Compact homomorphisms between Dales-Davie algebras

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Abstract. In this note we consider compact homomorphisms and endomorphisms between various Dales-Davie algebras. In particular, we obtain fairly complete results when the underlying set is the disc or the unit circle. Comparable results when the underlying set is the unit interval have been proven in previous papers [**5**], [**7**]. (For some related results on endomorphisms of these algebras for general perfect, compact plane sets see [**4**].)

This note is a follow up to previous results on endomorphisms of Dales-Davie algebras which are defined as follows [**2**]. Start with a perfect, compact plane set X. We say that a complex-valued function f defined on X is *complex-differentiable* at a point $a \in X$ if the limit

$$f'(a) = \lim_{z \to a,\ z \in X} \frac{f(z) - f(a)}{z - a}$$

exists. We call $f'(a)$ the *complex derivative* of f at a. Using this concept of derivative, we define the terms *complex–differentiable on* X, *continuously complex–differentiable on* X, and *infinitely complex–differentiable on* X in the obvious way. We denote the n-th complex derivative of f at a by $f^{(n)}(a)$, and we denote the set of infinitely differentiable functions on X by $D^\infty(X)$. Let X be a perfect compact subset of the complex plane, and let (M_n) be a sequence of positive numbers satisfying $M_0 = 1$ and $\dfrac{M_{n+m}}{M_n M_m} \geq \begin{pmatrix} n+m \\ n \end{pmatrix}$, m, n, non-negative integers.

A Dales-Davie algebra is an algebra of the form

$$D(X, M) = \{f \in D^\infty(X) : \|f\|_D = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_\infty}{M_n} < \infty\}.$$

In this note we will have some standing assumptions regarding the algebras $D(X, M)$ that we consider. First we only consider sets X for which $D(X, M)$ is complete for all weights (M_n). We say that such a set X is *good.* A sufficient

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condition for this is that X be uniformly regular.[1] Secondly, our weights (M_n) are *non-analytic*, i.e. $\lim_{n\to\infty}\left(\frac{n!}{M_n}\right)^{1/n}=0$.

As examples of weights (M_n), consider $M_n = n!(\log(n+2))^n$, and $M_n = n!^\alpha$, $\alpha \geq 1$. Of these, the non-analytic weights are $M_n = n!^\alpha$ for $\alpha > 1$ and $M_n = n!(\log(n+2))^n$. If X is uniformly regular, an algebra $D(X, M)$ is said to be *quasi-analytic* if for each $x_0 \in X$, if $f \in D(X, M)$ and $f^{(n)}(x_0) = 0$ for all non-negative n, then $f \equiv 0$. When X is a compact interval, the algebras $D(X, n!)$ and $D(X, n!(\log(n+2))^n)$ are quasi-analytic, while $D(X, n!^\alpha)$ is a regular Banach algebra for each $\alpha > 1$ (see, for example, [**1**, Theorem 4.4.23]).

Finally, we assume that the algebra $D(X, M)$ is *natural* meaning that the maximal ideal space of $D(X, M)$ is precisely X. In the non-analytic case a sufficient condition for a Dales-Davie algebra to be natural is that the set of rational functions with poles off X is dense in the algebra [**2**]. We will say that an algebra $D(X, M)$ is *good* if X is good, (M_n) is non-analytic and $D(X, M)$ is natural. We remark that examples of uniformly regular sets are the unit interval, the closure, $\overline{\Delta}$ of the unit disc Δ and Γ, the unit circle. It is sets X such as these and algebras $D(X, M)$ that we are most interested in, although we will be proving some results in considerably more generality. In the case of the interval the polynomials are dense in these algebras ([**8**]; see also [**1**, Theorem 4.4.15]). The polynomials are also dense in the case of the closed unit disc, as was proved in [**3**]. Exactly the same proof (using convolution with the Fejer kernel) proves that, for non-analytic (M_n), the rational functions are dense in $D(\Gamma, M)$. Thus for these three sets and non-analytic (M_n), the Dales-Davie algebras are good.

Suppose (M_n) is a weight sequence, and X and Y are perfect compact subsets of the plane such that $D(X, M)$ and $D(Y, M)$ are good. If $T : D(Y, M) \to D(X, M)$ is a unital homomorphism, then there exists a continuous function $\phi : X \to Y$ such that $Tf(x) = f(\phi(x))$ for all $x \in X$. The homomorphism T determines the continuous map ϕ and is determined by it. In particular, if T is a unital endomorphism of a Banach algebra $D(X, M)$, then T is determined by a continuous self-map ϕ of X.

For a perfect compact plane set X we sometimes require an additional condition on a map $\phi \in D^\infty(X)$: we say that ϕ is *analytic* if

$$\sup_k \left(\frac{\|\phi^{(k)}\|_\infty}{k!}\right)^{1/k} < \infty.$$

For non-analytic weights (M_n), such ϕ are always in $D(X, M)$.

Our aim is to prove some results about homomorphisms of $D(Y, M)$ into $D(X, M)$, and then to use them to obtain fairly complete results on compact endomorphisms of $D(X, M)$ when X is a geometrically nice set such as a disc or a circle.

In addition to the notions of good sets and algebras as defined above, we say that a compact plane set X is *locally good* if every point of X has a base of good neighbourhoods and we say that $D(X, M)$ is *locally good* if $D(X, M)$ is good and for each $x \in X$ there exists a good neighbourhood N of x in X such that the algebra

[1] A compact plane set X is *uniformly regular* if, for all $z, w \in X$, there is a rectifiable arc in X joining z to w, and the metric given by the geodesic distance between the points of X is uniformly equivalent to the Euclidean metric [**2**].

$D(N, M)$ is good. Note that the interval, the disc and the circle are certainly both good and locally good. Let $D(X, M)$ and $D(Y, M)$ be good and let ϕ be a map from X to Y. For f in $D(Y, M)$ we can look at $f \circ \phi$ and investigate whether it is in $D(X, M)$. In other words we can ask whether ϕ induces a homomorphism from $D(Y, M)$ to $D(X, M)$. Since these algebras are commutative semi-simple Banach algebras the homomorphism must be continuous. In addition we may ask whether the induced homomorphism is compact.

We start with the following lemma.

Lemma 1. *Let $D(X, M)$ and $D(Y, M)$ be good. Then ϕ induces a homomorphism $D(Y, M) \to D(X, M)$ if and only if every point z of X has a good neighbourhood N in X such that $D(N, M)$ is good and $\phi_{|N}$ induces a homomorphism from $D(Y, M)$ to $D(N, M)$. Further, under the same conditions ϕ induces a compact homomorphism if and only if every point z of X has a good neighbourhood N in X such that $D(N, M)$ is good and $\phi_{|N}$ induces a compact homomorphism from $D(Y, M)$ to $D(N, M)$.*

Proof. The first part follows easily from the fact that a complex valued function g on X is in $D(X, M)$ if and only if every point x of X has a good neighbourhood N in X such that $g_{|N}$ is in $D(N, M)$. The compactness follows from a similar result for convergent sequences: a sequence g_n in $D(X, M)$ converges if and only if every point x of X has a good neighbourhood N in X such that $g_n|_N$ converges in $D(N, M)$. □

The next lemma is a standard application of equicontinuity and dominated convergence for series.

Lemma 2. *Let $D(X, M)$ and $D(Y, M)$ be good. Suppose that $|\alpha| < 1$, $\alpha X \subseteq Y$ and $\phi(z) = \alpha z$ for all $z \in X$. Then ϕ induces a compact homomorphism from $D(Y, M)$ to $D(X, M)$.*

Proof. The result follows from the inequality of sup norms

$$||(f \circ \phi)^{(n)}||_X \leq |\alpha|^n ||f^{(n)}||_Y$$

and the usual equicontinuity argument. □

Theorem 3. *Suppose $D(X, M)$ and $D(Y, M)$ are good. Let $\phi \in D(X, M)$. If $\phi(X) \subseteq int(Y)$ then ϕ induces a compact homomorphism $D(Y, M) \to D(X, M)$.*

Proof. Since we are dealing with Banach algebras, the fact that ϕ induces a homomorphism follows from the holomorphic functional calculus. To show compactness, choose any α in $(0, 1)$. Set $Z = (1/\alpha)Y$. It is clear that $D(Z, M)$ is still good. Define $\phi_1 : X \to Z$ by $\phi_1(z) = (1/\alpha)\phi(z)$ and $\phi_2 : Z \to Y$ by $\phi_2(w) = \alpha w$. By the first part, ϕ_1 induces a (bounded) homomorphism from $D(Z, M)$ to $D(X, M)$. Then by Lemma 2, ϕ_2 induces a compact homomorphism from $D(Y, M)$ to $D(Z, M)$. The composite map is the homomorphism induced by ϕ, which is therefore compact. □

In [**5**] and [**7**] we proved that for $X = [0, 1]$ an analytic self-map ϕ induces an endomorphism of $D(X, M)$ if $\|\phi'\|_\infty < 1$, and if, further, $\dfrac{n^2 M_{n+1}}{M_n}$ is bounded, then $\|\phi'\|_\infty \leq 1$ is sufficient for ϕ to induce an endomorphism. Indeed, the computations using Faà di Bruno's formula for derivatives of composite functions can be seen to go through for arbitrary X. Thus we can state the following theorem.

THEOREM 4. *Suppose that $D(X,M)$ and $D(Y,M)$ are good and that $\phi : X \to Y$ is analytic on X (note this latter already implies $\phi \in D(X,M)$). If $||\phi'||_\infty < 1$ then ϕ induces a homomorphism. If, moreover, the sequence n^2M_n/M_{n+1} is bounded then $||\phi'||_\infty \leq 1$ is sufficient for ϕ to induce a homomorphism $D(Y,M) \to D(X,M)$.*

In contrast to this, for the weight $M_n = n!^{3/2}$ for which $\lim \dfrac{n^2M_n}{M_{n+1}} = \infty$, the map $\phi(x) = \dfrac{1+x^2}{2}$ does *not* induce an endomorphism of $D([0,1],M)$. ([**5**] Theorem 3.2)

The true story probably involves the order and type of the entire function $g(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{M_n}$.

THEOREM 5. *Suppose that $D(X,M)$ and $D(Y,M)$ are good and that ϕ is analytic $X \to Y$. If $||\phi'||_X < 1$, then ϕ induces a compact homomorphism from $D(Y,M)$ into $D(X,M)$.*

PROOF. The fact that ϕ induces a homomorphism is part of Theorem 4. For the compactness, we use a similar idea to that in Theorem 3. Choose $\beta > 1$ such that $\beta||\phi'||_\infty < 1$ and set $\alpha = 1/\beta$. Now define Z, ϕ_1, ϕ_2 as in the proof of Theorem 3 and note that $||\phi_1'||_\infty < 1$ so that ϕ_1 induces a homomorphism by Theorem 4. The rest is as before. □

THEOREM 6. ***(The mixed case)*** *Suppose now that $D(X,M)$ is locally good, $D(Y,M)$ is good and ϕ is analytic $X \to Y$. Further suppose that*

$$X = \{z \in X : |\phi'(z)| < 1\} \cup \phi^{-1}(int(Y)).$$

Then ϕ induces a compact homomorphism.

PROOF. Every point of X has a good neighbourhood N in X such that at least one of Theorem 3 and Theorem 5 apply to $\phi|_N$. Therefore $\phi|_N$ induces a compact homomorphism from $D(Y,M)$ to $D(N,M)$. The result then follows from Lemma 1. □

We remark that Theorems 4 through 6 require the inducing map ϕ to be analytic. We do not know whether or not this condition is redundant.

We now turn to cases where the sets X and Y are equal and thus the map ϕ induces an endomorphism of $D(X,M)$. We have just given sufficient conditions for ϕ to induce a compact endomorphism in the case that ϕ is analytic and $D(X,M)$ is locally good.

On the other hand the map ϕ is not required to be analytic for the next theorems regarding necessary conditions that ϕ induce a compact endomorphism when the set X is geometrically nice. The two sets we consider are $\overline{\Delta}$, the closure of the open unit disc Δ and Γ, the unit circle.

A key fact is the following. Let X be a compact, connected perfect subset of the complex numbers, (M_n) a weight sequence and $D(X,M)$ complete and natural. If the self-map ϕ of X induces a compact endomorphism of $D(X,M)$, then there exists a unique fixed point x_0 of ϕ and further at x_0, $|\phi'(x_0)| < 1$. (See Theorem 1.7 of [**6**] and Theorem 1.1 of [**5**].)

Lemma 7. *Suppose that (M_n) is a non-analytic weight sequence. If X is one of the sets $\overline{\Delta}$ or Γ and ϕ induces a compact endomorphism of $D(X, M)$, then $|\phi'(z)| < 1$ for all z such that $\phi(z)$ is on the boundary of X.*

Proof. Suppose $|\phi'(a)| \geq 1$ for some a on the unit circle with $\phi(a) = b$ which is also on the unit circle. Let $\psi(z) = \phi(az/b)$. Then clearly ψ induces a compact endomorphism, $\psi(b) = \phi(a) = b$ and $|\psi'(b)| = \left|\frac{a}{b}\phi'(a)\right| \geq 1$, a contradiction to the remarks just preceding the statement of the lemma. Hence if $\phi(z)$ is on the boundary of X, then $|\phi'(z)| < 1$. □

Theorem 8. *Suppose that $X = Y = \overline{\Delta}$ (where Δ is the open unit disk) and ϕ is an analytic self-map of X. Then ϕ induces a compact endomorphism of $D(\overline{\Delta}, M)$ if and only if*

$$\overline{\Delta} = \{z \in \overline{\Delta} : |\phi'(z)| < 1\} \cup \phi^{-1}(\Delta).$$

Proof. The if part follows from Theorem 6 and the necessity follows from the preceding lemma. □

Regarding $D(\Gamma, M)$, clearly $int(\Gamma) = \emptyset$. Hence the following holds.

Theorem 9. *An analytic self-map ϕ of Γ induces a compact endomorphism of $D(\Gamma, M)$ if $\|\phi'\|_\infty < 1$, and further, if any ϕ induces a compact endomorphism of $D(\Gamma, M)$, then necessarily, $\|\phi'\|_\infty < 1$.*

Some examples of analytic self-maps of the circle are $\phi(z) = \exp\big(c(z^2 - 1)/z\big)$ where c is real. Note that ϕ has the property that $\phi(1) = 1$ and $\|\phi'\|_\infty = |\phi'(1)| = 2|c|$. For fast growing (M_n) these ϕ induce endomorphisms of $D(\Gamma, M)$ for $|c| \leq 1/2$. For $|c| < 1/2$ these ϕ induce compact endomorphisms of $D(\Gamma, M)$ for all non-analytic (M_n). We wish to thank John Wermer for leading us to these examples.

Theorem 4 of [**4**] applied to $D(X, M)$ where $X = \overline{\Delta}$ or Γ gives that if an *analytic* self-map ϕ of X induces an endomorphism of $D(X, M)$ and if $\phi(z)$ is on the boundary of X for some z, then $|\phi'(z)| \leq 1$. In particular, if analytic ϕ induces an endomorphism of $D(\Gamma, M)$, then $\|\phi'\|_\infty \leq 1$. Using Lemma 7, or even the remark preceding it, we can show very quickly that the condition of analyticity on ϕ may be removed.

Theorem 10. *Let X be either $\overline{\Delta}$ or Γ and suppose that a self-map ϕ of X induces an endomorphism of $D(X, M)$. Then $|\phi'(z)| \leq 1$ for all z such that $\phi(z)$ is on the boundary of X.*

Proof. The proofs for both $X = \overline{\Delta}$ and $X = \Gamma$ start out the same. Suppose ϕ induces an endomorphism of $D(X, M)$ and for some $a, b \in \Gamma$, $\phi(a) = b$ and $|\phi'(a)| = A > 1$. Let $\phi_1(z) = \overline{b}\phi(az)$. Then ϕ_1 induces an endomorphism of $D(X, M)$, $\phi_1(1) = 1$ and $|\phi_1'(1)| = A > 1$.

(i) $X = \overline{\Delta}$. Choose a positive number C satisfying $A > 1 + C$ and let $p(z) = \frac{z+C}{1+C}$. Then p is an analytic self-map of $\overline{\Delta}$ with $\|p'\|_\infty < 1$. Using Theorem 5 we have that p induces a compact endomorphism of $D(\overline{\Delta}, M)$. Now let $\psi(z) = p(\phi_1(z))$. Clearly ψ induces a compact endomorphism of $D(\overline{\Delta}, M)$. However, $\psi(1) = 1$ and

$|\psi'(1)| = |p'(\phi_1(1))\phi_1'(1)| = \frac{1}{1+C}A > 1$, a contradiction to Lemma 7. Therefore if ϕ induces an endomorphism of $D(\overline{\Delta}, M)$ and $\phi(z)$ is on the boundary of $\overline{\Delta}$, then $|\phi'(z)| \leq 1$.

(ii) $X = \Gamma$. Let ϕ_1 and A be as before. Let $p(z) = \exp(\frac{1}{2A}\frac{z^2-1}{z})$ and set $\psi(z) = p(\phi_1(z))$. Clearly $\psi(1) = 1$. Since $\|p'\|_\infty < 1$, p induces a compact endomorphism of $D(\Gamma, M)$ and this, in turn, implies that ψ induces a compact endomorphism of $D(\Gamma, M)$. But $|\psi'(1)| = |p'(\phi_1(1))\phi_1'(1)| = \frac{1}{A}A = 1$, again a contradiction to Lemma 7. $\square$

We conclude with some remarks about the spectra of compact endomorphisms. The results are not unexpected in view of previous results along these lines and corresponding results for composition operators on spaces of analytic functions.

In [**5**] we proved that if a self-map ϕ of $[0,1]$ induced a compact endomorphism T of $D([0,1], M)$, then the spectrum of T, $\sigma(T)$, is given by

$$\sigma(T) = \{(\phi'(x_0))^n : n \in \mathbb{N}\} \cup \{0, 1\}$$

where x_0 is the fixed point of ϕ. In fact, the corresponding statement holds for compact endomorphisms T on any $D(X, M)$ when X is uniformly regular.

Indeed, the following facts which were used in proving the preceding can be shown to be valid for any $D(X, M)$ when X is connected.

(i) For each positive integer n, $(\phi'(x_0))^n$ is in $\sigma(T)$.
(ii) If $\lambda \neq 0, 1$, $\lambda \neq (\phi'(x_0))^n$ for all positive integers n, and $Tf = \lambda f$, then $f^{(\nu)}(x_0) = 0$ for all non-negative integers ν.

Thus if the fixed point x_0 of ϕ lies in the interior of X or if the algebra $D(X, M)$ is quasi-analytic, then (i) and (ii) are sufficient to show that

$$\sigma(T) = \{(\phi'(x_0))^n : n \in \mathbb{N}\} \cup \{0, 1\}.$$

In general, however, we need more to obtain the result and this is supplied by the next fact.

(iii) If $f^{(\nu)}(x_0) = 0$ for all ν and $Tf = \lambda f$ for some non-zero λ, then $f \equiv 0$.

The proof of (iii) in [**5**] for the interval used Taylor polynomial approximations.

However, for uniformly regular X, if $f^{(\nu)}(x_0) = 0$ for all ν then for some $C > 0$, $|f(z)| \leq C^{m+1}|z - x_0|^{m+1}\|f^{(m+1)}\|_\infty/m!$ for all m and z ([**2**, Lemma 1.5(iii)]). This is precisely what is needed to have the proof of (iii) carry over from $[0,1]$ to arbitrary uniformly regular X.

Consequently we have the following.

Theorem 11. *If (M_n) is non-analytic, X is uniformly regular and ϕ induces a compact endomorphism T of $D(X, M)$, then*

$$\sigma(T) = \{(\phi'(x_0))^n : n \in \mathbb{N}\} \cup \{0, 1\}$$

where x_0 is the fixed point of ϕ.

Certainly Theorem 11 holds when $X = \overline{\Delta}$ or Γ.

References

[1] H. G. Dales, *Banach Algebras and Automatic Continuity,* LMS Monographs 24, Clarenden Press, Oxford, 2000.

[2] H. G. Dales and A. M. Davie, *Quasianalytic Banach function algebras,* J. Funct. Anal. **13** (1973), 28–50.

[3] H.G. Dales and J.P. McClure, *Completion of normed algebras of polynomials,* J. Austral. Math. Soc. (A) **20** (1975), 504–510.

[4] J. F. Feinstein and H. Kamowitz, *Endomorphisms of algebras of infinitely differentiable functions on compact plane sets,* J. Funct. Anal. **173** (2000), 61–73.

[5] J. F. Feinstein and H. Kamowitz, *Compact endomorphisms of Banach algebras of infinitely differentiable functions,* J. Lond. Math. Soc. (to appear).

[6] H. Kamowitz, *Compact endomorphisms of Banach algebras,* Pac. J. Math **89** (1980), 313–325.

[7] H. Kamowitz, *Endomorphisms of Banach algebras of infinitely differentiable functions,* Proceedings of the 13th International Conference on Banach Algebras in Blaubeuren, Germany, Walter de Gruyter, Berlin, New York 1998.

[8] A. G. O'Farrell, *Polynomial approximation of smooth functions,*, J. London Math. Soc. (2) **28** (1983), 496–506.

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