space is a non-empty, compact, Hausdorff topological space. We denote by $\dim X$ the covering dimension of X; see [14] for details of many equivalent definitions of $\dim X$.

Let X be a non-empty set. The uniform norm on X is denoted by $|\cdot|_X$. Now let X be a compact space. The algebra of continuous, complex-valued functions on X is denoted by $C(X)$.

Let A be a unital algebra. Then $\mathrm{Inv}\, A$ denotes the invertible group of A. Now suppose that A is a unital Banach algebra. Then we say that A *has dense invertibles* if $\mathrm{Inv}\, A$ is dense in A. Let A be a commutative, unital Banach algebra, and let $a \in A$. We denote the spectrum of a by $\sigma(a)$, so that $\sigma(a) = \widehat{a}(\Phi_A)$. We set $\exp A = \{\exp a : a \in A\}$; in the case where A is commutative, $\exp A$ is exactly the component of $\mathrm{Inv}\, A$ containing the identity.

Let X be a compact space. A *Banach function algebra on X* is a unital subalgebra of $C(X)$ that separates the points of X and is a Banach algebra for a norm $\|\cdot\|$; such an algebra is a *uniform algebra* if it is closed in $(C(X), |\cdot|_X)$. Let A be a Banach function algebra on X. As usual, we identify X with the subset of Φ_A consisting of the evaluations at points of X, and we say that A is *natural* if $\Phi_A = X$. The *Shilov boundary* of A, denoted by Γ_A, is the minimum (non-empty) closed subset K of Φ_A such that $|f|_K = |f|_{\Phi_A}$ ($f \in A$). For $f \in C(X)$, the *zero set* is

$$Z_X(f) = \{x \in X : f(x) = 0\}.$$

Let A be a commutative, unital Banach algebra. Then A is *regular* if, for each proper, closed subset E of Φ_A and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $\psi(a) = 0$ ($\psi \in E$), and A is *approximately regular* if, for each proper, closed subset E of Φ_A and each $\varphi \in \Phi_A \setminus E$, there exists $a \in A$ with $\varphi(a) = 1$ and $|\psi(a)| < 1$ ($\psi \in E$). Let E be a closed subset of Φ_A. Then A_E is the closure in $(C(E), |\cdot|_E)$ of $\{\widehat{a} \mid E : a \in A\}$. We see that A is approximately regular if and only if each such uniform algebra A_E is natural on E. There are many uniform algebras that are approximately regular, but not regular. For example, let X be a compact subset of $\mathbb{C}$ with empty interior. Then $R(X)$ is always approximately regular [20, Proposition 4.7], but need not be regular (see, for example, [11]).

2. The new counter-example

We are now ready to construct our main example.

The topological boundary of a set $X \subseteq \mathbb{C}^n$ is denoted by ∂X. We shall use the notation $\underline{z} = (z, w)$ for a typical element of $\mathbb{C}^2$.

Let X be a compact subset of $\mathbb{C}^n$. The *polynomially convex hull* and the *rationally convex hull* of X are denoted by $\widehat{X}$ and $h_r(X)$, respectively. The algebras $P(X)$ and $R(X)$ are the uniform closures in $C(X)$ of the set of restrictions to X of the polynomials and of the rational functions with poles off X, respectively. It is

standard that $\Phi_{P(X)}$ and $\Phi_{R(X)}$ can be identified with $\widehat{X}$ and $h_r(X)$, respectively. With this identification, $P(\widehat{X})$ and $R(h_r(X))$ are equal to the sets of Gel'fand transforms of the elements of the algebras $P(X)$ and $R(X)$, respectively.

Theorem 2.1. *There exists a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ in $\mathbb{C}^2$ such that $(0,0) \in \widehat{Y}$, and yet $P(Y)$ has dense invertibles. In particular, setting $X = \widehat{Y}$, the uniform algebra $P(X)$ is natural on X and has dense invertibles, but $\Gamma_{P(X)}$ is a proper subset of X.*

Proof Let $\mathcal{F}$ be the family of all non-constant polynomials p in two variables with coefficients in $\mathbb{Q}+i\mathbb{Q}$ such that $p(\overline{\mathbb{D}}^2) \subseteq \overline{\mathbb{D}}$. This is a countable family, and the family $\{p|Y : p \in \mathcal{F}\}$ is guaranteed to be dense in the unit ball of $P(Y)$ whenever Y is a compact subset of $\overline{\mathbb{D}}^2$. We shall construct a compact set $Y \subseteq \partial\overline{\mathbb{D}}^2$ such that $(0,0) \in \widehat{Y}$ and such that, for each $p \in \mathcal{F}$, the spectrum of $p|Y$ with respect to $P(Y)$ has empty interior. From this it quickly follows that Y has the desired properties.

Choose a countable, dense subset $\{\zeta_i : i = 1, 2, \ldots\}$ of $\mathbb{D}$ which does not meet the countable set $\{p(0,0) : p \in \mathcal{F}\}$. Define sets $E_{i,p}$ for $i \in \mathbb{N}$ and $p \in \mathcal{F}$ by

$$E_{i,p} = \{\underline{z} \in \overline{\mathbb{D}}^2 : p(\underline{z}) = \zeta_i\}.$$

Each $E_{i,p}$ is compact, and there are only countably many such sets. Enumerate those pairs $(i, p) \in \mathbb{N} \times \mathcal{F}$ for which $E_{i,p}$ is non-empty as $(i_j, p_j)_{j=1}^{\infty}$, and then set $K_j = E_{i_j, p_j}$ $(j \in \mathbb{N})$. For notational convenience, set $a_j = \zeta_{i_j}$, so that

$$K_j = \{\underline{z} \in \overline{\mathbb{D}}^2 : p_j(\underline{z}) = a_j\}.$$

Note that there may be repeats in the sequence (a_j), and that the sets K_j need not be pairwise disjoint, but this will not matter.

Define polynomials G_j for $j \in \mathbb{N}$ by

$$G_j = \frac{p_j - a_j}{p_j(0,0) - a_j} \, .$$

Then $G_j|K_j = 0$ and $G_j(0,0) = 1$ for each $j \in \mathbb{N}$.

We now define inductively a sequence (F_j) of entire functions on $\mathbb{C}^2$ which also satisfy the conditions that $F_j|K_j = 0$ and $F_j(0,0) = 1$ for each $j \in \mathbb{N}$.

We begin with $F_1 = G_1$. Thus $F_1|K_1 = 0$ and $F_1(0,0) = 1$.

Now suppose that $j \in \mathbb{N}$, and assume that $F_1, \ldots, F_j$ have been defined so as to satisfy $F_m|K_m = 0$ and $F_m(0,0) = 1$ $(m = 1, \ldots, j)$. Set

(1) $$L_j = \{\underline{z} \in \overline{\mathbb{D}}^2 : \operatorname{Re} F_j(\underline{z}) \leq 1/2\},$$

and $H_{j+1}(\underline{z}) = \exp(F_j(\underline{z}) - 1)$. We see that $|H_{j+1}|_{L_j} \leq \exp(-1/2) < 1$ and that $H_{j+1}(0,0) = 1$. Choose $N \in \mathbb{N}$ large enough so that

$$|H_{j+1}^N G_{j+1}|_{L_j} < 1/4 \, .$$

Since $G_{j+1}|K_{j+1} = 0$, we then also have

$$|H_{j+1}^N G_{j+1}|_{L_j \cup K_{j+1}} < 1/4 \, .$$

Set $F_{j+1} = H_{j+1}^N G_{j+1}$. Clearly $F_{j+1}|K_{j+1} = 0$ and $F_{j+1}(0,0) = 1$. The inductive construction now proceeds.

Note that this construction also produces a sequence (L_j) of compact subsets of $\overline{\mathbb{D}}^2$ as specified in (1). We have $K_j \subseteq \operatorname{int} L_j$ $(j \in \mathbb{N})$ (here and below the interior is taken in $\overline{\mathbb{D}}^2$). Moreover, for each $j \in \mathbb{N}$, we have

$$|F_{j+1}|_{L_j \cup K_{j+1}} < 1/4\,,$$

and hence $L_j \cup K_{j+1} \subseteq \operatorname{int} L_{j+1}$.

For each $j \in \mathbb{N}$, consider the variety $W_j = \{\underline{z} \in \mathbb{C}^2 : F_j(\underline{z}) = 1\}$. Note that $(0,0) \in W_j$ and $W_j \cap L_j = \emptyset$. Set $V_j = W_j \cap \overline{\mathbb{D}}^2$. Since W_j is the zero set of an entire function on $\mathbb{C}^2$ which vanishes at $(0,0)$, it follows (using, for example, [13, Theorem C5, Chapter I]) that $V_j \cap \partial \overline{\mathbb{D}}^2 \neq \emptyset$.

Finally, set $M_j = \{\underline{z} \in \overline{\mathbb{D}}^2 : \operatorname{Re} F_j(\underline{z}) \geq 1/2\}$. We now see that the following three facts hold.

(a) The sets $\operatorname{int} L_j$ are nested increasing, and

$$\bigcup_{j=1}^{\infty} K_j \subseteq \bigcup_{j=1}^{\infty} \operatorname{int} L_j\,.$$

(b) We have $V_j \subseteq M_i$ $(j \geq i)$.

(c) For each $j \in \mathbb{N}$, there is a polynomial h_j such that $|h_j|_{M_j} < |h_j(\underline{z})|$ $(\underline{z} \in K_j)$. Here (a) and (b) are clear from the definitions and properties above. For (c), note that

$$|\exp(-F_j)|_{M_j} \leq \exp(-1/2) < 1 = \exp(-F_j(\underline{z})) \quad (\underline{z} \in K_j)\,,$$

and so we may take h_j to be a suitable partial sum of the power series for $\exp(-F_j)$.

We now consider the space of all non-empty, closed subsets of $\overline{\mathbb{D}}^2$, with the Hausdorff metric. (For more details on this metric, see, for example, [8, Appendix A.1].) This metric space is compact, so the sequence (V_j) has a convergent subsequence, say $V_{j_k} \to V \subseteq \overline{\mathbb{D}}^2$. We see that $(0,0) \in V$ because $(0,0)$ is in all of the sets V_j. By property (b) above, $V \subseteq M_i$ for each $i \in \mathbb{N}$. Finally, as noted above, all of the sets V_j have non-empty intersection with $\partial \overline{\mathbb{D}}^2$, and so the same is true of V. Set $Y = V \cap \partial \overline{\mathbb{D}}^2$, a non-empty, compact set. We shall show that this set Y has the desired properties.

We first show that $\widehat{Y} = \widehat{V}$ (and so, in particular, that $(0,0) \in \widehat{Y}$). For this it is enough to show that, for every polynomial p, we have $|p|_V = |p|_Y$. Given a polynomial p, assume for contradiction that there is a $\underline{z}_0 \in V \setminus \partial \overline{\mathbb{D}}^2$ such that $|p(\underline{z}_0)| > |p|_Y$. Then there are disjoint open sets U_1 and U_2 in $\mathbb{C}^2$ with

$$\underline{z}_0 \in U_1 \subseteq \mathbb{D}^2\,,$$

with $Y \subseteq U_2$, and such that $|p(\underline{z})| > |p|_{U_2}$ for all $\underline{z} \in U_1$. By the definition of V, there must be a $j \in \mathbb{N}$ such that $V_j \cap U_1 \neq \emptyset$ and $V_j \cap \partial \overline{\mathbb{D}}^2 \subseteq U_2$. Thus $\max\{|p(\underline{z})| : \underline{z} \in V_j\}$ is attained at a point of $\mathbb{D}^2$, and not at any point of $V_j \cap \partial \overline{\mathbb{D}}^2$. This contradicts the maximum principle [13, Theorem B16, Chapter III] on V_j. Thus we must have $\widehat{Y} = \widehat{V}$, as claimed.

Finally, let $p \in \mathcal{F}$. We wish to show that the spectrum of $p|Y$ in $P(Y)$ has empty interior, i.e., that the set $p(\widehat{Y}) = p(\widehat{V})$ has empty interior. Assume for contradiction that this is not the case. Then there must be some $i \in \mathbb{N}$ with $\zeta_i \in p(\widehat{V})$, and there is some $\underline{z}_0 \in \widehat{V}$ with $p(\underline{z}_0) = \zeta_i$. In particular, $\underline{z}_0 \in E_{i,p}$, and so there is a $j \in \mathbb{N}$

with $\zeta_i = a_j$, $p = p_j$, and $E_{i,p} = K_j$. Thus $\underline{z}_0 \in K_j \cap \widehat{V}$. Since $V \subseteq M_j$, it follows from (c), above, that the polynomial h_j satisfies

$$|h_j|_V \leq |h_j|_{M_j} < |h_j(\underline{z}_0)|,$$

and this contradicts the fact that $\underline{z}_0 \in \widehat{V}$.

We have proved that Y has the desired properties.

Finally, we set $X = \widehat{Y}$. The required properties of $P(X)$ are now immediate from the standard theory discussed earlier. $\qquad\square$

Remarks Let X and Y be as in Theorem 2.1, and set $A = P(X)$.

(a) The coordinate projections $\pi_1 : (z, w) \mapsto z$ and $\pi_2 : (z, w) \mapsto w$ are polynomials and are clearly in $\mathcal{F}$. Thus the projections of X on the two coordinate planes have empty interior, and so $\mathrm{int}\, X = \emptyset$.

(b) Since A is natural on X and $A \neq C(X)$, we must have $\dim X > 0$. By [14, Theorem IV 3], since $\mathrm{int}\, X = \emptyset$, we must have $\dim X \leq 3$. In fact, $P(X)$ clearly has 2 generators, and so, by [7, Remark 2.2], $\dim X \leq 2$. Thus $\dim X$ is equal to either 1 or 2. We *conjecture* that $\dim X = 2$; if so, as we shall note below, $C(X)$ does not have dense invertibles.

(c) Set $E = \pi_1(X)$. Since E has empty interior, the Shilov boundary of $R(E)$ is E. By considering functions in A of the form $f : (z, w) \mapsto r(z)p(w)$, where $r \in R(E)$ and p is a polynomial, we see easily that $X \cap (\overline{\mathbb{D}} \times \mathbb{T}) \subseteq \Gamma_A$. Applying the same argument to the second coordinate projection, we obtain $X \cap \partial \overline{\mathbb{D}}^2 \subseteq \Gamma_A$, and hence we must have $\Gamma_A = Y = X \cap \partial \overline{\mathbb{D}}^2$.

The following result shows that the example constructed above has some further unusual properties. This result is probably known, but we know of no explicit reference. We use the term *clopen* to describe sets which are both open and closed.

Theorem 2.2. *Let A be a uniform algebra with compact character space Φ_A, and suppose that A has dense invertibles. Then every component of every* (*non-empty*) *zero set for A meets Γ_A.*

Proof Set $X = \Gamma_A$. Assume for contradiction that $f \in A$ has a non-empty zero set $Z(f)$ which has a component $K \subseteq \Phi_A \setminus X$. Then K is the intersection of the family of all relatively clopen subsets of $Z(f)$ which contain K, and so there is a relatively clopen subset E of $Z(f)$ with $K \subseteq E \subseteq \Phi_A \setminus X$. We may then choose an open subset U of $\Phi_A \setminus X$ with $E \subseteq U$ and such that $Y := \partial U \subseteq \Phi_A \setminus Z(f)$.

Set $\delta = \inf\{|f(x)| : x \in Y\}$, so that $\delta > 0$, and then choose an invertible element $g \in A$ with $|g - f|_X < \delta/2$. Then $|g^{-1}|_K > 2/\delta > |g^{-1}|_Y$, which contradicts the local maximum modulus principle [17, Theorem 9.8]. $\qquad\square$

Let X and Y be as constructed in Theorem 2.1. Let $\underline{z} \in \widehat{X}$, and suppose that p is a polynomial with $p(\underline{z}) = 0$. Then, since $P(X)$ has dense invertibles, the zero set of p must meet Y. It follows that $\underline{z}$ is in the rational hull $h_r(Y)$ of Y. Thus we have $h_r(Y) = \widehat{Y}$ and $P(Y) = R(Y)$.

From the fact that $(0, 0) \in \widehat{Y} = h_r(Y)$ we quickly deduce that, for all $a \in \mathbb{C}$, there is some $(z, w) \in Y$ with $z = aw$. Set $U = \{(z, w) \in X : w \neq 0\}$. Then the image of U under the rational function z/w is the whole of $\mathbb{C}$. This suggests that the set X is fairly large, but does not by itself contradict the possibility that $\dim X = 1$.

We conclude this section with another example with some even stronger properties.

Theorem 2.3. *There is a uniform algebra A on a compact metric space such that every point of Φ_A is a one-point Gleason part and such that the invertible elements are dense in A, but $\Gamma_A \neq \Phi_A$.*

Proof Taking the example above as our base algebra, we may form a system of root extensions (as in [3] and [17, §19]) to obtain a new uniform algebra A on a compact metric space such that $\{f^2 : f \in A\}$ is dense in A, and hence every point of Φ_A is a one-point Gleason part. Since the base algebra has proper Shilov boundary, the same is true for A [3]. Finally, since the base algebra has dense invertibles, so does A [9, pp. 2837–2838]. $\qquad\square$

3. Topological stable rank

We now discuss the topological stable rank and some related ranks of a commutative Banach algebra. Indeed, there is a variety of different types of stable rank; for more details, history and open questions, see, for example, [2, 7, 15].

Let A be a commutative, unital Banach algebra, and let $a_1, \ldots, a_n \in A$. Set $a = (a_1, \ldots, a_n) \in A^n$. Then the *joint spectrum* of a is denoted by $\sigma(a)$, so that $\sigma(a) = \widehat{a}(\Phi_A)$. The element a is *unimodular* if $\sum_{i=1}^n Aa_i = A$; we denote the set of unimodular elements of A^n by $U_n(A)$. Thus $a \in U_n(A)$ if and only if $0 \notin \sigma(a)$ if and only if $\widehat{a_1}, \ldots, \widehat{a_n}$ have no common zero on Φ_A.

We begin with the definitions of the (Bass) stable rank and topological stable rank, given just for commutative Banach algebras.

Definition 3.1. Let A be a commutative, unital Banach algebra.

The *(Bass) stable rank* of A is the least $n \in \mathbb{N}$ with the property that, for all $(a_1, a_2, \ldots, a_{n+1}) \in U_{n+1}(A)$, there exists $(b_1, b_2, \ldots, b_n) \in A^n$ such that

$$(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1}) \in U_n(A),$$

or ∞ if no such n exists.

The *topological stable rank* of A is the least $n \in \mathbb{N}$ such that $U_n(A)$ is dense in A^n, or ∞ if no such n exists.

The stable rank and the topological stable rank of A are denoted by $\mathrm{sr}(A)$ and $\mathrm{tsr}(A)$, respectively.

We see immediately that a commutative, unital Banach algebra A has dense invertibles if and only if $\mathrm{tsr}(A) \leq 1$.

We now quote some standard results concerning (topological) stable ranks.

Let A be a commutative, unital Banach algebra. For each $n \in \mathbb{N}$, we have $\mathrm{sr}(A) \leq n$ if and only if the natural map $U_n(A) \to U_n(A/I)$ induced by the natural projection $A \to A/I$ is a surjection for every closed ideal I in A [4, Theorem 4]. Further,

$$(2) \qquad\qquad\qquad\qquad \mathrm{sr}(A) \leq \mathrm{tsr}(A) \,;$$

this is [4, Theorem 3] and [15, Theorem 2.3]. This inequality is strict for the disk algebra A, where $\mathrm{sr}(A) = 1$ and $\mathrm{tsr}(A) = 2$ [5, 6].

Let X be a compact space, and set $d = \dim X$. Then

$$(3) \qquad\qquad\qquad \mathrm{sr}(C(X)) = \mathrm{tsr}(C(X)) = [d/2] + 1 \,,$$

where $[t]$ denotes the greatest integer less than or equal to t [21]. In particular, $C(X)$ has dense invertibles if and only if $\dim X \in \{0, 1\}$.

Let A be a commutative, unital Banach algebra. Then

$$(4) \qquad \operatorname{sr}(A) \leq \operatorname{sr}(C(\Phi_A)).$$

This is [6, Corollary 1.6]; it depends on a deep generalization of the Arens–Royden theorem given in [18]. This inequality is strict in many cases; for example, if A is the disk algebra, then $\operatorname{sr}(A) = 1$, whereas $\operatorname{sr}(C(\overline{\mathbb{D}})) = 2$.

Now let A be a regular Banach function algebra on $X = \Phi_A$. Then it is proved in [6, Corollary 1.5] that

$$(5) \qquad \operatorname{sr}(A) \geq \operatorname{sr}(C(X)),$$

and so, by (4), $\operatorname{sr}(A) = \operatorname{sr}(C(X))$. It then follows from (2), (3), and (5) that

$$(6) \qquad \operatorname{tsr}(A) \geq \operatorname{tsr}(C(\Phi_A)).$$

We shall strengthen this last result by giving an elementary proof that applies to approximately regular, rather than regular, algebras. This proof uses the standard version of the Arens–Royden theorem [12, Theorem 7.2].

We first give a topological lemma.

Lemma 3.2. *Let X be a compact space, let E be a closed subset of X, and let $n \in \mathbb{N}$. Suppose that $(f_1, \ldots, f_n) \in C(E)^n$, that $(h_1, \ldots, h_n) \in U_n(C(X))$, and that there exist open subsets $U_1, \ldots, U_n$ of X satisfying the following properties:*

(i) $\bigcup_{i=1}^{n} U_i = X$;

(ii) $0 \notin h_i(U_i)$ $(i = 1, \ldots, n)$;

(iii) $h_i \mid E \cap U_i = f_i \mid E \cap U_i$ $(i = 1, \ldots, n)$.

Then there exists $(g_1, \ldots, g_n) \in U_n(C(X))$ such that $g_i \mid E = f_i$ $(i = 1, \ldots, n)$.

Proof Choose closed subsets $X_1, \ldots, X_n$ of X such that $X_i \subseteq U_i$ $(i = 1, \ldots, n)$ and $\bigcup_{i=1}^{n} X_i = X$, and then, for $i = 1, \ldots, n$, define g_i on $X_i \cup E$ so that $g_i \mid E = f_i$ and $g_i \mid X_i = h_i \mid X_i$. Clearly $g_i \in C(X_i \cup E)$. Next extend g_i to a function, also called g_i, in $C(X)$. We note that, by (ii), $Z_X(g_i) \cap X_i = \emptyset$ $(i = 1, \ldots, n)$. Then $(g_1, \ldots, g_n)$ has the required properties. $\qquad \square$

Theorem 3.3. *Let A be an approximately regular, commutative, unital Banach algebra. Then $\operatorname{tsr}(A) \geq \operatorname{tsr}(C(\Phi_A))$.*

Proof We may suppose that $\operatorname{tsr}(A) < \infty$; set $n = \operatorname{tsr}(A)$. Recall that we have $\operatorname{sr}(C(\Phi_A)) = \operatorname{tsr}(C(\Phi_A))$. We shall show that $\operatorname{sr}(C(\Phi_A)) \leq n$.

Let I be a closed ideal in $C(\Phi_A)$. Then we can identify $C(\Phi_A)/I$ with $C(E)$ for a certain closed subset E of $X := \Phi_A$. Let $(f_1, \ldots, f_n) \in U_n(C(E))$, so that $\bigcap_{i=1}^{n} Z_E(f_i) = \emptyset$. For $i = 1, \ldots, n$, choose closed subsets $F_1, \ldots, F_n$ of X such that $F_i \cap Z_E(f_i) = \emptyset$ $(i = 1, \ldots, n)$ and $X = \bigcup_{i=1}^{n} \operatorname{int}_X F_i$. Set

$$E_i = E \cap F_i \quad (i = 1, \ldots, n).$$

Let $i \in \{1, \ldots, n\}$. Since A is approximately regular, A_{E_i} is a natural Banach function algebra on E_i, and so, by the standard Arens–Royden theorem, there exist $b_i \in \operatorname{Inv}(A_{E_i})$ and $r_i \in C(E_i)$ with $f_i = b_i \exp r_i$ in $C(E_i)$. Since $\operatorname{Inv}(A_{E_i})$ is open in A_{E_i}, we may suppose that $b_i = \widehat{a_i} \mid E_i$ for some $a_i \in A$. Since $U_n(A)$ is dense in

A^n, we may further suppose that $(a_1, \ldots, a_n) \in U_n(A)$. We may also extend r_i so that $r_i \in C(X)$, whilst preserving the equalities

$$f_i \mid E_i = (b_i \exp r_i) \mid E_i = (\widehat{a_i} \exp r_i) \mid E_i \,.$$

Set $h_i = \widehat{a_i} \exp r_i \in C(X)$ $(i = 1, \ldots, n)$. Then $(h_1, \ldots, h_n) \in U_n(C(X))$, and $h_i \mid E_i = f_i \mid E_i$ $(i = 1, \ldots, n)$. For each $i \in \{1, \ldots, n\}$, define H_i to be the closure of the set

$$\{x \in E : f_i(x) \neq h_i(x)\}\,,$$

so that $\bigcap_{i=1}^n H_i = \emptyset$ by the choice of $F_1, \ldots, F_n$. Set

$$U_i = X \setminus (Z_X(h_i) \cup H_i) \quad (i = 1, \ldots, n)\,,$$

so that each U_i is an open subset of X.

We verify that clauses (i), (ii), and (iii) of Lemma 3.2 are satisfied. Clearly we have $0 \notin h_i(U_i)$ and $h_i \mid E \cap U_i = f_i \mid E \cap U_i$ for each $i \in \{1, \ldots, n\}$.

To show that $\bigcup_{i=1}^n U_i = X$, we must see that $S := \bigcap_{i=1}^n (Z_X(h_i) \cup H_i) = \emptyset$. Clearly $S \subseteq E$ because $\bigcap_{i=1}^n Z_X(h_i) = \emptyset$. Assume that $x \in S$, say

$$x \in E \cap \mathrm{int}_X F_i \subseteq \mathrm{int}_E E_i$$

for some $i \in \{1, \ldots, n\}$. We have $x \in Z_X(h_i) \cup H_i$. Since $(\mathrm{int}_E E_i) \cap H_i = \emptyset$, we have $x \notin H_i$. Thus $h_i(x) = 0$. Since $h_i \mid E_i = f_i \mid E_i$, we have $f_i(x) = 0$. However $Z_E(f_i) \cap E_i = \emptyset$, and so this is the required contradiction.

It follows from Lemma 3.2 that there exists $(g_1, \ldots, g_n) \in U_n(C(X))$ such that $g_i \mid E = f_i$ $(i = 1, \ldots, n)$. This shows that $\mathrm{sr}(C(X)) \leq n$, as required. $\qquad\square$

Corollary 3.4. *Let A be an approximately regular, commutative, unital Banach algebra with dense invertibles. Then $C(\Phi_A)$ has dense invertibles.* $\qquad\square$

4. Open questions

We conclude with some open questions.

(1) Let X be as constructed in Theorem 2.1, and set

$$M = \{f \in P(X) : f(0,0) = 0\}\,.$$

What can be said about M^2 and $\overline{M^2}$?

(2) Let A be a uniform algebra with the property that $\exp A$ is dense in A. Can the Shilov boundary of A be proper? Must A be approximately regular? Must A be $C(X)$?

It is conceivable (but unlikely) that $\exp A$ is dense in A for the example $A = P(X)$ of Theorem 2.1. This is true if and only if $\exp A = \mathrm{Inv}\, A$, because A has dense invertibles. By [12, Chapter III, Corollary 7.4], $\exp A = \mathrm{Inv}\, A$ if and only if $H^1(X, \mathbb{Z})$, the first Čech cohomology group of X with integer coefficients, is trivial. Thus, for our example, $\exp A$ is dense in A if and only if X is *simply coconnected* in the sense of [17, Definition 29.24].

(3) Let A be a uniform algebra with character space X. Suppose that $C(X)$ has dense invertibles. Must A have dense invertibles? Does it help to assume that A is regular/approximately regular/has Shilov boundary X? More generally (as asked in [7]), is it always true that $\mathrm{tsr}(A) \leq \mathrm{tsr}(C(X))$, and hence $\mathrm{tsr}(C(\Phi_A)) = \mathrm{tsr}(A)$ in the case where A is approximately regular?

We offer the following caution. Assume that one can prove that A has dense invertibles whenever A is approximately regular and $C(\Phi_A)$ has dense

invertibles. Then we would have a solution to the famous 'Gel'fand problem': Is there a natural uniform algebra A on $\mathbb{I}$ such that $A \neq C(\mathbb{I})$? Indeed, suppose that A is a natural uniform algebra on $\mathbb{I}$. By [20, Lemma 3.1], A is approximately regular. By our assumption, A has dense invertibles, and so $A = C(\mathbb{I})$ by [9, Corollary 1.8].

(4) Let A be a uniform algebra with dense invertibles. Does it follow that $C(\Phi_A)$ has dense invertibles? More generally (as asked in [7]), must we have $\mathrm{tsr}(C(\Phi_A)) \leq \mathrm{tsr}(A)$.

  Let X be as constructed in Theorem 2.1, and set $A = P(X)$. We have conjectured that $\dim X = 2$. If so, then we would have a negative answer to the first question because $C(\Phi_A)$ would not have dense invertibles, but A does.

(5) Let A be an approximately regular, commutative, unital Banach algebra. Is it true that $\mathrm{sr}(A) \geq \mathrm{sr}(C(\Phi_A))$?

(6) Our final questions concern the existence of topological disks in the character space of uniform algebras.

  (a) Let A be a uniform algebra such that $\Gamma_A \neq \Phi_A$. Does $\Phi_A \setminus \Gamma_A$ contain a homeomorphic copy of $\overline{\mathbb{D}}$?

  (b) Let K be a compact subset of $\mathbb{C}^n$ such that $\widehat{K} \neq K$. Does $\widehat{K} \setminus K$ contain a homeomorphic copy of $\overline{\mathbb{D}}$?

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