A NOTE ON IDEAL SPACES OF BANACH ALGEBRAS

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Abstract In a previous paper the second author introduced a compact topology τ_r on the space of closed ideals of a unital Banach algebra A. If A is separable then τ_r is either metrizable or else neither Hausdorff nor first countable. Here it is shown that τ_r is Hausdorff if A is $C^1[0,1]$, but that if A is a uniform algebra then τ_r is Hausdorff if and only if A has spectral synthesis. An example is given of a strongly regular, uniform algebra for which every maximal ideal has a bounded approximate identity, but which does not have spectral synthesis.

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Introduction

Following the work of Beckhoff [1], [2], [3], the second author introduced in [15] the topology τ_r on the lattice $Id(A)$ of closed, two-sided ideals of a unital Banach algebra A (throughout this note, all Banach algebras will be assumed to be unital). This topology is defined using the quotient norms, and is invariant under a change to an equivalent norm on A. If A is a commutative Banach algebra, then τ_r coincides, on the maximal ideal space of A, with the Gelfand topology, while if A is a C*-algebra τ_r coincides with the compact, Hausdorff topology τ_s on $Id(A)$. It was shown in [15] that τ_r is always compact, and that for separable Banach algebras a dichotomy occurs: either τ_r is metrizable, or else τ_r is neither Hausdorff nor first countable. The first possibility is known to occur for finite-dimensional Banach algebras, TAF-algebras, and Banach algebras with spectral synthesis, but the question of whether the second possibility ever occurs was left open.

Two cases of particular interest are the disc algebra, and the algebra $C^1[0,1]$. In both cases a complete description of the set of closed ideals is available, but there is some difficulty in describing the quotient norms. The purpose of this paper is to show that τ_r is Hausdorff for $C^1[0,1]$, but that if A is a uniform algebra then τ_r is Hausdorff on $Id(A)$ if and only if A has spectral synthesis. Thus for the disc algebra τ_r is neither Hausdorff nor first countable. An example is given of a strongly regular, uniform algebra for which every maximal ideal has a bounded approximate identity, but which does not have spectral synthesis.

Let us now define the topology τ_r, which is the join of two weaker topologies. The first is easily defined: τ_u is the weakest topology on $Id(A)$ for which all the norm functions $I \mapsto \|a + I\|$ $(a \in A,\ I \in Id(A))$ are upper semi-continuous. The other topology τ_n can

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be described in various different ways, but none is particularly easy to work with. A net (I_α) in $Id(A)$ is said to have the *normality property* with respect to $I \in Id(A)$ if $a \notin I$ implies that $\liminf \|a + I_\alpha\| > 0$. Let τ_n be the topology whose closed sets N have the property that if (I_α) is a net in N with the normality property relative to $I \in Id(A)$ then $I \in N$. It follows that if (I_α) is a net in $Id(A)$ having the normality property relative to $I \in Id(A)$ then $I_\alpha \to I$ (τ_n). Any topology for which convergent nets have the normality property with respect to each of their limits (such a topology is said to have the *normality property*) is necessarily stronger than τ_n, but it is not clear that τ_n need have the normality property. Indeed the following is true. Let τ_r be the topology on $Id(A)$ generated by τ_u and τ_n. Then τ_r is always compact [15; 2.3], and τ_r is Hausdorff if and only if τ_n has the normality property [15; 2.12]. Since, as we shall show, τ_r is not Hausdorff for uniform algebras without spectral synthesis, it follows that τ_n does not have the normality property in these algebras.

We shall need the following lemma.

Lemma 0.1 *Let (I_α) be a net in $Id(A)$, either decreasing or increasing, and correspondingly either set $I = \bigcap I_\alpha$ or $I = \overline{\bigcup I_\alpha}$. Then $I_\alpha \to I$ (τ_r).*

Proof. In either case, it is easy to see that (I_α) has the normality property with respect to I, and converges to I (τ_u). Hence $I_\alpha \to I$ (τ_r). Q.E.D.

1. Uniform algebras

In this section we show that if A is a uniform algebra then τ_r is Hausdorff if and only if A has spectral synthesis. An example is given of a strongly regular, uniform algebra in which every maximal ideal has a bounded approximate identity, but which does not have spectral synthesis.

Let A be a Banach function algebra, with Shilov boundary $\Gamma(A)$. For a non-empty, closed subset F of $Max(A)$ let $I(F)$ be the ideal of elements of A which vanish on F, and let $J(F)$ the ideal of elements vanishing in a neighbourhood of F in $Max(A)$. If $F \subseteq \Gamma(A)$ let $L(F)$ be the ideal which is the closure of set of elements vanishing in a neighbourhood of F in $\Gamma(A)$. It is not clear whether $L(F)$ need always be the closure of $J(F)$, but of course this is so whenever $Max(A) = \Gamma(A)$. Recall that A has *spectral synthesis* if $J(F)$ is dense in $I(F)$ for each non-empty, closed subset F of $Max(A)$, and that A is *strongly regular* if $J(\{x\})$ is dense in $I(\{x\})$ for each $x \in Max(A)$. If $I(\{x\}) = L(\{x\})$ for each $x \in \Gamma(A)$ then A is *strongly regular on* $\Gamma(A)$; in fact, this implies that $\Gamma(A) = Max(A)$, see [17; Lemma] (for uniform algebras) and [8] (for the general case: the elementary argument used is due to Mortini, and may be found in [13]).

Now let A be a uniform algebra. A subset X of $Max(A)$ is a *peak set* if there is an $f \in A$ such that $f(x) = 1$ for all $x \in X$, and $|f(x)| < 1$ for all $x \notin X$. An intersection of peak sets is called a *p-set*, and a singleton p-set is a *p-point*. We shall use the facts that the union of two p-sets is a p-set [10; II.12.8], and the set of p-points is a dense subset of $\Gamma(A)$ [16; 7.24].

Lemma 1.1 *Let A be a uniform algebra, X a closed subset of $Max(A)$, and F a p-set in $Max(A)$. Suppose that $f \in I(X)$. Then $\|f + I(X \cup F)\| = \sup\{|f(y)| : y \in F\}$.*

Proof. It is trivial that $\|f + I(X \cup F)\| \geq \sup\{|f(y)| : y \in F\}$. For the converse, there exists, by [16; 7.22], an $h \in fA \subseteq I(X)$ such that $h|_F = f|_F$ and $\|h\| = \sup\{|f(y)| : y \in F\}$. Hence $h - f \in I(X \cup F)$, and the result follows. Q.E.D.

Theorem 1.2 *Let A be a uniform algebra. Then τ_r is Hausdorff on $Id(A)$ if and only if A has spectral synthesis.*

Proof. If A has spectral synthesis then τ_r is Hausdorff on $Id(A)$ by [15; 3.1].

If A does not have spectral synthesis, there are two possibilities. Either $Max(A) = \Gamma(A)$, in which case $J(F)$ is dense in $L(F)$ for every closed subset F of $Max(A)$, so by assumption there is a closed subset X of $Max(A)$ such that $I(X) \neq L(X)$. Otherwise $Max(A) \neq \Gamma(A)$, so by [17; Lemma] A is not strongly regular on $\Gamma(A)$. In other words, there exists $x \in \Gamma(A)$ such that $I(\{x\}) \neq L(\{x\})$. Thus in either event there is a closed subset X of $\Gamma(A)$ such that $I(X) \neq L(X)$.

Let $(N_\alpha)_\alpha$ be a net of decreasing, closed neighbourhoods of X in $\Gamma(A)$ such that $\bigcap N_\alpha = X$. Then $(I(N_\alpha))_\alpha$ is an increasing net in $Id(A)$, and $I(N_\alpha) \subseteq L(X)$, for each α, so $I := \overline{\bigcup_\alpha I(N_\alpha)} \subseteq L(X)$. For each α, let $(F_{\beta(\alpha)})_{\beta(\alpha)}$ be the increasing net of p-sets in N_α (recall that the union of two p-sets is a p-set [10; II.12.8]). Then $(I(F_{\beta(\alpha)}))_{\beta(\alpha)}$ is a decreasing net in $Id(A)$, and the density of the set of p-points in $\Gamma(A)$ [16; 7.24] implies that $\bigcap_{\beta(\alpha)} I(F_{\beta(\alpha)}) = I(N_\alpha)$. Hence $I(F_{\beta(\alpha)}) \overset{\beta(\alpha)}{\longrightarrow} I(N_\alpha)$ (τ_r) by Lemma 0.1. But $I(N_\alpha) \to I$ (τ_r), also by Lemma 0.1, so if $(I(F_\gamma))_\gamma$ denotes the 'diagonal' net, see [11; §2, Theorem 4], then $I(F_\gamma) \to I \subseteq L(X)$ (τ_r).

Suppose that $f \notin I(X)$. Then there is an open subset O of $\Gamma(A)$, meeting X, and an $\epsilon > 0$ such that $|f(x)| > \epsilon$ for all $x \in O$. By the density of p-points in $\Gamma(A)$, there is, for each α, a p-point in $O \cap N_\alpha$, so that there is a $\beta_0(\alpha)$ such that $\|f + I(F_{\beta(\alpha)})\| > \epsilon$ for all $\beta(\alpha) \geq \beta_0(\alpha)$. Hence the 'diagonal' net $I(F_\gamma) \to I(X)$ (τ_n). On the other hand, if $f \in I(X)$ then a simple topological argument shows that there exists α_0 such that for $\alpha \geq \alpha_0$, $N_\alpha \subseteq \{x \in \Gamma(A) : |f(x)| < \epsilon\}$. Thus for $\alpha \geq \alpha_0$,

$$\|f + I(F_{\beta(\alpha)})\| \leq \|f + I(X \cup F_{\beta(\alpha)})\| < \epsilon$$

for all $\beta(\alpha)$, by Lemma 1.1. Hence $I(F_\gamma) \to I(X)$ (τ_u), using [15; 2.1], and so

$$I(F_\gamma) \to I(X) \quad (\tau_r).$$

Since $I \neq I(X)$, τ_r is not Hausdorff. Q.E.D.

In particular, τ_r is neither Hausdorff nor first countable if A is the disc algebra.

It seems to be an open question whether there are any uniform algebras with spectral synthesis, other than the trivial examples of commutative C*-algebras. Examples are given in [7] of non-trivial, strongly regular uniform algebras for which every maximal ideal has a bounded approximate identity. However, this is not enough to ensure spectral synthesis, as we shall now see.

We begin by restating some results from [7], which use Cole's systems of root extensions [4].

Proposition 1.3 *Let A_1 be a normal uniform algebra on a compact, Hausdorff space X_1. Then there is a uniform algebra A on a compact, Hausdorff space X, a surjective continuous map π from X onto X_1 and a bounded linear map $S : A \to A_1$ with the following properties.*

(a) The uniform algebra A is strongly regular and every maximal ideal in A has a bounded approximate identity.

(b) For every $f \in A_1$, $f \circ \pi$ is in A.

(c) If $x \in X_1$ and $g \in A$ with g constantly equal to some complex number c on $\pi^{-1}(\{x\})$ then $(Sg)(x) = c$. In particular, for all $f \in A_1$, $S(f \circ \pi) = f$.

If X_1 is metrizable, then in addition to the above properties we may also insist that X is metrizable.

Surprisingly, perhaps, these properties are not enough to guarantee that A has spectral synthesis. Indeed, the following result shows that the method of [7] cannot produce a uniform algebra with spectral synthesis unless it starts with such an algebra in the first place.

Proposition 1.4 *Let A_1, X_1, A, and X be as in Proposition 1.3. If A has spectral synthesis, then A_1 has spectral synthesis.*

Proof. Let S, π be as in Proposition 1.3. Note that A and A_1 are both normal, so that $X = Max(A)$, and $X_1 = Max(A_1)$. Suppose that A has spectral synthesis. Let E be a closed subset of X_1. Let $f \in A_1$ with $f(E) \subseteq \{0\}$. We show that f is a uniform limit of functions in A_1 each of which vanishes in a neighbourhood of E.

Set $F = \pi^{-1}(E)$. Certainly $f \circ \pi$ vanishes on F. Since A has spectral synthesis, there is a sequence of functions $g_n \in A$ each vanishing in a neighbourhood of F, and such that g_n tends uniformly to $f \circ \pi$ on X. We then have that $S(g_n)$ tends uniformly to f in A_1. Moreover it follows from Proposition 1.3(c), and some elementary topology, that each of the functions $S(g_n)$ vanishes in a neighbourhood of E. The result now follows. Q.E.D.

In fact the same argument shows, more generally, that it is impossible to obtain a uniform algebra with spectral synthesis from one without merely by taking a system of root extensions.

Theorem 1.5 *There exists a strongly regular, uniform algebra A on a compact, metric space such that every maximal ideal of A has a bounded approximate identity, but such that A does not have spectral synthesis.*

Proof. Let A_1 be a normal, uniform algebra on a compact, metric space X_1 such that A_1 is not strongly regular (the uniform algebra constructed by O'Farrell in [14] would do). Certainly A_1 does not have spectral synthesis. The result now follows immediately from Propositions 1.3 and 1.4. Q.E.D.

2. The Banach algebra $C^1[0,1]$

In this section we show that if A is the Banach algebra $C^1[0,1]$ then τ_r is Hausdorff on $Id(A)$. It is interesting that it is possible to identify τ_r without having a complete description of the quotient norms of $C^1[0,1]$.

Let $A = C^1[0,1]$ be the Banach algebra of continuously differentiable, complex functions on $[0,1]$, with norm given by $\|f\| = \|f\|_\infty + \|f'\|_\infty$ where $\|\,.\,\|_\infty$ is the supremum norm, and f' is the derivative of f. It is well known that each closed ideal I of A can be described uniquely in the form

$$I = \{f \in A : f(x) = 0 \ (x \in C),\ f'(x) = 0 \ (x \in E)\},$$

where C is a closed subset of $[0,1]$ and E is a closed subset of C containing the limit points of C. For topological reasons, however, we shall use a slightly different description.

For $(x,y) \in [0,1] \times [0,1]$ define a functional $\phi_{x,y}$ by

$$\phi_{x,y}(f) = \frac{f(x) - f(y)}{x - y}$$

if $x \neq y$, and by $\phi_{x,y}(f) = f'(x)$ if $x = y$. With C and E as above, let D be the closed subset of $[0,1]^2$ given by

$$D = \{(x,y) \in C \times C : \text{either } x \neq y, \text{ or } x \in E\}.$$

It is straightforward to check that the ideal I above can now be described as

$$I = \{f \in A : f(x) = 0 \ (x \in C), \ \phi_{x,y}(f) = 0 \ ((x,y) \in D)\}.$$

Furthermore, the pair (C,D) of closed sets has the following two properties: (i) $D \subseteq C \times C$, (ii) D contains all the non-diagonal points of $C \times C$. On the other hand, if C and D are any closed subsets of $[0,1]$ and $[0,1]^2$, respectively, having these two properties, then there is a unique ideal $I(C,D) \in Id(A)$ given by

$$I(C,D) = \{f \in A : f(x) = 0 \ (x \in C), \ \phi_{x,y}(f) = 0 \ ((x,y) \in D)\}.$$

For $Y = [0,1]$ or $Y = [0,1]^2$, let $H(Y)$ be the set of closed subsets of Y, equipped with the Fell topology (the topology induced by the Hausdorff metric) which is compact and Hausdorff [9]. Let K be the subset of $H([0,1]) \times H([0,1]^2)$ consisting of pairs of closed sets (C,D) having the properties (i) and (ii). By the remarks above, there is a bijective correspondence between K and $Id(A)$. This is the description of $Id(A)$ which we shall use.

Lemma 2.1 *K is a closed subset of $H([0,1]) \times H([0,1]^2)$, with the product Fell topology.*

Proof. Let $((C_\alpha, D_\alpha))_\alpha$ be a net in K with limit $(C,D) \in H([0,1]) \times H([0,1]^2)$. If $(x,y) \in D$, there is a net $((x_\alpha, y_\alpha))_\alpha$ with $(x_\alpha, y_\alpha) \in D_\alpha$ for each α, such that $(x_\alpha, y_\alpha) \to (x,y)$. By property (i), $x_\alpha, y_\alpha \in C_\alpha$ for each α, and $x_\alpha \to x$, $y_\alpha \to y$, so $x, y \in C$. Hence $D \subseteq C \times C$.

On the other hand, suppose that $x, y \in C$ with $x \neq y$. Then there are nets (x_α) and (y_α) with $x_\alpha, y_\alpha \in C_\alpha$ for each α and $x_\alpha \to x$, $y_\alpha \to y$. By passing to subnets we may assume that $x_\alpha \neq y_\alpha$ for each α. Hence $(x_\alpha, y_\alpha) \in D_\alpha$ by property (ii), and $(x_\alpha, y_\alpha) \to (x,y)$, so $(x,y) \in D$.

This establishes that (i) and (ii) hold for the pair (C,D), so $(C,D) \in K$. Q.E.D.

Thus K is a compact, Hausdorff space in the restriction of the product Fell topology. Let τ be the corresponding compact, Hausdorff topology on $Id(A)$, induced by the bijective correspondence $(C,D) \leftrightarrow I(C,D)$ between K and $Id(A)$. We shall show that τ coincides with τ_r.

Let $I = I(C,D)$ be a closed ideal of A, and define a semi-norm r_I on A by

$$r_I(f) = \sup\{|f(x)| : x \in C\} + \sup\{|\phi_{x,y}(f)| : (x,y) \in D\} \qquad (f \in A).$$

Note that $\ker r_I = I$, and that if $I_\alpha \to I$ (τ) then $r_{I_\alpha}(f) \to r_I(f)$ for all $f \in A$. Let q_I be the quotient semi-norm $q_I(f) = \|f + I\|$.

Lemma 2.2 *Let $I = I(C, D)$ be a closed ideal of A, and let $f \in A$. Then $q_I(f) \geq r_I(f)$.*

Proof. Let $h \in A$ with $h - f \in I$. Then $(h - f)(x) = 0$, for $x \in C$, and $\phi_{x,y}(h - f) = 0$, for $(x, y) \in D$. Hence $\|h\|_\infty \geq \sup\{|f(x)| : x \in C\}$, by common sense, and $\|h'\|_\infty \geq \sup\{|\phi_{x,y}(f)| : (x, y) \in D\}$, by the Mean Value Theorem [6; 8.5.4] (recall that we are dealing with complex functions). Thus $\|h\| \geq r_I(f)$. Q.E.D.

Let $S_1(A)$ denote the set of semi-norms ρ on A satisfying $\rho(f) \leq \|f\|$ for all $f \in A$. Then $S_1(A)$ is a compact, Hausdorff space with respect to the topology of pointwise convergence [1]. Note that if (ρ_α) is a net in $S_1(A)$ and $\rho_\alpha \to \rho \in S_1(A)$ then $\ker \rho_\alpha \to \ker \rho$ (τ_u).

Theorem 2.3 *The topologies τ and τ_r coincide on $Id(A)$. Hence τ_r is a compact, Hausdorff topology.*

Proof. First we show that τ has the normality property. Suppose that $I_\alpha = I_\alpha(C_\alpha, D_\alpha) \to I = I(C, D)$ (τ), and that $f \notin I$. If there exists $x \in C$ such that $|f(x)| = \delta > 0$, then eventually the set $\{y \in [0, 1] : |f(y)| > \delta/2\} \cap C_\alpha$ is non-empty, so eventually $\|f + I_\alpha\| \geq r_{I_\alpha}(f) > \delta/2$ by Lemma 2.2. Otherwise there exists $(x, y) \in D$ such that $|\phi_{x,y}(f)| = \delta > 0$, so eventually the set $\{(z, w) \in [0, 1]^2 : |\phi_{z,w}(f)| > \delta/2\} \cap D_\alpha$ is non-empty, so eventually $\|f + I_\alpha\| > \delta/2$, again by Lemma 2.2. Thus we have shown that τ has the normality property for the net (I_α) with respect to I.

Next we show that $\tau \geq \tau_u$. Suppose, for a contradiction, that $I_\alpha = I_\alpha(C_\alpha, D_\alpha) \to I = I(C, D)$ (τ), but that $I_\alpha \not\to I$ (τ_u). We may assume, by passing to a subnet, if necessary, and using the compactness of $S_1(A)$, that (q_{I_α}) is a convergent net in $S_1(A)$, with limit σ, say, and that $I \neq \ker \sigma$. Set $J = I(F, G) = \ker \sigma$, and note that since $q_{I_\alpha}(f) \geq r_{I_\alpha}(f)$ $(f \in A)$ for all α, it follows that $\sigma(f) = \lim q_{I_\alpha}(f) \geq \lim r_{I_\alpha}(f) = r_I(f)$. Hence $J \subseteq I$, so $F \supseteq C$ and $G \supseteq D$. Suppose that $x \in F \setminus C$, and let $f \in A$ be such that $f(x) = 1$, and f vanishes in an open neighbourhood N of C. Then $D \subseteq C \times C \subseteq N \times N$, so $\phi_{y,z}(f) = 0$ for all $(y, z) \in D$. Hence $f \in I$, but $f \notin J$ so $\sigma(f) > 0$. Since $I_\alpha \to I$ (τ), eventually $C_\alpha \subseteq N$, so eventually $f \in I_\alpha$. Hence $0 = \lim q_{I_\alpha}(f) = \sigma(f) > 0$, a contradiction. Thus $F = C$.

Now suppose that $(x, y) \in G \setminus D$. Then $x = y$, by property (ii), since $C = F$, so x is an isolated point of C, by the first description of $Id(A)$. Let $f \in A$ such that $f(x) = 0$, $f'(x) \neq 0$, and f vanishes on a closed set M, disjoint from x, but containing $C \setminus \{x\}$ in its interior. Thus $f \in I \setminus J$. Since $I_\alpha \to I$ (τ), there exists α_0 such that for $\alpha \geq \alpha_0$, $C_\alpha \cap ([0, 1] \setminus M)$ is a singleton $\{x_\alpha\}$. For fixed $\alpha \geq \alpha_0$ and any $\epsilon > 0$ there exists $g \in A$

such that $g(x_\alpha) = f(x_\alpha)$, $g(M) = 0$, and

$$\|g\| < |f(x_\alpha)| + \frac{|f(x_\alpha)|}{d(x_\alpha, M)} + \epsilon,$$

where $d(x_\alpha, M)$ is the distance from x_α to the set M. Since ϵ is arbitrary and $g - f \in I_\alpha$ it follows that

$$q_{I_\alpha}(f) \leq |f(x_\alpha)| + \frac{|f(x_\alpha)|}{d(x_\alpha, M)}.$$

But $f(x_\alpha) \to f(x) = 0$, and $d(x_\alpha, M) \to d(x, M) > 0$, so $\sigma(f) = \lim q_{I_\alpha}(f) = 0$. Hence $f \in J$, a contradiction. It follows that $G = D$, so $I = J$, another contradiction. Thus $I_\alpha \to I$ (τ_u) after all.

We have shown that τ is a compact (Hausdorff) topology stronger than τ_u, and such that τ-convergent nets have the normality property. Hence $\tau = \tau_r$ by [15; 2.11]. Q.E.D.

Remarks. (i) $C^1[0,1]$ is, with an equivalent norm, isometrically isomorphic to a quotient of a uniform algebra [5]. It was shown in [15; 2.9] that if I is a closed ideal of a Banach algebra A then the natural map from $Id(A/I)$ onto the set $\{J \in Id(A) : J \supseteq I\}$ is a τ_r–τ_r homeomorphism.

(ii) Khanin [12] has introduced the class of D-algebras, of which $C^1[0,1]$ and commutative Banach algebras with spectral synthesis are examples. It would be interesting to know if τ_r is Hausdorff on $Id(A)$ for all D-algebras A.

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