Trivial Jensen measures without regularity

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Abstract In this note we construct Swiss cheeses X such that $R(X)$ is non-regular but such that $R(X)$ has no non-trivial Jensen measures. We also construct a non-regular uniform algebra with compact, metrizable character space such that every point of the character space is a peak point.

In [Co] Cole gave a counterexample to the peak point conjecture by constructing a non-trivial uniform algebra A with compact, metrizable character space Φ_A such that every point of Φ_A is a peak point for A. This uniform algebra was obtained from an example of McKissick [M] by a process of repeatedly adjoining square roots. Because McKissick's algebra is regular, so is this first example constructed by Cole (see [F2] and [Ka]). This leads to the following question: Let A be a uniform algebra with compact, metrizable character space Φ_A such that every point of Φ_A is a peak point for A. Must A be regular?

In this note we construct an example of a Swiss cheese X for which the uniform algebra $R(X)$ is non-regular, but such that $R(X)$ has no non-trivial Jensen measures. We then apply Cole's construction to this example to produce an example of a non-regular uniform algebra with compact, metrizable character space such that every point of the character space is a peak point.

We begin by recalling some standard facts about Jensen measures.

Notation For a commutative Banach algebra A, we denote by Φ_A the character space of A. Now suppose that A is a uniform algebra on a compact space X. For $x \in X$, let M_x and J_x be the ideals of functions in A vanishing at x, and in a neighbourhood of x, respectively.

Definition 1 Let A be a uniform algebra on a compact space X, and let $\phi \in \Phi_A$. Then a *Jensen* measure for ϕ is a regular, Borel probability measure μ on X such that, for all $f \in A$,

$$\log|\phi(f)| \leq \int_X \log|f(x)| \mathrm{d}\mu(x)$$

(where $\log(0)$ is defined to be $-\infty$).

Let $x \in X$. We say that x is a *point of continuity* for A if there is no point y of $X \setminus \{x\}$ satisfying $M_x \supseteq J_y$.

It is standard (see, for example, [G, p.33]) that every $\phi \in \Phi_A$ has a Jensen measure supported on X, and that each such measure *represents* ϕ, i.e., for all $f \in A$,

$$\phi(f) = \int_X f(x) \mathrm{d}\mu(x).$$

It is elementary to see that if x is a point of continuity for A then the only Jensen measure for the evaluation character at x which is supported on X is the point mass at x. The converse is, in general, false as is easily seen by considering the disc algebra: here the only representing measures for points of the unit circle are point masses, while clearly there are no points of continuity.

In the case where $X = \Phi_A$, the usual definition of *regularity* of A is that, for each closed subset E of X and each $x \in X \setminus E$ there is an $f \in A$ such that $f(x) = 1$ and f is identically 0 on E. This is easily seen to be equivalent to the fact that every point of X is a point of continuity for A. The first example of a regular uniform algebra which was non-trivial (i.e. not equal to the algebra of all continuous functions on a compact space) was given in [M]. This example was modified by O'Farrell in [O] to give an example of a regular uniform algebra which has a non-zero, continuous point derivation of infinite order. When regularity fails, it usually fails fairly drastically: see [FS] for results on non-regularity.

It is clear that if A is a regular uniform algebra, then there are no non-trivial Jensen measures supported on X. The examples constructed in this note show that the converse is false: there are non-regular uniform algebras for which there are no non-trivial Jensen measures.

There are many examples already known of Swiss cheeses X for which $R(X)$ has no non-trivial Jensen measures: as well as McKissick's original example of a non-trivial regular $R(X)$ ([M], but see also [Ko]), other examples can be found in [H], [Bro, pp.193–195] and also (in view of Theorem 3.8 of [G], as explained on page 64 of [G]) in [W]. It is not clear whether or not $R(X)$ is regular for these latter examples.

In this note we shall construct a compact plane set X such that $R(X)$ has no non-trivial Jensen measures, but such that $R(X)$ is definitely not regular.

We shall make heavy use of the following version of McKissick's Lemma, given in [Ko].

Lemma 2 *Let D be an open disc in $\mathbf{C}$ and let $\epsilon > 0$. Then there is a sequence $\Delta_k (k \in \mathbf{N})$ of (pairwise disjoint) open discs with each $\Delta_k \subseteq D$ such that the sum of the radii of the Δ_k is less than ϵ and such that, setting $U = \bigcup_{k \in \mathbf{N}} \Delta_k$, there is a sequence f_n of rational functions with poles only in U and such that f_n converges uniformly on $\mathbf{C} \setminus U$ to a function F such that $F(z) = 0$ for all $z \in \mathbf{C} \setminus D$ while $F(z) \neq 0$ for all $z \in D \setminus U$.*

We also need the following elementary lemma, which follows from Cauchy's integral formula applied to a sequence of compact sets obtained by deleting finitely many open discs from the closed unit disc. These estimates are well-known: for an explicit proof of the estimate for the first derivative, see, for example, [F1].

Notation For a bounded, complex-valued function f defined on a non-empty set S we shall denote by $|f|_S$ the uniform norm of f on S, that is

$$|f|_S = \sup\{|f(x)| : x \in S\}.$$

For a compact plane set X, $R_0(X)$ is the set of restrictions to X of rational functions with poles off X. (So $R(X)$ is the uniform closure of $R_0(X)$.)

Lemma 3 *Let D_n be a sequence of open discs in $\mathbf{C}$ (not necessarily pairwise disjoint), and set $X = \bar{\Delta} \setminus \bigcup_{n=1}^{\infty} D_n$. Suppose that $z \in X$. Let s_n denote the distance from D_n to z and r_n the radius of D_n. We also set $r_0 = 1$ and $s_0 = 1 - |z|$. Suppose that $s_n > 0$ for all n. Then, for all $f \in R_0(X)$ and $k \geq 0$, we have*

$$|f^{(k)}(z)| \le k! \sum_{j=0}^{\infty} \frac{r_j}{{s_j}^{k+1}} |f|_S.$$

We shall also require the following famous lemma due originally to Denjoy [D], under some additional hypotheses which were shown to be unnecessary in [C].

Lemma 4 *Let f be an infinitely differentiable function on an interval I such that*

$$\sum_{k=0}^{\infty} \frac{1}{|f^{(k)}|_I^{1/k}} = \infty. \qquad (*)$$

Suppose that there is an $x \in I$ such that $f^{(k)}(x) = 0$ for all $k \ge 0$. Then f is constantly 0 *on I.*

Any algebra of infinitely differentiable (complex-valued) functions on I satisfying condition $(*)$ is thus a *quasianalytic* algebra, in the sense of Denjoy and Carleman.

We are now able to construct the desired Swiss cheese. The method for ensuring quasianalyticity in the following theorem is based on the method used by Brennan in [Bre]. However, we use McKissick's Lemma and the Cauchy estimates above, and do not need to appeal to the theory of the Bergman kernel.

Theorem 5 *Set $I = [-1/2, 1/2]$. There is a Swiss cheese X with $I \subseteq X$ such that every point of $X \setminus I$ is a point of continuity for $R(X)$ but such that every $f \in R(X)$ is infinitely differentiable on I and satisfies condition $(*)$ there.*

Proof The Swiss cheese X is constructed inductively, deleting a countable collection of discs at each stage, and applying the above lemmas. First we set $\delta_n = \frac{1}{n+2}$ for $n \in \mathbf{N}$, and we set

$$K_n = \{z \in \mathbf{C} : \mathrm{dist}(z, I) \le \delta_k\}.$$

Let $\mathcal{S}_n$ be the set of all discs with rational radius which are centred on points of $\mathbf{Q} + i\mathbf{Q}$ and which do not meet K_n. Enumerate $\mathcal{S}_n$ as $\{D_{n,1}, D_{n,2}, \ldots\}$. Set $X_0 = \bar{\Delta}$ (the closed unit disc).

Now set $\epsilon_1 = 1/4$. For each $k \in \mathbf{N}$, apply McKissick's lemma to the discs $D_{1,k}$ to obtain a countable family of open discs $\mathcal{F}_{1,k}$ and a sequence of rational functions $f_{1,k,n}$ as in that lemma, such that the sum of the radii of the discs in $\mathcal{F}_{1,k}$ is less than $\frac{\epsilon_1}{2^k}$, and, with $U_{1,k} = \bigcup\{\Delta : \Delta \in \mathcal{F}_{1,k}\}$, the functions $f_{1,k,n}$ have poles only in $U_{1,k}$ and converge to a function $F_{1,k}$ uniformly on $\mathbf{C} \setminus U_{1,k}$, where $F_{1,k}$ vanishes identically on $\mathbf{C} \setminus D_{1,k}$ and is nowhere zero on $D_{1,k} \setminus U_{1,k}$. Set $X_1 = X_0 \setminus \bigcup_{k \in \mathbf{N}} U_{1,k}$. Then, for $f \in R_0(X_1)$, we can use Lemma 3 to estimate the derivatives of f on I: for $z \in I$ and $n \in \mathbf{N}$ we have

$$|f^{(n)}(z)| \le \frac{n!(1+\epsilon_1)}{\delta_1^{n+1}} |f|_{X_1}.$$

Set

$$A_{1,n} = \frac{2n!(1+\epsilon_1)}{\delta_1^{n+1}}.$$

Note that $\sum A_{1,n}^{-1/n}$ diverges. Choose N_1 such that $\sum_{n=1}^{N_1} A_{1,n}^{-1/n} \geq 1$. We now move to the next stage of the construction. Choose $\epsilon_2 > 0$ small enough that $n!((1+\epsilon_1)/\delta_1^{n+1} + \epsilon_2/\delta_2^{n+1}) < A_{1,n}$ for $1 \leq n \leq N_1$. For each $k \in \mathbf{N}$, apply McKissick's lemma to the discs $D_{2,k}$ to obtain a countable family of open discs $\mathcal{F}_{2,k}$ and a sequence of rational functions $f_{2,k,n}$ as in that lemma, such that the sum of the radii of the discs in $\mathcal{F}_{2,k}$ is less than $\epsilon_2/2^k$, and, with $U_{2,k} = \bigcup\{\Delta : \Delta \in \mathcal{F}_{2,k}\}$, the functions $f_{2,k,n}$ have poles only in $U_{2,k}$ and converge to a function $F_{2,k}$ uniformly on $\mathbf{C} \setminus U_{2,k}$, where $F_{2,k}$ vanishes identically on $\mathbf{C} \setminus D_{2,k}$ and is nowhere zero on $D_{2,k} \setminus U_{2,k}$. Set $X_2 = X_1 \setminus \bigcup_{k \in \mathbf{N}} U_{2,k}$. Applying Lemma 3 we see that, for $f \in R_0(X_2)$, $z \in I$ and $n \in \mathbf{N}$, $|f^{(n)}(z)| \leq n!((1+\epsilon_1)/\delta_1^{n+1} + \epsilon_2/\delta_2^{n+1})|f|_{X_2}$. Set $A_{2,n} = A_{1,n}$ for $1 \leq n \leq N_1$, and

$$A_{2,n} = 2n! \left(\frac{1+\epsilon_1}{\delta_1^{n+1}} + \frac{\epsilon_2}{\delta_2^{n+1}} \right)$$

for $n > N_1$. Again we see that $\sum_{n=1}^{\infty} A_{2,n}^{-1/n} = \infty$, so choose $N_2 > N_1$ such that $\sum_{n=N_1+1}^{N_2} A_{2,n}^{-1/n} \geq 1$. Now choose $\epsilon_3 > 0$ such that $n!((1+\epsilon_1)/\delta_1^{n+1} + \epsilon_2/\delta_2^{n+1} + \epsilon_3/\delta_3^{n+1}) < A_{2,n}$ for $1 \leq n \leq N_2$. We now proceed to choose families of discs and functions as before. The inductive process is now clear, and produces a decreasing family of compact sets X_n, arrays of functions $F_{n,k}$ and positive real numbers $A_{n,k}$ for $n, k \in \mathbf{N}$ and a strictly increasing sequence of positive integers N_n such that $A_{n,k} = A_{j,k}$ whenever $1 \leq j \leq n$ and $1 \leq k \leq N_n$. We may now define A_k by setting $A_k = A_{n,k}$ whenever $k \leq N_n$ (this is well defined by above). We have $\sum_{k=1}^{\infty} {A_k}^{-1/k} = \infty$. Set $X = \bigcap_{n=1}^{\infty} X_n$. Then we claim X is a Swiss cheese with the required properties. First we note that for $f \in R_0(X)$, $z \in I$ and $k \in \mathbf{N}$ we have $|f^{(k)}(z)| \leq A_k |f|_X$. It thus follows that every element of $R(X)$ is infinitely differentiable on I and satisfies condition $(*)$ there. Finally, for any point $z \in X \setminus I$, we see that there is some n with $z \in \underline{X \setminus K_n}$. Let $w \in X$ with $w \neq z$. Then there is some disc $D_{n,k}$ with $z \in D_{n,k}$ and $w \in X \setminus \overline{D_{n,k}}$. Then $F_{n,k}|X$ is in $R(X)$ and is non-zero at z, but vanishes in a neighbourhood of w. This shows that J_w is not a subset of M_z. Thus each such point z is a point of continuity for $R(X)$, as required. Q.E.D.

Corollary 6 *Let X be the Swiss cheese constructed in the preceding Theorem. Then $R(X)$ has no non-trivial Jensen measures, but $R(X)$ is not regular.*

Proof Certainly there are no non-trivial Jensen measures for points of $X \setminus I$. Also, $R(X)$ is not regular, since none of the points of I are points of continuity for $R(X)$. It remains to show that there are no non-trivial Jensen measures for points of I. However, from the standard theory of Jensen interior (or fine interior), see page 319 of [GL] or numerous papers of Debiard and Gaveau, it is clear that if $R(X)$ has any non-trivial Jensen measures, then the set of points which have non-trivial Jensen measures is fairly large: certainly it could not be contained in the interval I. The result follows. Q.E.D.

We now show that there are non-regular uniform algebras with compact, metrizable character space for which every point of the character space is a peak point (which is equivalent to saying that every maximal ideal has a bounded approximate identity). We shall do this by applying the following result to the algebra constructed in Corollary 6. This result is a combination of results from [Co] and [F2] (see also [Ka]), and is based on Cole's systems of root extensions for uniform algebras.

Proposition 7 *Let A_1 be a uniform algebra on a compact, Hausdorff space $X_1 = \Phi_{A_1}$ such that the only Jensen measures for characters of A_1 are point masses. Then there is a uniform algebra A on a compact, Hausdorff space $X = \Phi_A$, a surjective continuous map π from X onto X_1 and a bounded linear map $S : A \to A_1$ with the following properties.*

(a) Every maximal ideal in A has a bounded approximate identity.

(b) For every $f \in A_1$, $f \circ \pi$ is in A.

(c) If $x \in X_1$ and $g \in A$ with g constantly equal to some complex number c on $\pi^{-1}(\{x\})$ then $(Sg)(x) = c$. In particular, for all $f \in A_1$, $S(f \circ \pi) = f$.

(d) If A_1 is regular then so is A.

If X_1 is metrizable, then in addition to the above properties we may also insist that X is metrizable.

Note that it is easy, using (c), to see that the converse to (d) also holds: if A is regular, then so is A_1.

Corollary 8 *There is a uniform algebra A on a compact, metrizable topological space $X = \Phi_A$ such that every point of X is a peak point and such that A is not regular.*

Proof Let X_1 be the Swiss cheese constructed in Theorem 5, and set $A_1 = R(X_1)$ (so A_1 is not regular, but the only Jensen measures for A_1 are point masses). Now apply Proposition 7, and the remark following it, to produce a non-regular, uniform algebra A on a compact, metrizable topological space $X = \Phi_A$ such that every maximal ideal of A has a bounded approximate identity. This algebra has the required properties. Q.E.D.

We conclude with some open questions. An alternative way to construct the example of Corollary 8 is to use Basener's simple construction of non-trivial uniform algebras for which every point of the character space is a peak point (announced in [Ba], but see pages 202-203 of [S] for the details). It is easy to see, as with Cole's construction, that Basener's example is non-regular whenever the original algebra $R(X)$ is non-regular. This leads to two related questions:

Question 1. Is Basener's example ever regular?

Question 2. Is Basener's example always regular when the original algebra $R(X)$ is?

Our final question concerns quasianalyticity for $R(X)$. There are several alternative definitions in print for a collection of functions to be quasianalytic. In [Bre], Brennan gave an example of a Swiss cheese X such that, for all $p \geq 2$, no non-zero element of $R^p(X)$ vanishes almost everywhere on a subset of X with positive area. Here $R^p(X)$ is the closure of $R(X)$ in $L^p(X)$ (area measure).

Let X be a compact plane set with positive area. We shall say that $R(X)$ is *quasianalytic* if the only function in $R(X)$ whose zero set has positive area is the constant function 0.

Question 3. Let X be a compact plane set with positive area. Is it possible for $R(X)$ to be quasianalytic and yet to have no non-trivial Jensen measures?

The example constructed in Theorem 5 above shows that some elements of quasianalyticity may be introduced, but it is not clear whether this method can be extended to answer Question 3. We may also ask the same question using a different form of quasianalyticity, for example insisting that the only function in $R(X)$ which vanishes identically on a non-empty, relatively open subset of X is the zero function.

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