

Regularity points and Jensen measures for $R(X)$

by

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Abstract. We discuss two types of “regularity point”, points of continuity and R-points for Banach function algebras, which were introduced by the first author and Somerset (2000). We show that, even for the natural uniform algebras $R(X)$ (for compact plane sets X), these two types of regularity point can be different. We then give a new method for constructing Swiss cheese sets X such that $R(X)$ is not regular, but has no non-trivial Jensen measures. The original construction appears in the first author’s previous work. Our new approach to constructing such sets is more general, and allows us to obtain additional properties. In particular, we use our construction to give an example of such a Swiss cheese set X with the property that the set of points of discontinuity for $R(X)$ has positive area.

1. Introduction. In [16], the first author and Somerset studied the failure of regularity for Banach function algebras in terms of two types of “regularity point”: points of continuity and R-points. (These technical terms and others are defined in the next section.) They gave examples of Banach function algebras where points of continuity do not coincide with R-points, but these algebras are not natural. There do not appear to be any examples in the literature of natural Banach function algebras where the two types of regularity point are different. In Section 3 we give examples to show that they can differ even for natural uniform algebras such as $R(X)$, for suitable compact plane sets X . These examples are inspired by an example in [15], where the first author and Mortini also studied these types of regularity points. (In that paper, these points were also described as regularity points of types I and II.) We show that it is possible for $R(X)$ to have exactly one point of continuity while having no R-points, and it is also possible for $R(X)$ to have exactly one R-point while having no points of continuity.

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In Section 4 we give a new way to construct Swiss cheese sets X such that $R(X)$ has no non-trivial Jensen measures, but is not regular. Such examples were first constructed in [11] by the first author. Our new construction is more general, and allows us to obtain additional properties. In particular, we show that in such examples, the “exceptional set” of points where regularity fails (the set of points of discontinuity, and also the set of non-R-points) can have positive area. We also provide a more elementary argument to show that there are no non-trivial Jensen measures for our Swiss cheese sets X , as an alternative to the method used in [11], which was based on the deeper theory of the fine topology. In the next section we describe the background in more detail, and introduce the definitions, terminology and preliminary results we need.

2. Preliminaries. Throughout this note, we shall only consider unital, complex algebras, and we require all algebra norms to be both unital ($\|1\| = 1$) and submultiplicative. By a *compact plane set* we mean a *non-empty*, compact subset of the complex plane. For $F \subseteq \mathbb{C}$, we denote the interior of F by $\text{int } F$.

Let X be a non-empty, compact Hausdorff space. We denote by $C(X)$ the algebra of all complex-valued continuous functions on X . For each $f \in C(X)$ and each non-empty subset E of X , we denote the uniform norm of the restriction of f to E by $\|f\|_E$. In particular, we denote by $\|\cdot\|_X$ the uniform norm on X . When endowed with the norm $\|\cdot\|_X$, $C(X)$ is a Banach algebra. A *Banach function algebra on X* is a subalgebra A of $C(X)$ such that A separates the points of X and contains the constant functions, and such that A has a complete algebra norm. It is standard that if A is a Banach function algebra on X under some algebra norm, then this algebra norm dominates $\|\cdot\|_X$. We say that A is a *uniform algebra on X* if A is a Banach function algebra on X whose norm is $\|\cdot\|_X$. Let A be a Banach function algebra on X . We often identify points of X with the corresponding evaluation characters on A . We say that A is *natural* (on X) if these are the only characters on A .

Let A be a natural Banach function algebra on X , and let $x \in X$. We denote by J_x the ideal of functions f in A such that x is in the interior of the zero set of f , $f^{-1}(\{0\})$. We denote by M_x the ideal of functions f in A such that $f(x) = 0$. Recall that, for an ideal I in A , the *hull* of I , denoted by $h(I)$, is defined by

$$h(I) = \bigcap_{f \in I} \{z \in X : f(z) = 0\}.$$

We say that x is a *point of continuity* (for A) if, for all $y \in X \setminus \{x\}$, we have $J_y \not\subseteq M_x$; we say that x is an *R-point* (for A) if, for all $y \in X \setminus \{x\}$, we have

$J_x \not\subseteq M_y$. We say that A is *regular* if, for every closed subset F of X and every $y \in X \setminus F$, there exists $f \in A$ with $f(y) = 1$ and $f(F) \subseteq \{0\}$.

We note that the name *point of continuity* is related to the *hull kernel topology* on X . Let F be a subset of X . The *kernel* of F , denoted by $k(F)$, is an ideal of A defined by

$$k(F) = \{f \in A : f(F) \subseteq \{0\}\}.$$

The hull kernel topology on X is defined by declaring the closed sets to be subsets $F \subseteq X$ such that $h(k(F)) = F$ [17, p. 12]. The hull kernel topology is weaker than the original topology on X , and they coincide if and only if the natural Banach function algebra A is regular. As shown in [16, Lemma 2.1], a point $x \in X$ is a point of continuity if and only if the identity map from X with the hull kernel topology to X with the original topology is continuous at x , and this holds if and only if all functions in A are hull-kernel-continuous at x .

It is standard (by the above, for example) that the natural Banach function algebra A is regular if and only if every point of X is a point of continuity. This is also equivalent to the condition that every point of X is an R-point [15, 16].

Let $x \in X$. Then we note that x is a point of continuity if and only if, for all $y \in X \setminus \{x\}$, we have $x \notin h(J_y)$; x is an R-point if and only if $h(J_x) = \{x\}$.

Let A be a uniform algebra on X , and let φ be a character on A . A *representing measure* μ for φ is a regular, Borel probability measure supported on X such that

$$\varphi(f) = \int_X f d\mu \quad (f \in A).$$

A *Jensen measure* μ for φ is a representing measure for φ such that

$$\log |\varphi(f)| \leq \int_X \log |f(w)| d\mu(w) \quad (f \in A),$$

where we follow the usual convention that $\log(0) = -\infty$.

It is standard (see, for example, [2, p. 114] or [17, p. 33]) that every character on A has at least one Jensen measure.

Let $x \in X$. We say that a representing measure for x is *trivial* if it is the point mass measure at x . As noted in [11], if x is a point of continuity, then there are no non-trivial Jensen measures for x . In fact the elementary argument that proves this actually establishes the following lemma. (The first part of this result is essentially stated in [17, p. 33].)

LEMMA 2.1. *Let A be a natural uniform algebra on X , and let $x \in X$. Suppose that x has a non-trivial Jensen measure μ , and let F be the closed*

support of μ . Then, for all $y \in F \setminus \{x\}$, we have $J_y \subseteq M_x$. Thus x is not a point of continuity for A , and no point of $F \setminus \{x\}$ can be an R -point for A .

Let X be a compact plane set. We denote by $P(X)$ the set of those functions $f \in C(X)$ which can be uniformly approximated on X by polynomial functions; by $R(X)$ the set of those $f \in C(X)$ which can be uniformly approximated on X by rational functions with no poles on X ; and by $A(X)$ the set of those $f \in C(X)$ which are holomorphic on the interior of X . It is standard that when endowed with the uniform norm, each of $P(X)$, $R(X)$ and $A(X)$ is a uniform algebra on X , and $R(X)$ and $A(X)$ are always natural on X . It is clear that $P(X) \subseteq R(X) \subseteq A(X)$, and Mergelyan's theorem [17, p. 48] asserts that if $\mathbb{C} \setminus X$ is connected, then $P(X) = R(X) = A(X)$.

The algebras $P(X)$ and $R(X)$ are defined similarly when X is a compact subset of \mathbb{C}^n for $n > 1$, where the theory is rather different. However, in this note we will focus on compact plane sets.

For more details concerning the above definitions, see standard texts on uniform algebras [2, 17, 25], and for commutative Banach algebras, see [6].

In this note, by a *Swiss cheese set*, we mean a compact plane set with empty interior obtained by deleting a sequence of open discs from a closed disc, where the sum of the radii of the deleted discs is finite. Note that, in the literature, Swiss cheese sets are often simply called Swiss cheeses, and some authors require additional conditions, or fewer conditions, on these sets.

Swiss cheese sets were first introduced by Roth [24], who required (in addition to our conditions) that the closures of the deleted open discs be pairwise disjoint subsets of the interior of the closed disc from which they were deleted. See [5] for an interesting discussion of Roth's life and work, including her work on Swiss cheese sets. Roth's Swiss cheese sets were the first known examples of compact plane sets X with empty interior such that $R(X) \neq C(X)$. Since then there have been numerous applications of Swiss cheese sets in the literature. We now discuss a few of these applications. We refer the reader to the general texts mentioned above for the definitions of those terms mentioned here which are not central to the subject matter of this note.

In [22], McKissick constructed a Swiss cheese set X such that $R(X)$ is non-trivial and regular. This was the first known example of a non-trivial regular uniform algebra. For a simpler approach to McKissick's example, see [20] or [25, p. 344]. In [4], Cole made use of the properties of McKissick's example in order to construct his famous counterexample to the peak point conjecture. A simpler counterexample was constructed by Basener [1], again based on McKissick's example. Basener's example was later modified by

Izzo [19] to construct polynomially convex compact subsets X of the unit sphere in \mathbb{C}^3 such that $P(X) \neq C(X)$.

A number of examples of the first author are based on Swiss cheese constructions. For example, in [9], a Swiss cheese set X was constructed such that $R(X)$ has no non-zero, bounded point derivations but $R(X)$ is not weakly amenable: this gave a counterexample to a conjecture of S. E. Morris.

In [12], the first author and Heath investigated when it was possible to modify Swiss cheese sets X in order to obtain examples with better topological properties, while preserving some desired properties of $R(X)$. In that paper, the term ‘‘Swiss cheese’’ was used in a more abstract way than the traditional usage: during the modification process it was important to keep track of the large closed disc and the collection of open discs to be deleted, rather than just the associated Swiss cheese set.

We refer the reader to [13, 14] for some further discussion and applications of Swiss cheese sets.

Note that if $R(X)$ is regular, there are no non-trivial Jensen measures supported on X . In [11], the first author constructed Swiss cheese sets X such that $R(X)$ has no non-trivial Jensen measures, but is not regular. In the examples, the line segment $I = [-1/2, 1/2]$ is contained in X , and only the points in $X \setminus I$ are points of continuity. The proof in [11] that $R(X)$ has no non-trivial Jensen measures uses the theory of fine interior (see [7] or [18, p. 319]). In Section 4, we use a more elementary approach based on regularity points.

3. R-points and points of continuity for $R(X)$. In this section we construct a compact plane set X such that $R(X)$ has exactly one R-point but has no points of continuity. We also construct a compact plane set X such that $R(X)$ has exactly one point of continuity but no R-points.

Before we proceed to examples where R-points differ from points of continuity, we give the following lemma, which establishes a relationship between these two types of regularity points.

LEMMA 3.1. *Let X be a compact Hausdorff space, let A be a natural Banach function algebra on X , and let $x \in X$. Suppose that there is an open neighbourhood U of x such that each point in $U \setminus \{x\}$ is a point of continuity for A . Then x is an R-point for A .*

Proof. Since each $y \in U \setminus \{x\}$ is a point of continuity, we have $J_x \not\subseteq M_y$ and thus $y \notin h(J_x)$. This shows $h(J_x) \cap (U \setminus \{x\}) = \emptyset$. Since $h(J_x)$ is connected [16, Theorem 3.2], we conclude that $h(J_x) = \{x\}$ and x is an R-point. ■

In particular, the above lemma holds for $R(X)$.

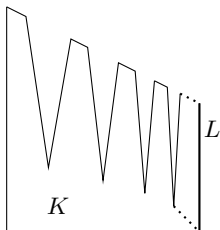


Fig. 1. A compact plane set K , where the slits accumulate on the right

The building blocks of our examples are closures of *slit domains*, such as the compact plane set K shown in Figure 1. The slits cut in K accumulate on the whole of the line segment forming the right edge of K , which we denote by L .

Note that if $f \in R(K) = A(K)$ and f is constant on the line segment forming the *left* edge of K , then an easy application of the reflection principle (see, for example, [21, Chapter IX, Theorem 2.1]) shows that f is constant on K . On the other hand, as a result of the following proposition, there exists a non-constant $f \in R(K)$ which restricts to a constant function on the *right* edge L of K .

PROPOSITION 3.2. *Let K and L be as described above (and shown in Figure 1). Then the Riemann map ψ from $\text{int } K$ onto the open unit disc can be extended to give a continuous map, still called ψ , from K onto the closed unit disc, such that $\psi(z) \neq \psi(w)$ for all $z \in K \setminus L$ and $w \in K \setminus \{z\}$, and ψ is constant on L .*

We remark that this proposition is a special case of a more general theorem on prime ends explained in [23, Section 2.4]. A similar compact set K was used in [15, Example 4.2].

Let K_n be a sequence of such compact plane sets side by side, touching, tapering, and accumulating on the right at a point x_0 , as shown in Figure 2. Set

$$X = \bigcup_{n=1}^{\infty} K_n \cup \{x_0\}.$$

This is our first example. Note that, for the remainder of this note, unless otherwise specified, the ideals J_x and M_x considered will always be the ideals in the relevant algebra $R(X)$ under consideration.

THEOREM 3.3. *Let X be the compact plane set constructed above. Then $\{x_0\}$ is the only R -point for $R(X)$, while $R(X)$ has no points of continuity.*

Proof. For each $n \in \mathbb{N}$, let L_n be the line segment forming the right edge of K_n , and let F_n be the trapezoid-shaped compact domain obtained by filling the slits in K_n .

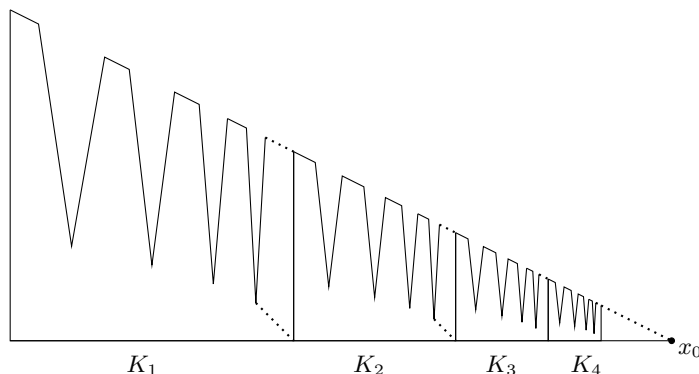


Fig. 2. A compact set X , where $R(X)$ admits exactly one R-point (the point x_0), but no points of continuity

Since $\mathbb{C} \setminus X$ is connected, $R(X) = A(X)$ by Mergelyan's theorem. Let $n \in \mathbb{N}$, let $x \in K_n$, and let $f \in J_x$. Then f vanishes identically on K_n , which forces f to vanish identically on K_m for all $m \geq n$, and hence also at x_0 . This shows that $J_x \subseteq M_y$ for all $y \in \bigcup_{m \geq n} K_m \cup \{x_0\}$. Since n is arbitrary, it follows that no points in X can be points of continuity for $R(X)$, and no points in $X \setminus \{x_0\}$ can be R-points.

Again let $x \in K_n$ for some $n \in \mathbb{N}$. We show that $J_{x_0} \not\subseteq M_x$. Set

$$F = K_{n+1} \cup \bigcup_{\ell=1}^n F_\ell.$$

Then $x \in F$, and x is not on L_{n+1} . Note that F is the closure of a slit domain with right hand segment L_{n+1} . By Proposition 3.2, the Riemann map ψ from $\text{int } F$ to the open unit disc can be extended to give a continuous map, still denoted by ψ , from F onto the closed unit disc. Note that $\psi(z) \neq \psi(w)$ for $z \in F \setminus L_{n+1}$, and $w \in F \setminus \{z\}$. The map ψ is constant on L_{n+1} ; we denote this constant value by c , and set

$$f(z) = \begin{cases} \psi(z) - c & \text{if } z \in F, \\ 0 & \text{if } z \in X \setminus F. \end{cases}$$

Then it is clear that the restriction of f to X is in $A(X) = R(X)$ and $f \in J_{x_0} \setminus M_x$. Thus $J_{x_0} \not\subseteq M_x$. This shows that x_0 is an R-point for $R(X)$. By the above, x_0 is the only R-point. ■

Next we modify the example constructed above to show that there exist compact plane sets X such that $R(X)$ has exactly one point of continuity, but no R-points. Let K be the closure of a slit domain, with the slits accumulating on the whole of the line segment forming the *left* edge, L , of K ,

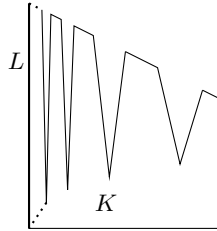


Fig. 3. A compact plane set K , where the slits accumulate on the left

as shown in Figure 3. As in Proposition 3.2, the Riemann mapping ψ from $\text{int } K$ onto the open unit disc can be extended to give a continuous map, still denoted by ψ , from K onto the closed unit disc, such that $\psi(z) \neq \psi(w)$ for all $z \in K \setminus L$ and $w \in K \setminus \{z\}$, and ψ is constant on L . Note that this time, if $f \in A(K)$ and f is constant on the line segment forming the *right* edge of K , then f is constant on K .

Let K_n be a sequence of such compact plane sets side by side, touching, tapering, and accumulating on the right at a point x_0 , as shown in Figure 4.

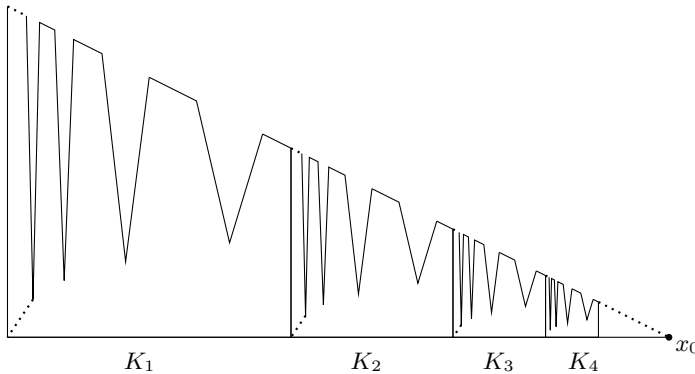


Fig. 4. A compact set X , where $R(X)$ admits exactly one point of continuity (the point x_0), but no R-points

Set

$$X = \{x_0\} \cup \bigcup_{n=1}^{\infty} K_n.$$

This is our second example.

THEOREM 3.4. *Let X be the compact plane set constructed above. Then $\{x_0\}$ is the only point of continuity for $R(X)$, while $R(X)$ has no R-points.*

Proof. For each $n \in \mathbb{N}$, let L_n be the line segment forming the left edge of K_n , and let F_n be the trapezoid-shaped compact domain obtained by filling the slits in K_n .

Since $\mathbb{C} \setminus X$ is connected, $R(X) = A(X)$ by Mergelyan's theorem. Let $x \in X$, and let $f \in J_x$. Then f vanishes on an open neighbourhood of a point in some K_n . This forces f to vanish on all K_m with $m \leq n$, and shows that $J_x \subseteq M_y$ for all $y \in \bigcup_{\ell=1}^n K_\ell$. Thus x is not an R-point. Let $x \in K_m$ for some m , and let $y \in K_n$ for some $n > m$. If $f \in J_y$, then f vanishes identically on all K_ℓ for $\ell \leq n$. This forces $f \in M_x$, which shows that x is not a point of continuity. Next we show that x_0 is a point of continuity for $R(X)$. Let $y \in X \setminus \{x_0\}$; then $y \in K_n$ for some n . Set

$$F = K_{n+2} \cup \left(\bigcup_{\ell=n+3}^{\infty} F_\ell \right) \cup \{x_0\}.$$

Then F is the closure of a slit domain, where the slits accumulate on the left edge L_{n+2} , but where the right edge of F is degenerate (this does not affect the properties that we need here). By Proposition 3.2, the Riemann map ψ from $\text{int } F$ onto the open unit disc can be extended to give a continuous map, still denoted by ψ , from F onto the closed unit disc. Note that $\psi(z) \neq \psi(w)$ for $z \in F \setminus L_{n+2}$, and $w \in F \setminus \{z\}$. Also, ψ is constant on L_{n+2} . We denote this constant value by c , and set

$$f(z) = \begin{cases} \psi(z) - c & \text{if } z \in F, \\ 0 & \text{if } z \in X \setminus F. \end{cases}$$

Then it is clear that the restriction of f to X is in $A(X) = R(X)$ and $f \in J_y \setminus M_{x_0}$. This shows that $J_y \not\subseteq M_{x_0}$, and thus x_0 is a point of continuity. By the above, x_0 is the only point of continuity for $R(X)$. ■

4. Trivial Jensen measures without regularity. In [11], the first author constructed a Swiss cheese set X such that $R(X)$ has no non-trivial Jensen measures, but is not regular. In this section we give a new way to construct such sets. Our new approach also allows us to obtain some additional properties.

We begin with a lemma concerning representing measures for $R(X)$.

LEMMA 4.1. *Let X be a compact plane set, let F be a non-empty closed subset of X , and let $x \in F$. Suppose that no bounded component of $\mathbb{C} \setminus F$ is contained in X , and there exists a non-trivial representing measure μ for x with respect to $R(X)$, whose closed support is contained in F . Then μ is also a non-trivial representing measure for x with respect to $R(F)$, and $R(F) \neq C(F)$.*

Proof. Since no bounded component of $\mathbb{C} \setminus F$ is contained in X , the algebra of functions that are restrictions of functions in $R(X)$ to F is dense in $R(F)$ by Runge's theorem [17, p. 28]. It follows easily that μ is also a

non-trivial representing measure for x with respect to $R(F)$. Since $C(F)$ has no non-trivial representing measures, we must have $R(F) \neq C(F)$. ■

We remark that the compact subset F mentioned in the above lemma must have positive area (by the Hartogs–Rosenthal theorem [2, p. 161]), and F cannot have empty interior and connected complement in \mathbb{C} (by Mergelyan’s theorem, or Lavrent’ev’s theorem [17, p. 48]). We also remark that in the above lemma, the condition that no bounded component of $\mathbb{C} \setminus F$ be contained in X cannot be omitted. To see this, consider the disc algebra, which is equal to $R(\bar{\Delta})$, where Δ is the open unit disc. Let F be the union of the unit circle and the origin. Then there is a non-trivial representing measure μ supported on F for the origin with respect to $R(\bar{\Delta})$, while $R(F) = C(F)$ by the Hartogs–Rosenthal theorem. Of course this measure μ is *not* a representing measure with respect to $R(F)$.

In our construction below, we will need a lemma of McKissick. The following version of McKissick’s lemma was given by Körner [20]. A detailed explanation of this lemma, when the deleted discs are not required to be pairwise disjoint, is given in [25, p. 344].

LEMMA 4.2. *Let D be an open disc in \mathbb{C} and let $\varepsilon > 0$. Then there is a sequence Δ_k ($k \in \mathbb{N}$) of pairwise disjoint open discs contained in D with sum of radii less than ε and such that, if we set $U = \bigcup_{k \in \mathbb{N}} \Delta_k$, there is a sequence f_n of rational functions with poles only in U and such that f_n converges uniformly on $\mathbb{C} \setminus U$ to a function F such that $F(z) = 0$ for all $z \in \mathbb{C} \setminus D$ while $F(z) \neq 0$ for all $z \in D \setminus U$.*

We shall also require the following lemma of Denjoy [8] and Carleman [3]. The original lemma is stated for closed intervals in \mathbb{R} , but there is no difficulty in replacing a closed interval by a line segment in \mathbb{C} . Note that we define derivatives here as limits of quotients using points in the line segment, and all line segments in this note are non-degenerate. We use the usual convention that $1/0 = \infty$ in order to cover the easy special case where the function is a polynomial.

LEMMA 4.3. *Let f be an infinitely differentiable function on a line segment I in \mathbb{C} such that*

$$(4.1) \quad \sum_{k=1}^{\infty} \frac{1}{|f^{(k)}|_I^{1/k}} = \infty.$$

Suppose that there is an $x \in I$ such that $f^{(k)}(x) = 0$ for all $k \geq 0$. Then f is constantly 0 on I .

We also need the following well-known result concerning derivatives of rational functions. (For an explicit proof of this estimate in the case of the first derivative, see the proof of [10, Lemma 2.11].)

LEMMA 4.4. *Let D_n be a sequence of open discs in \mathbb{C} (not necessarily pairwise disjoint), and set $X = \bar{\Delta} \setminus \bigcup_{n=1}^{\infty} D_n$. Suppose that $z \in X$. Let s_n denote the distance from D_n to z , and r_n the radius of D_n . Also set $r_0 = 1$ and $s_0 = 1 - |z|$. Suppose that $s_n > 0$ for all n . Then, for all rational functions f with no poles on X and $k \geq 0$, we have*

$$|f^{(k)}(z)| \leq k! \sum_{j=0}^{\infty} \frac{r_j}{s_j^{k+1}} |f|_X.$$

Let K be a non-empty, compact subset of Δ . Our new approach allows us to construct Swiss cheese sets $X \supseteq K$ such that $R(X)$ has the kind of properties we want. Usually we will be interested in sets K which contain some line segments.

In the following theorem (only) we drop the requirement that Swiss cheese sets must have empty interior. Thus the theorem remains valid when $\text{int } K \neq \emptyset$. The sum of the radii of the deleted discs will still be finite here, though this is not particularly important in view of the other properties of the Swiss cheese set we construct.

THEOREM 4.5. *Let K be a non-empty, compact subset of Δ . Then there is a Swiss cheese set X with $K \subseteq X$ such that every point of $X \setminus K$ is both a point of continuity and an R -point for $R(X)$, but for every closed line segment I contained in K , every $f \in R(X)$ is infinitely differentiable on I and satisfies condition (4.1).*

Proof. Let \mathcal{B} be the set of all open discs B in the plane such that the centre of B is in $\mathbb{Q} + \mathbb{Q}i$, the radius of B is in $\mathbb{Q} \cap (0, \infty)$, and $\bar{B} \subseteq \bar{\Delta} \setminus K$. Let (B_n) be a sequence enumerating the countable set \mathcal{B} . For each $n \in \mathbb{N}$, let d_n be the distance from B_n to K , and choose $\varepsilon_n > 0$ small enough so that $\varepsilon_n \leq d_n^{k+1} (\log(k+3))^k / 2^n$ for all $k \in \mathbb{N}$. Note that, for all $k \in \mathbb{N}$,

$$(4.2) \quad \sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_n^{k+1}} \leq (\log(k+3))^k.$$

Now we apply Lemma 4.2 to each B_n , with ε in the lemma replaced by ε_n . We denote the sequence of open discs obtained from the lemma by $(D_{n,m})_{m \in \mathbb{N}}$, and we denote the radius of $D_{n,m}$ by $r_{n,m}$. Note that for each $n \in \mathbb{N}$ we have $\sum_{m=1}^{\infty} r_{n,m} < \varepsilon_n$. Set

$$X = \bar{\Delta} \setminus \bigcup_{n,m \in \mathbb{N}} D_{n,m}.$$

We have

$$\sum_{n,m \in \mathbb{N}} r_{n,m} < \sum_{n \in \mathbb{N}} \varepsilon_n < \infty,$$

so the sum of the radii of the deleted discs is finite.

Let $z \in K$, and let f be a rational function with no poles on X . For $n, m \in \mathbb{N}$, let $s_{n,m}$ be the distance from z to $D_{n,m}$. Note that $s_{n,m} \geq d_n$, since $D_{n,m} \subseteq B_n$. Let d_0 be the distance from K to the unit circle. For each non-negative integer k , by Lemma 4.4, we have

$$\begin{aligned}
 (4.3) \quad |f^{(k)}(z)| &\leq k! \left(\frac{1}{(1-|z|)^{k+1}} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{r_{n,m}}{s_{n,m}^{k+1}} \right) |f|_X \\
 &\leq k! \left(\frac{1}{d_0^{k+1}} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{r_{n,m}}{d_n^{k+1}} \right) |f|_X \\
 &\leq k! \left(\frac{1}{d_0^{k+1}} + \sum_{n=1}^{\infty} \frac{\varepsilon_n}{d_n^{k+1}} \right) |f|_X \\
 &\leq k! \left(\frac{1}{d_0^{k+1}} + (\log(k+3))^k \right) |f|_X,
 \end{aligned}$$

by (4.2). Let I be a closed line segment in K , and let $f \in R(X)$. Then it follows from (4.3) that f is infinitely differentiable on I and, for each non-negative integer k , we have

$$|f^{(k)}|_I \leq k! \left(\frac{1}{d_0^{k+1}} + (\log(k+3))^k \right) |f|_X.$$

Now we pick $N \in \mathbb{N}$ large enough such that $(\log(k+3))^k \geq 1/d_0^{k+1}$ for all $k \geq N$. Then we have

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{1}{|f^{(k)}|_I^{1/k}} &\geq \sum_{k=N}^{\infty} \frac{1}{|f^{(k)}|_I^{1/k}} \geq \sum_{k=N}^{\infty} \frac{1}{(2|f|_X)^{1/k} (k!)^{1/k} \log(k+3)} \\
 &\geq \sum_{k=N}^{\infty} \frac{1}{(2|f|_X)^{1/k} k \log(k+3)} = \infty.
 \end{aligned}$$

Thus f satisfies condition (4.1) on I .

Let $z \in X \setminus K$. We show that z is a point of continuity for $R(X)$. Let $w \in X$ with $w \neq z$. Then we can find some B_n with $z \in B_n$ and $w \in X \setminus \bar{B}_n$. By Lemma 4.2, we can find $f \in R(X)$ with $f(z) \neq 0$ and such that f vanishes on the whole of $X \setminus \bar{B}_n$. Then $f \in J_w \setminus M_z$, and so $J_w \not\subseteq M_z$. Thus z is a point of continuity for $R(X)$.

Since every point of $X \setminus K$ is a point of continuity for $R(X)$, by Lemma 3.1 all of these points are also R-points for $R(X)$. ■

Next we give a corollary to show that, for suitable K , we can eliminate all the non-trivial Jensen measures for $R(X)$, where X is as constructed in the above theorem.

COROLLARY 4.6. *Let K be a compact subset of Δ . Suppose that $\text{int } K = \emptyset$, $\mathbb{C} \setminus K$ is connected, and K contains at least one closed line segment. Let X be the Swiss cheese set constructed in Theorem 4.5, and let I be a line segment contained in K . Then $R(X)$ has no non-trivial Jensen measures, but no point of I is an R-point or a point of continuity for $R(X)$, and so $R(X)$ is not regular.*

Proof. Let I be a closed line segment in K . By Lemma 4.3, if $f \in R(X)$ vanishes on a neighbourhood of a point x in I , then f vanishes on I . Thus no point of I can be an R-point or a point of continuity.

Certainly there are no non-trivial Jensen measures for points in $X \setminus K$. It remains to show that for each $z \in K$ there are no non-trivial Jensen measures for z . Recall that every point of $X \setminus K$ is an R-point for $R(X)$.

Let $z \in K$. Suppose, for contradiction, that μ is a non-trivial Jensen measure for z with closed support F . By Mergelyan's theorem, we have $R(K) = C(K)$, and so, by Lemma 4.1, F is not contained in K . Let w be a point in $F \setminus K$. Then w is an R-point for $R(X)$. However, this contradicts Lemma 2.1. The result follows. ■

We remark that [11, Corollary 6] is now the special case of Corollary 4.6 where we take K to be the closed line segment $[-1/2, 1/2]$. We also remark that an alternative approach to the last part of our proof is to note that the fine interior of K must be empty, and then use the approach from [11]. Our approach appears to be more elementary, although Lemma 3.1 uses [16, Theorem 3.2], and that result uses the Shilov idempotent theorem. However, for compact plane sets X with empty interior, the special case of [16, Theorem 3.2] for $R(X)$ can be proved without appealing to the Shilov idempotent theorem.

Finally, we give a concrete example where K has positive area and empty interior, and such that the Swiss cheese set X constructed in Theorem 4.5 has the property that the set of points of continuity for $R(X)$ is precisely $X \setminus K$, and this is also equal to the set of R-points for $R(X)$. For this purpose, we use *fat Cantor* sets. A fat Cantor set, also known as a *Smith–Volterra–Cantor* set, is a compact subset of \mathbb{R} that is nowhere dense and has positive length. The Cartesian product of a fat Cantor set and a closed interval can be regarded as a compact subset K of \mathbb{C} with positive area such that $\mathbb{C} \setminus K$ is connected and K has no interior points. Moreover, K is the union of the closed line segments which are contained in K .

The following theorem is now an immediate consequence of Theorem 4.5 and Corollary 4.6.

THEOREM 4.7. *Let $F \subseteq [-1/2, 1/2]$ be a fat Cantor set, and set*

$$K = \{z \in \mathbb{C} : \text{Re}(z) \in F, \text{Im}(z) \in [-1/2, 1/2]\}.$$

Then there is a Swiss cheese set X with $K \subseteq X$ such that every point of $X \setminus K$ is a point of continuity and an R -point for $R(X)$, every point in K is a non- R -point and a point of discontinuity, and $R(X)$ has no non-trivial Jensen measures.

We end this note by asking whether Lemma 3.1 remains true if we interchange R -point with point of continuity.

QUESTION 4.8. Let X be a compact Hausdorff space, let A be a natural Banach function algebra on X , and let $x \in X$. Suppose that there is an open neighbourhood U of x such that each point in $U \setminus \{x\}$ is an R -point for A . Is x necessarily a point of continuity for A ?

We remark that if the algebra A is local (or even 2-local), then Question 4.8 has a positive answer. This is because by [16, Proposition 4.1] the set $F_x = \{y \in X : J_y \subseteq M_x\}$ is connected if A is 2-local. If each $y \in U \setminus \{x\}$ is an R -point for A , then $y \notin F_x$, and the connectedness of F_x forces $F_x = \{x\}$, which is equivalent to x being a point of continuity. In particular, the answer to Question 4.8 is positive for $R(X)$.

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References

- [1] R. Basener, *On rationally convex hulls*, Trans. Amer. Math. Soc. 182 (1973), 353–381.
- [2] A. Browder, *Introduction to Function Algebras*, Benjamin, New York, 1969.
- [3] T. Carleman, *Sur un théorème de M. Denjoy*, C. R. Acad. Sci. Paris 174 (1922), 121–124.
- [4] B. Cole, *One-point parts and the peak point conjecture*, Ph.D. thesis, Yale Univ., 1968.
- [5] U. Daepf, P. Gauthier, P. Gorkin and G. Schmieder, *Alice in Switzerland: the life and mathematics of Alice Roth*, Math. Intelligencer 27 (2005), no. 1, 41–54.
- [6] H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.) 24, Clarendon Press, New York, 2000.
- [7] A. Debiard et B. Gaveau, *Frontière de Jensen d’une algèbre de fonctions*, C. R. Acad. Sci. Paris Sér. A-B 280 (1975), Aii, A85–A87.
- [8] A. Denjoy, *Sur les fonctions quasi-analytiques de variable réelle*, C. R. Acad. Sci. Paris 173 (1921), 1329–1331.
- [9] J. F. Feinstein, *A counterexample to a conjecture of S. E. Morris*, Proc. Amer. Math. Soc. 132 (2004), 2389–2397.
- [10] J. F. Feinstein, *Point derivations and prime ideals in $R(X)$* , Studia Math. 98 (1991), 235–246.

- [11] J. F. Feinstein, *Trivial Jensen measures without regularity*, *Studia Math.* 148 (2001), 67–74.
- [12] J. F. Feinstein and M. J. Heath, *Swiss cheeses, rational approximation and universal plane curves*, *Studia Math.* 196 (2010), 289–306.
- [13] J. F. Feinstein, S. Morley and H. Yang, *Abstract Swiss cheese space and classicalisation of Swiss cheeses*, *J. Math. Anal. Appl.* 438 (2016), 119–141.
- [14] J. F. Feinstein, S. Morley and H. Yang, *Swiss cheeses and their applications*, in: *Function Spaces in Analysis*, *Contemp. Math.* 645, Amer. Math. Soc., 2015, 99–118.
- [15] J. F. Feinstein and R. Mortini, *Partial regularity and t -analytic sets for Banach function algebras*, *Math. Z.* 271 (2012), 139–155.
- [16] J. F. Feinstein and D. W. B. Somerset, *Non-regularity for Banach function algebras*, *Studia Math.* 141 (2000), 53–68.
- [17] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, NJ, 1969.
- [18] T. W. Gamelin and T. J. Lyons, *Jensen measures for $R(K)$* , *J. London Math. Soc.* (2) 27 (1983), 317–330.
- [19] A. Izzo, *Failure of polynomial approximation on polynomially convex subsets of the sphere*, *Bull. London Math. Soc.* 28 (1996), 393–397.
- [20] T. W. Körner, *A cheaper Swiss cheese*, *Studia Math.* 83 (1986), 33–36.
- [21] S. Lang, *Complex Analysis*, 4th ed., *Grad. Texts in Math.* 103, Springer, New York, 1999.
- [22] R. McKissick, *A nontrivial normal sup norm algebra*, *Bull. Amer. Math. Soc.* 69 (1963), 391–395.
- [23] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992.
- [24] A. Roth, *Approximationseigenschaften und Strahlengrenzwerte meromorpher und ganzer Funktionen*, *Comment. Math. Helv.* 11 (1938), 77–125.
- [25] E. L. Stout, *The Theory of Uniform Algebras*, Bogden & Quigley, Tarrytown-on-Hudson, NY, 1971.

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