# Quasicompact and Riesz endomorphisms of Banach algebras

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**Abstract**

Let $B$ be a unital commutative semi-simple Banach algebra. We study endomorphisms of $B$ which are also quasicompact operators or Riesz operators. Clearly compact and power compact endomorphisms are Riesz and hence quasicompact. Several general theorems about quasicompact endomorphisms are proved, and these results are then applied to the question of when quasicompact or Riesz endomorphisms of certain algebras are necessarily power compact.

## Introduction

Let $B$ be a unital commutative semi-simple Banach algebra. An endomorphism $T : B \to B$ is a linear operator which also preserves multiplication. An endomorphism $T$ is called unital if $T1 = 1$. Since the Banach algebra is assumed to be semi-simple, it follows from very early theory that $T$ is necessarily bounded. It is interesting to see what can be deduced if various operator theoretic properties are imposed on the endomorphism. In this note we consider endomorphisms which are also quasicompact or Riesz operators (as defined below).

We assume some familiarity with Fredholm operators and the Calkin algebra, but include specific definitions below for convenience. A useful general reference for this theory is [13]. However the reader should be warned that the definition of *essential spectrum* given there ([13, p. 222]) is not the usual one, which we give below. (This does not affect the value of the *essential spectral radius*, which is equal to the *Fredholm radius* $r_\Phi$ defined in [13].)

**Notation** Let $E$ be a Banach space. We denote the sets of bounded operators and compact operators on $E$ by, respectively, $\mathcal{B}(E)$ and $\mathcal{K}(E)$.

**Definition** For a bounded operator $T$ on an infinite-dimensional Banach space $E$, the *essential spectrum* of $T$ is the set of complex numbers $\lambda$ such that $\lambda I - T$ is not a Fredholm operator. This is also equal to the spectrum of $T + \mathcal{K}(E)$ in the Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$. Accordingly, the *essential spectral radius* of $T$, denoted by $r_e(T)$, is given by the formula

$$r_e(T) = \lim_{n\to\infty} \mathrm{dist}(T^n, \mathcal{K}(E))^{1/n} = \inf\{\mathrm{dist}(T^n, \mathcal{K}(E))^{1/n} : n \in \mathbf{N}\}.$$

We say that $T$ is a *Riesz operator* if $\lambda I - T$ is Fredholm for all non-zero complex numbers $\lambda$. Thus $T$ is Riesz if and only if $r_e(T) = 0$. We say that $T$ is *quasicompact* if $r_e(T) < 1$. Thus $T$ is quasicompact if and only if there exists $n \in \mathbf{N}$ with $\mathrm{dist}(T^n, \mathcal{K}(E)) < 1$.

A useful property of a bounded linear operator $T$ on a Banach space is that each spectral element of $T$ which lies in the unbounded component of the complement of the essential spectrum of $T$ is an eigenvalue of finite multiplicity. Further, if there are infinitely many of them, then they cluster only on the essential spectrum. (See, for example, [13, Theorem 51.1].)

An operator $T$ on a Banach space is said to be power compact if there exists a positive integer $N$ such that $T^N$ is compact. Certainly we have the implications: compact $\Longrightarrow$ power compact $\Longrightarrow$ Riesz $\Longrightarrow$ quasicompact. One of the questions considered is for which algebras is every quasicompact endomorphism or every Riesz endomorphism necessarily power compact.

## Part 1

If $B$ is a unital commutative semi-simple Banach algebra with maximal ideal space $X$, and $T$ is a unital endomorphism of $B$, then $\phi := T^*_{|X}$ is a w*-continuous selfmap of $X$ such that for all $f \in B$ and $x \in X$, $\widehat{Tf}(x) = \hat{f}(\phi(x))$. In this case we say that $T$ is induced by $\phi$ or that $\phi$ induces $T$.

It was shown in [14] that if $\phi$ induces a compact endomorphism of the unital commutative semi-simple Banach algebra $B$ and the maximal ideal space $X$ of $B$ is connected, then $\bigcap_{n=0}^{\infty} \phi_n(X) = \{x_0\}$ for some $x_0 \in X$. Here $\phi_n$ denotes the nth iterate of $\phi$.

We will give a short proof showing that a stronger result holds even for quasicompact endomorphisms. First we need a lemma concerning the spectra of quasicompact endomorphisms. (The special case of power compact endomorphisms is part of [16, Korollar 2.21].)

Recall that every commutative semi-simple Banach algebra may be regarded (via the Gelfand transform) as an algebra of continuous functions on its maximal ideal space.

**Lemma 1.1:** Let $B$ be a unital commutative semi-simple Banach algebra with connected maximal ideal space $X$, and let $T$ be a unital quasicompact endomorphism of $B$. Then 1 is an eigenvalue of $T$ with multiplicity 1 and eigenspace $\mathbf{C} \cdot 1$, and $\sigma(T)$ (the spectrum of $T$) is contained in $\{\lambda : |\lambda < 1\} \cup \{1\}$.

**Proof:** Let $B, X, T$ be as described. Clearly 1 is an eigenvalue of $T$, and the corresponding eigenspace contains the constant functions. Since the maximal ideal space of $B$ is connected, the eigenspace $N(I-T) = \{f \in B : Tf = f\}$ is a finite-dimensional unital semi-simple complex Banach algebra with no idempotents other than 0 and 1. Thus $N(I-T) = \mathbf{C} \cdot 1$, as claimed.

Since $r_e(T) < 1$ and all of the spectral elements of magnitude greater than $r_e(T)$ are eigenvalues, the fact that the set of eigenvalues is closed under powers implies that $\sigma(T) \subset \{\lambda : |\lambda| \leq 1\}$.

Now suppose that there exists $\lambda$ such that $|\lambda| = 1$, $\lambda \neq 1$ and $\lambda$ is an eigenvalue of $T$. Let $g$ be a non-zero function in $B$ with $Tg = \lambda g$. Since $r_e(T) < 1$, there are only finitely many eigenvalues of $T$ on the unit circle; otherwise there would be a cluster point on the unit circle. Again, the eigenvalues are closed under powers, whence we have that for some positive integer $N$, $\lambda^N = 1$, and hence $T^N g = g$. However, $T^N$ is also a quasicompact endomorphism and so (by the first part) $g$ is a constant function. This contradicts the choice of $g$ and $\lambda$, and the result follows. $\square$

If $B$ is a unital commutative semi-simple Banach algebra with maximal ideal space $X$, for $x, y \in X$ we let $\|x-y\| = \sup\{|\hat{f}(x) - \hat{f}(y)| : f \in B, \|f\| \leq 1\}$. That is, $\|x-y\|$ is the norm of $x-y$ regarded as an element of the dual space $B^*$ of $B$. Further, for $\varepsilon > 0$ and $a \in X$, we let $B(a, \varepsilon) = \{x \in X : \|x-a\| < \varepsilon\}$.

**Theorem 1.2:** Suppose that $B$ is a unital commutative semi-simple Banach algebra with connected maximal ideal space $X$, and $T$ is a quasicompact endomorphism of $B$ induced by the selfmap $\phi$ of $X$. Then the following hold.

(i) The operators $T^n$ converge in operator norm to a rank-one unital endomorphism $S$ of $B$, and there exists $x_0 \in X$ such that, for all $f \in B$, $Sf = \hat{f}(x_0)1$. This point $x_0$ is the unique fixed point of $\phi$.

(ii) For each $\varepsilon > 0$, there exists a positive integer $N$ such that $\phi_N(X) \subset B(x_0, \varepsilon)$.

(iii) $\bigcap \phi_n(X) = \{x_0\}$.

**Proof:** Let $B, X, T, \phi$ be as described in the statement. Since the maximal ideal space $X$ is connected, the conclusions of Lemma 1.1 apply. We claim that the result will follow once we have shown that $T^n$ converges in norm. To see this, suppose that $T^n$ converges in norm to an operator $S$. Then $S$ is an endomorphism of $B$ and $TS = S$. Hence, $SB$ is contained in the eigenspace of $T$ corresponding to the eigenvalue 1, and so $S$ is a rank-one endomorphism of the form $S : f \to \hat{f}(x_0)1$ for some $x_0 \in X$. From this it follows that $(T^*)^n(X)$ converges normwise down to $\{x_0\}$, and the remaining parts of (i), (ii) and (iii) are immediate.

We now show that $T^n$ converges. From the Fredholm theory, $(T-I)B$ is closed and further it is easy to see that $1 \notin (T-I)B$ and $B = \mathbf{C}\cdot 1 \oplus (T-I)B$. Since $\sigma(T) \subset \{\lambda : |\lambda| < 1\} \cup \{1\}$ it follows that the spectral radius of $T_{|(T-I)B}$ is less than 1; hence $\sum_{n=0}^{\infty} \|T^n_{|(T-I)B}\|$ converges. Therefore $\sum_{n=0}^{\infty} \|T^{n+1} - T^n\| = \sum_{n=0}^{\infty} \|T^n(I-T)\|$ also converges and, consequently, $T^n$ converges in norm, as required. $\square$

**Remark** Clearly, if $T^n$ converges in norm to a rank-one operator then $T$ is quasicompact. In particular, (i) above is also sufficient for the endomorphism $T$ to be quasicompact. However neither (ii) nor (iii) alone is sufficient, as we shall see in Part 3.

**Corollary 1.3:** Suppose that $B$ is a unital commutative semi-simple Banach algebra with connected maximal ideal space $X$. Let $T$ be a quasicompact endomorphism induced by a selfmap $\phi$ of $X$. Suppose that $\{x_0\} = \bigcap_{n=0}^{\infty} \phi_n(X)$.

(a) If $x_0$ is an isolated point of $X$ in the norm topology, then there exists a positive integer $N$ such that $T^N$ is the rank-one endomorphism $f \mapsto \hat{f}(x_0)1$.

(b) In particular, if $B$ has no non-zero point derivations at $x_0$, then the conclusion of (a) applies.

**Proof:** (a) Since $x_0$ is an isolated point in the norm topology, there exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) = \{x_0\}$ and the result follows from Theorem 1.2.

(b) Theorem 1.6.2 of [1] asserts that if there are no non-zero point derivations at $x_0$, then $x_0$ is an isolated point of $X$ in the norm topology. The result follows. $\square$

Note that if $B$ is a unital commutative semi-simple Banach algebra with connected maximal ideal space $X$, and every point of $X$ is isolated in the norm topology, then Corollary 1.3 shows that every quasicompact endomorphism of $B$ is power compact. In particular this holds if $B$ is the uniform algebra $C(X)$.

Part 2: Dales-Davie algebras

Let $X$ be a perfect compact subset of the complex plane and let $D^\infty(X)$ denote the set of infinitely differentiable functions on $X$. Suppose, too, that $(M_n)$ is a sequence of positive numbers satisfying $M_0 = 1$ and $\dfrac{M_{n+m}}{M_n M_m} \geq \dbinom{n+m}{m}$. Finally, let

$$D(X, M) = \{f \in D^\infty(X) : \|f\|_D = \sum_{n=0}^{\infty} \frac{\|f^{(n)}\|_\infty}{M_n} < \infty\}.$$

With pointwise addition and multiplication, $D(X, M)$ is a normed algebra. We call such algebras Dales-Davie algebras. See [3] and [4] for examples and basic facts about Dales-Davie algebras, and [5], [6] and [15] for some results about endomorphisms of these algebras. We will also assume that a weight sequence $(M_n)$ is nonanalytic meaning that $\lim_{n\to\infty}(n!/M_n)^{1/n} = 0$. Suppose that $(M_n)$ is nonanalytic, $D(X, M)$ is a Banach algebra, and that the maximal ideal space of $D(X, M)$ is precisely $X$. In such cases, every unital endomorphism $T$ of $D(X, M)$ has the form $Tf(x) = f(\phi(x))$ for some continuous selfmap $\phi$ of $X$. As a final definition, an infinitely-differentiable selfmap $\phi$ of a compact subset of the plane is called analytic if

$$\sup_k \left( \frac{\|\phi^{(k)}\|_\infty}{k!} \right)^{1/k} < \infty.$$

**Lemma 2.1:** Let $X$ be a connected perfect compact subset of the complex plane, $(M_n)$ a non-analytic weight sequence and $D(X, M)$ a Banach algebra with maximal ideal space $X$. Suppose that $T$ is a quasicompact endomorphism of $D(X, M)$ induced by the selfmap $\phi$ of $X$. If $x_0$ is the fixed point of $\phi$, then $|\phi'(x_0)| < 1$.

**Proof:** Assume that $D(X, M)$, $T$, $\phi$ and $x_0$ are as described. Let $f(x) = x$. From Theorem 1.2, $\|T^n f - f(x_0)1\| \to 0$, or equivalently, $\|\phi_n - x_0 1\|_{D(X,M)} \to 0$. However, $\|\phi_n - x_0 1\|_{D(X,M)} \geq \dfrac{\|\phi_n'\|_\infty}{M_1} \geq \dfrac{|\phi_n'(x_0)|}{M_1} = \dfrac{|\phi'(x_0)|^n}{M_1}$. Therefore, $|\phi'(x_0)| < 1$. □

Suppose that $X$ is a connected perfect compact subset of the complex plane and $(M_n)$ is a nonanalytic weight sequence. Suppose further that $D(X, M)$ is a Banach algebra with maximal ideal space $X$. It was shown in [6] that if $T$ is an endomorphism induced by an analytic selfmap $\phi$ of $X$ and if $\|\phi'\|_\infty < 1$, then $T$ is a compact endomorphism. The next theorem follows easily from this and Lemma 2.1.

**Theorem 2.2:** Let $X$ be a connected perfect compact subset of the complex plane and $(M_n)$ a nonanalytic weight sequence. If $D(X, M)$ is a Banach algebra with maximal ideal space $X$, then every quasicompact endomorphism induced by an analytic selfmap $\phi$ of $X$ is power compact.

**Proof:** Suppose that $T$ is a quasicompact endomorphism of $D(X, M)$ induced by an analytic selfmap $\phi$ of $X$ and let $x_0$ be the fixed point of $\phi$. From Lemma 2.1, $|\phi'(x_0)| = c_1 < 1$. Let $0 \leq c_1 < c < 1$ and let $\mathcal{U}$ be a neighborhood of $x_0$ such that $|\phi'(t)| \leq c$ for all $t \in \mathcal{U}$. Then there exists a positive integer $N$ for which $\phi_N(x) \in \mathcal{U}$ for all $x \in X$. Note that, for $n > N$,

$$|\phi_n'(x)| = |\phi'(\phi_{n-1}(x)) \cdots \phi'(\phi_{N+1}(x))\phi'(\phi_N(x)) \cdots \phi'(\phi(x))\phi'(x)|.$$

Hence if $n > N$, then, for all $x \in X$, $|\phi_n'(x)| \leq c^{n-N}\|\phi'\|_\infty^N$. Thus $\|\phi_n'\|_\infty < 1$ for large $n$, whence $\phi_n$ induces a compact endomorphism of $D(X, M)$ for large $n$. Therefore $T$ is power compact. $\square$

Further, the spectrum $\sigma(T)$ of a quasicompact endomorphism $T$ of $D(X, M)$ is easy to determine in many cases. Suppose that $X$ is uniformly regular, meaning that for all $z, w \in X$, there is a rectifiable arc in $X$ joining $z$ to $w$, and the metric given by the geodesic distance between the points of $X$ is uniformly equivalent to the Euclidean metric. (It was shown in [4] that uniform regularity is sufficient for $D(X, M)$ to be complete, although this condition is not necessary.) Suppose further that the maximal ideal space of $D(X, M)$ is precisely $X$. In this case, the proofs of Theorem 2.4 of [6] when $X = [0, 1]$ and Theorem 11 of [7] when $X$ is uniformly regular show that

$$\sigma(T) = \{\phi'(x_0)^n : \text{n is a positive integer}\} \cup \{0, 1\}$$

holds whenever the non-zero spectrum contains only eigenvalues. Thus we have the following theorem.

**Theorem 2.3:** Let $X$ be a uniformly regular compact subset of the complex plane and $(M_n)$ a nonanalytic weight sequence. Suppose that $D(X, M)$ is a Banach algebra with maximal ideal space $X$. If $T$ is a Riesz endomorphism of $D(X, M)$ induced by $\phi$ and if $x_0$ is the fixed point of $\phi$, then

$$\sigma(T) = \{\phi'(x_0)^n : \text{n is a positive integer}\} \cup \{0, 1\}.$$

If the inducing map $\phi$ is analytic, the result holds for quasicompact endomorphisms. Also, each nonzero element in $\sigma(T)$ has multiplicity 1.

## Part 3: $C^1[0,1]$

We next look at the Banach algebra $C^1[0,1]$ for examples of quasicompact endomorphisms which are not Riesz and Riesz endomorphisms which are not power compact. Throughout this section we abbreviate $\mathcal{K}(C^1[0,1])$ by $\mathcal{K}$.

**Lemma 3.1:** Let $T$ be a unital endomorphism of $C^1[0,1]$ induced by a selfmap $\phi$ of $[0,1]$. Then for each fixed point $\bar{x}$ of $\phi$ and each $K \in \mathcal{K}$, we have $\|T - K\| \geq |\phi'(\bar{x})|$.

**Proof:** Suppose that $\bar{x}$ is a fixed point of $\phi$. The lemma is certainly true if $\phi'(\bar{x}) = 0$. Therefore consider the case where $\phi'(\bar{x}) \neq 0$. We will use an argument which was employed by L. Zheng ([17], Lemma 2) in connection with Riesz composition operators of $H^\infty$ of the unit disk. Choose, as we may, $x_n \in [0,1]$ and $f_n \in C^1[0,1]$ with $x_n \to \bar{x}$, $f_n'(\phi(x_n)) = -1$, $f_n'(\bar{x}) = 1$ and $\|f_n\|_{C^1} \leq 1 + \frac{1}{n}$. Suppose that $K \in \mathcal{K}$. Then there exist $g \in C^1[0,1]$ and a subsequence $\{f_{n_j}\}$ such that $Kf_{n_j} \to g$. Therefore,

$$\|f_{n_j}\|\|T - K\| \geq \|Tf_{n_j} - Kf_{n_j}\|_{C^1} \geq \|Tf_{n_j} - g\|_{C^1} - \|Kf_{n_j} - g\|_{C^1},$$

and so

$$\|T - K\| \geq \limsup_j \|Tf_{n_j} - g\|_{C^1}.$$

Now

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} =$$

$$\limsup_j \left( \sup_x |f_{n_j}(\phi(x)) - g(x)| + \sup_x |f_{n_j}'(\phi(x))\phi'(x) - g'(x)| \right).$$

First evaluating at $x_{n_j}$, we get

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} \geq \limsup_j [0 + |-\phi'(x_{n_j}) - g'(x_{n_j})|] =$$

$$\limsup_j [|\phi'(x_{n_j}) + g'(x_{n_j})|] = |\phi'(\bar{x}) + g'(\bar{x})|.$$

Then evaluating at $\bar{x}$, we get

$$\limsup_j \|Tf_{n_j} - g\|_{C^1} \geq \limsup_j |f_{n_j}'(\bar{x})\phi'(\bar{x}) - g'(\bar{x})| = |\phi'(\bar{x}) - g'(\bar{x})|.$$

Adding we have that $\|T - K\| \geq \limsup_j \|Tf_{n_j} - g\|_{C^1} \geq |\phi'(\bar{x})|$. □

**Theorem 3.2:** Suppose that $T$ is a unital endomorphism of $C^1[0,1]$ induced by a selfmap $\phi$ of $[0,1]$ for which $\bigcap_{n=0}^{\infty} \phi_n([0,1]) = \{x_0\}$ for some $x_0 \in [0,1]$. Then the essential spectral radius $r_e(T)$ is equal to $|\phi'(x_0)|$.

**Proof:** Clearly $\phi \in C^1[0,1]$, and $x_0$ is a fixed point of $\phi$.

We first show that $r_e(T) \leq |\phi'(x_0)|$. To this end let $\alpha > |\phi'(x_0)|$ and let $Lf = f(x_0)1$. Clearly $L \in \mathcal{K}$. Since $\bigcap_{n=0}^{\infty} \phi_n([0,1]) = \{x_0\}$, there exists a positive number $M$ such that $|\phi_n'(t)| < M\alpha^n$ for all $n$ and all $t \in [0,1]$. Hence for each $f \in C^1[0,1]$,

$$|f(\phi_n(x)) - f(x_0)| \leq \|f'\|_\infty \|\phi_n'\|_\infty |x - x_0| \leq M\|f'\|_\infty \alpha^n$$

for all $x$, and

$$|f'(\phi_n(x))\phi_n'(x)| \leq \|f'\|_\infty |\phi_n'(x)| \leq M\|f'\|_\infty \alpha^n$$

for all $x$. Thus $\|T^n f - Lf\|_{C^1} \leq 2M\|f\|_{C^1}\,\alpha^n$, for all $f \in C^1[0,1]$ and this, in turn, implies that $\limsup_n \|T^n - L\|^{1/n} \leq \alpha$. Since this holds for all $\alpha > |\phi'(x_0)|$ and $r_e(T) \leq \limsup_n \|T^n - L\|^{1/n}$, we conclude that $r_e(T) \leq |\phi'(x_0)|$.

We now show that $r_e(T) \geq |\phi'(x_0)|$. For each positive integer $m$, $T^m$ is an endomorphism which is induced by $\phi_m$. Clearly $x_0$ is a fixed point of $\phi_m$. Applying Lemma 3.1 to $T^m$, we have $\|T^m - K\| \geq |\phi'(x_0)|^m$ for all positive integers $m$ and all $K \in \mathcal{K}$. Thus $r_e(T) = \lim_{m\to\infty} (\mathrm{dist}(T^m, \mathcal{K})^{1/m} \geq |\phi'(x_0)|$.

The result now follows. $\square$

It was shown in ([14], Theorem 2.3) that every unital compact endomorphism of $C^1[0,1]$ has the form $Lf = f(x_0)1$ for some $x_0 \in [0,1]$. Thus an endomorphism $T$ of $C^1[0,1]$ is power compact if and only if the inducing map $\phi$ has an iterate which is constant.

We now have the following.

**Corollary 3.3:** For $C^1[0,1]$ there exist quasicompact endomorphisms which are not Riesz, Riesz endomorphisms which are not power compact, and power compact endomorphisms which are not compact. Moreover, endomorphisms of $C^1[0,1]$ satisfying (ii) and (iii) of Theorem 1.2 need not be quasicompact.

**Proof:** Respective examples of inducing $\phi$ for the first part are $\phi_1(x) = (x + x^2)/3$, $\phi_2(x) = x^2/2$ and

$$\phi_3(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1/2; \\ (x - 1/2)^2, & \text{otherwise.} \end{cases}$$

For the last part, the endomorphism induced by $\phi_4(x) = (1 + x^2)/2$ is easily seen to satisfy (ii) and (iii) of Theorem 1.2, yet $\phi_4$ does not induce a quasicompact endomorphism of $C^1[0, 1]$ since $\phi_4(1) = \phi_4'(1) = 1$. □

Part 4: Uniform algebras

Thus far we have examples of algebras for which every quasicompact endomorphism is power compact, as well as an example of an algebra with a non-power compact Riesz endomorphism. As we now show, every quasicompact endomorphism of the disk algebra or $H^\infty(\Delta)$, $\Delta$ the unit disk, is power compact. One might think that this was the case for every uniform algebra. However, in this section we also construct an example of a uniform algebra whose maximal ideal space is connected and which has a non-power compact Riesz endomorphism.

We will assume some knowledge of the standard theory of uniform algebras; for more details we refer the reader to [11].

Let $A$ be a uniform algebra with maximal ideal space $X$ and let $T$ be an endomorphism of $A$ induced by a selfmap $\phi$ of $X$. It is standard that $T$ is compact if and only if $\phi(X)$ is a norm (Gleason) compact subset of $X$ ([12, Theorem 1]). Note, however, that the 'if' part fails for more general Banach function algebras.

It follows from our earlier Theorem 1.2 that if $\phi$ induces a quasicompact endomorphism $T$ of a uniform algebra with connected maximal ideal space $X$, then for some positive integer $N$, $\phi_N(X)$ is a Gelfand (and hence also Gleason) closed subset of a closed norm ball of radius less than 2. This ball is, of course, contained in exactly one Gleason part.

Recall that a uniform algebra $A$ is a *unique representing measure algebra* (*URM-algebra*) if every character of $A$ has a unique representing measure on the Shilov boundary of $A$. The disk algebra, $H^\infty(\Delta)$, and the trivial uniform algebras $C(X)$ are all examples of URM-algebras. It is standard that the Gleason parts of URM-algebras are either one point parts or else analytic disks. In particular, for these algebras, every closed Gleason ball of radius less than 2 is Gleason compact.

From the above discussion we now see immediately that if $A$ is a URM-algebra with connected maximal ideal space, then every quasicompact endomorphism of $A$ is power compact.

See [16] for related results and examples concerning compact endomorphisms. Further, similar results, using different techniques, can be found in [10].

In view of the above results, in order to construct non-power compact Riesz endomorphisms of uniform algebras, we should look at examples where the small closed Gleason balls are not Gleason compact. Klein [16] considered such uniform algebras for similar reasons. We begin by proving a lemma which gives a sufficient condition for an endomorphism of a uniform algebra to be Riesz.

**Lemma 4.1:** Let $A$ be a uniform algebra on a compact space $X$ and let $x_0 \in X$. Let $\phi$ be a self-map of $X$ which induces an endomorphism $T$ of $A$, and as usual let $\phi_n$ be the $n$th iterate of $\phi$. Using the norm (Gleason) distance in the maximal ideal space, set $C_n = \sup\{\|\phi_n(x) - x_0\| : x \in X\}$. Suppose that $C_n^{1/n} \to 0$ as $n \to \infty$. Then $T$ is a Riesz endomorphism of $A$.

**Proof:** Consider the distance from $T^n$ to the compact endomorphism $L$ given by $Lf = f(x_0)1$. For $f \in A$ and $x \in X$ we have $(T^n f - Lf)(x) = f(\phi_n(x)) - f(x_0)$ and it then follows from this that $\|T^n - L\| \leq C_n$. Thus if $C_n^{1/n} \to 0$ as $n \to \infty$, then $\|T^n - L\|^{1/n} \to 0$ as $n \to \infty$, and the result follows. $\square$

This condition is far from necessary, as is shown by, for example, the compact endomorphism of the disk algebra induced by the self-map $\phi(z) = z/2$.

We remark that in the case when $X$ is connected, a modification of the proof of Lemma 4.1, combined with Theorem 1.2, shows that the existence of an $x_0$ such that $C_n < 1$ for some $n$ is a necessary and sufficient condition for $T$ to be quasicompact. In particular, if $C_n < 1$ for some $n$ then in fact we have $C_n \to 0$ as $n \to \infty$.

We now use Lemma 4.1 to construct a non-power compact Riesz endomorphism of a Banach algebra of analytic functions on the unit ball of a Banach space.

For our example let $\mathcal{X}$ denote the closed unit ball of the complex Banach space $E = \ell^\infty(\mathbf{N}^2)$, with the weak * topology. Then $\mathcal{X}$ is compact and metrizable. Elements of $\mathcal{X}$ will be denoted by $x = (x_{j,k})_{j,k=1}^\infty$. Next define $\mathcal{A}$ as the uniform algebra on $\mathcal{X}$ generated by the coordinate projections $p_{j,k}$. Then $\mathcal{A}$ is a subalgebra of the uniform algebra of all continuous functions on $\mathcal{X}$ which are disk algebra functions in each variable separately.

We consider the norm metric $d_\infty$ on the Banach space $E$, and also the Gleason distance (from the norm of $\mathcal{A}^*$) on $\mathcal{X}$. Further, let $x^0$ be the zero element of $E$. Obviously $x^0 \in \mathcal{X}$. This element will be the fixed point of the example we construct below.

The algebra $\mathcal{A}$ is an isomorphic copy of the well-known infinite polydisk algebra, which was also used by Klein in [16]. (See [2] for a deeper investigation of the properties of this algebra.)

A variety of related algebras were considered by Galindo, Gamelin, and

Lindström [9] in relation to weakly compact homomorphisms. It is easy to see from their work that weakly compact endomorphisms need not be Riesz.

Although we do not use this fact, it is standard that the maximal ideal space of $\mathcal{A}$ is $\mathcal{X}$. This is because every character is determined by what it does to all of the coordinate functionals, and must agree at all of these with some evaluation character at a point of $\mathcal{X}$.

We start with a lemma to help us estimate the Gleason distance from $x^0$.

**Lemma 4.2:** Let $\mathcal{A}$ and $\mathcal{X}$ be as described.

(i) If $\alpha \in \mathbf{C}$ with $|\alpha| \leq 1$, then the selfmap $\psi$ of $\mathcal{X}$ defined by $\psi(x) = \alpha x$ induces an endomorphism of $\mathcal{A}$.

(ii) For each $x \in \mathcal{X}$, the Gleason distance from $x$ to $x^0$ is at most $2d_\infty(x, x^0)$.

**Proof:** (i) For all $j, k$ we have $p_{j,k} \circ \psi = \alpha p_{j,k} \in \mathcal{A}$. Thus the closed subalgebra $\{f \in C(\mathcal{X}): f \circ \psi \in \mathcal{A}\}$ of $C(\mathcal{X})$ contains all the $p_{j,k}$ and hence all of $\mathcal{A}$, as required.

(ii) Let $x \in \mathcal{X}$ and assume that $x \neq x^0$. Set $R = 1/d_\infty(x, x^0)$. Let $f$ be a function in the unit ball of the dense subalgebra of $A$ generated by 1 and the coordinate projections $p_{j,k}$ and consider the function $g$ defined on the closed unit disk by $g(z) = f(zRx) - f(x^0)$. This is simply a polynomial in $z$ which vanishes at 0, and $\|g\|_\infty \leq 2\|f\|_\infty$. Note the norm of $g$ is taken on the closed unit disk, so by Schwarz's Lemma we must have $|g(z)| \leq 2\|f\|_\infty |z|$ for $|z| \leq 1$. In particular, setting $z = 1/R$ gives us $|f(x) - f(x^0)| \leq 2\|f\|_\infty / R \leq 2d_\infty(x, x^0)$. The result now follows. $\Box$

**Theorem 4.3:** With $\mathcal{A}$ and $\mathcal{X}$ as above, there is a Riesz endomorphism of $\mathcal{A}$ which is not power compact.

**Proof:** We define a certain 'weighted shift' $\phi$ on $\mathcal{X}$ and show that this induces an endomorphism with the desired properties.

For $x \in \mathcal{X}$, we define $\phi$ by

$$(\phi(x))_{j,k} = x_{j,k+1}/(k+1).$$

Note that

$$d_\infty(\phi_n(x), x^0) \leq 1/(n+1)! \tag{$*$}$$

for all $n$ and all $x \in \mathcal{X}$, and in particular $\phi$ is a selfmap of $\mathcal{X}$.

We next show that $\phi$ induces an endomorphism of $\mathcal{A}$. By definition of $\phi$ we have, for all $j, k$, $p_{j,k} \circ \phi = p_{j,k+1}/(k+1)$. Thus (as for $\psi$ above) $f \circ \phi \in \mathcal{A}$ for all $f \in \mathcal{A}$. Let $T$ be the endomorphism induced by $\phi$. It follows easily from the preceding two lemmas and $(*)$ that $T$ is Riesz. To show that $T$ is not power compact, let $n \in \mathbf{N}$ and note that for all $j$, $(T^n p_{j,1}) = p_{j,n+1}/(n+1)!$. Since $(T^n p_{j,1})_{j=1}^{\infty}$ has no convergent subsequence, $T^n$ is not compact for all $n$, whence the Riesz endomorphism $T$ is not power compact. $\square$

Since weakly compact endomorphisms of the disk algebra are always compact [8] it follows that there are many Riesz endomorphisms of the disk algebra which are not weakly compact. (An easy example of this is given by the inducing map $\phi(z) = (1 - z)/2$.) Since the disk algebra is a URM-algebra, such endomorphisms are necessarily power compact. It is easy to see, however, that the Riesz endomorphism constructed in Theorem 4.3 has the property that no iterate of it is weakly compact.

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