SPECTRAL SYNTHESIS AND TOPOLOGIES ON IDEAL SPACES FOR BANACH *-ALGEBRAS

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Abstract This paper continues the study of spectral synthesis and the topologies τ_∞ and τ_r on the ideal space of a Banach algebra, concentrating on the class of Banach *-algebras, and in particular on L^1-group algebras. It is shown that if a group G is a finite extension of an abelian group then τ_r is Hausdorff on the ideal space of $L^1(G)$ if and only if $L^1(G)$ has spectral synthesis, which in turn is equivalent to G being compact. The result is applied to nilpotent groups, $[\mathrm{FD}]^-$-groups, and Moore groups. An example is given of a non-compact, non-abelian group G for which $L^1(G)$ has spectral synthesis. It is also shown that if G is a non-discrete group then τ_r is not Hausdorff on the ideal lattice of the Fourier algebra $A(G)$.

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1. Introduction

The goal of ideal theory for Banach algebras is, on the one hand to describe the set $Id(A)$ of closed two-sided ideals of a Banach algebra A, and on the other hand to use knowledge of the ideal structure to obtain information about the algebra itself. This usually involves representing the algebra as a bundle of quotient algebras over a topological base-space. Standard examples include the Gelfand theory for commutative Banach algebras and the theory of continuous bundles of C*-algebras. Another familiar example is the representation of a C*-algebra as a bundle of C*-algebras over its primitive ideal space with the hull-kernel topology.

In trying to describe the closed ideals of a Banach algebra A, the first question is whether it is sufficient to know about the primitive ideals of A. This is the question of 'spectral synthesis', originally studied for commutative Banach algebras, then for L^1-group algebras and other Banach *-algebras, and more recently also for the Haagerup tensor product of C*-algebras [1], [13]. On the other hand, the representing of a Banach algebra as a bundle of quotient algebras involves the study of topologies on the space $Id(A)$, and its subsets. This is an area which has received considerable attention in recent years, with the work of Archbold [2], Beckhoff [6], [7], [8], and others [3], [19], [34].

The two main topologies that have been introduced are τ_∞ and τ_r. The first, τ_∞, was defined by Beckhoff [6], using the various continuous norms and seminorms which the algebra can carry. It is always compact on $Id(A)$, but seldom Hausdorff. In fact, for a commutative Banach algebra A, τ_∞ is Hausdorff on $Id(A)$ if and only if A has spectral synthesis [13; 1.8]. Since the topology τ_∞ is also Hausdorff for C*-algebras, and since C*-algebras also have a form of spectral synthesis, the third author was led to introduce a definition of spectral synthesis for a general Banach algebra A, and to investigate the relation between this and τ_∞ [33], [13]. It is known that if A is a separable, unital PI-Banach algebra then A has spectral synthesis if and only if τ_∞ is Hausdorff on $Id(A)$, and that for a general Banach algebra A the Hausdorffness of τ_∞ on $Id(A)$ implies a weak form of spectral synthesis. Conversely, if A is separable and has a strong form of spectral synthesis then τ_∞ is a T_1-topology. Thus spectral synthesis and the Hausdorffness of τ_∞ seem to be closely related, and possibly identical.

The second topology that has been introduced is τ_r [34]. This is also compact on $Id(A)$, and is Hausdorff whenever τ_∞ is [34; 3.1.1]. It was shown in [34; 2.11] that if there is a compact Hausdorff topology on a subspace of $Id(A)$, which is related to the quotient norms in a useful way, then that topology necessarily coincides with the restriction of τ_r. Thus, for instance, τ_r coincides with the Gelfand topology on the maximal ideal space of a unital commutative Banach algebra. There are a number of cases when τ_r is Hausdorff but τ_∞ is not, e.g. for TAF-algebras [34] and for the Banach algebra $C^1[0,1]$ [12]. For uniform algebras, however, it turned out that τ_r is Hausdorff if and only if τ_∞ is Hausdorff [12]. Thus for uniform algebras without spectral synthesis, such as the disc algebra, there is no useful compact Hausdorff topology on the space of closed ideals.

The purpose of this paper is to continue the study of spectral synthesis and the topologies τ_∞ and τ_r, concentrating on the class of Banach *-algebras. We are interested in the questions of whether spectral synthesis is equivalent to the Hausdorffness of τ_∞ for these algebras, and whether τ_r can be Hausdorff when spectral synthesis fails. In Section 2 we employ general techniques of Banach *-algebra theory. The main result of the section is that spectral synthesis is equivalent to the Hausdorffness of τ_∞ for a class of Banach *-algebras which includes the L^1-algebras of $[\mathrm{FC}]^-$-groups. In Section 3 we employ group-theoretic techniques. We give an example of a non-compact, non-abelian group G for which $L^1(G)$ has spectral synthesis, but we show that for nilpotent groups, $[\mathrm{FD}]^-$-groups, and Moore groups, spectral synthesis for $L^1(G)$ is equivalent to the compactness of G, and furthermore that if G fails to be compact then τ_r fails to be Hausdorff on $Id(L^1(G))$. We also show that if G is a non-discrete group then τ_r is not Hausdorff on the ideal lattice of the Fourier algebra $A(G)$.

We now give the definitions of the various topologies on $Id(A)$, starting with the lower topology τ_w. Let A be a Banach algebra. A subbase for τ_w on $Id(A)$ is given by the sets $\{I \in Id(A) : I \not\supseteq J\}$ as J varies through the elements of $Id(A)$. Thus the restriction of τ_w to the set of closed prime ideals is simply the hull-kernel topology. Next we define τ_∞. For each $k \in \mathbf{N}$, let $S_k = S_k(A)$ denote the set of seminorms ('seminorm' means 'algebra seminorm' in this paper) ρ on A satisfying $\rho(a) \leq k\|a\|$ for all $a \in A$. Then S_k is a compact, Hausdorff space [6]. We say that $\rho \geq \sigma$, for $\rho, \sigma \in S_k$, if $\rho(a) \leq \sigma(a)$ for all $a \in A$. The point of this upside-down definition is that if $\rho \geq \sigma$ then $\ker \rho \supseteq \ker \sigma$. Clearly if $\rho, \sigma \in S_k$ the seminorm $\rho \wedge \sigma$ defined by $(\rho \wedge \sigma)(a) = \max\{\rho(a), \sigma(a)\}$ is the greatest seminorm less than both ρ and σ in the order structure. Thus S_k is a lattice. The topology τ_∞ is defined on $Id(A)$ as follows [6]: for each k let $\kappa_k : S_k \to Id(A)$ be the map $\kappa_k(\rho) = \ker \rho$, and let τ_k be the quotient topology of κ_k on $Id(A)$. Then $\tau_\infty = \bigcap_k \tau_k$. Clearly each τ_k is compact, so τ_∞ is compact.

Next we define the topology τ_r, which is the join of two weaker topologies. The first is easily defined: τ_u is the weakest topology on $Id(A)$ for which all the norm functions $I \mapsto \|a+I\|$ ($a \in A$, $I \in Id(A)$) are upper semi-continuous. The other topology τ_n can be described in various different ways, but none is particularly easy to work with. A net (I_α) in $Id(A)$ is said to have the *normality property* with respect to $I \in Id(A)$ if $a \notin I$ implies that $\liminf \|a + I_\alpha\| > 0$. Let τ_n be the topology whose closed sets N have the property that if (I_α) is a net in N with the normality property relative to $I \in Id(A)$ then $I \in N$. It follows that if (I_α) is a net in $Id(A)$ having the normality property relative to $I \in Id(A)$ then $I_\alpha \to I$ (τ_n). Any topology for which convergent nets have the normality property with respect to each of their limits (such a topology is said to have the *normality property*) is necessarily stronger than τ_n, but τ_n itself need not have the normality property. Indeed the following is true. Let τ_r be the topology on $Id(A)$ generated by τ_u and τ_n. Then τ_r is always compact [34; 2.3], and τ_r is Hausdorff if and only if τ_n has the normality property [34; 2.12]. It is a useful fact that for $I \in Id(A)$, $Id(A/I)$ is τ_∞- τ_∞ and τ_r- τ_r homeomorphic to the subset $\{J \in Id(A) : J \supseteq I\}$ of $Id(A)$ [6; Prop. 5], [34; 2.9].

The following simple lemma is taken from [12; 0.1].

Lemma 1.1 *Let A be a Banach algebra. Let (I_α) be a net in $Id(A)$, either decreasing or increasing, and correspondingly either set $I = \bigcap I_\alpha$ or $I = \overline{\bigcup I_\alpha}$. Then $I_\alpha \to I$ (τ_r).*

Now let A be a Banach algebra and let $Prim(A)$ be the space of primitive ideals of A (i.e. the kernels of algebraicially irreducible representations of A) equipped with the hull-kernel topology. Let $Prime(A)$ be the space of proper closed prime ideals of A, and let $Prim^s(A)$ be the space of semisimple prime ideals of A (such ideals are automatically closed), both spaces also being equipped with the hull-kernel topology. The notation, and the importance of $Prim^s(A)$, is explained in [13].

The paper [33] contained a definition of 'spectral synthesis', but unfortunately that definition was slightly too restrictive, and was replaced in [13] by the following definition.

Definition of spectral synthesis A Banach algebra A has *spectral synthesis* if it has the following properties:

(i) $Prim^s(A)$ is locally compact.

(ii) τ_w has the normality property on $Prim(A)$ (or equivalently on $Prim^s(A)$).

(iii) $Id(A)$ is isomorphic to the lattice of open subsets of $Prim(A)$, under the correspondence $I \leftrightarrow \{P \in Prim(A) : P \not\supseteq I\}$. Equivalently, every proper, closed ideal of A is semisimple.

Remarks (a) It was noted in [13], see [33; 1.1], that if A has spectral synthesis then τ_w has the normality property on the whole of $Id(A)$.

(b) The definition just given coincides with the standard one for the class of commutative Banach algebras [13; 1.7]. Recall that a (possibly non-unital) commutative Banach algebra A has spectral synthesis (usual definition) if the map $I \mapsto \{P \in Prim(A) : P \supseteq I\}$ sets up a 1–1 correspondence between closed ideals of A and Gelfand closed subsets of $Prim(A)$. This is equivalent to requiring that the hull-kernel and Gelfand topologies coincide on $Prim(A)$, and that every closed ideal of A is semisimple.

2. Banach *-algebras

In this section we consider spectral synthesis and the topologies τ_∞ and τ_r within the class of Banach *-algebras. We show that our definition of spectral synthesis coincides with the usual definition for a large subclass which probably contains all the cases of interest. For a smaller class, which contains the L^1-algebras of $[\mathrm{FC}]^-$-groups, we are able to show that spectral synthesis is equivalent to the Hausdorffness of τ_∞.

Let A be a *-semisimple Banach *-algebra with C*-envelope $C^*(A)$ and self-adjoint part A_{sa}. Let $Prim_*(A)$ be the set of kernels of topologically irreducible $*$-representations of A on Hilbert space, with the hull-kernel topology (such representations are automatically continuous [10; p.196]). The map $P \mapsto P \cap A$ ($P \in Prim(C^*(A))$) maps $Prim(C^*(A))$ continuously onto $Prim_*(A)$, but is not a homeomorphism in general. If it is a homeomorphism then A is said to be **-regular*. A Banach *-algebra A is said to be *hermitian* (or *symmetric*) if every self-adjoint element of A has real spectrum. This has the implication that every primitive ideal of A is the kernel of a topologically irreducible $*$-representation on a Hilbert space, or in other words, that $Prim(A) \subseteq Prim_*(A)$, see [25; pp.50-51].

Let A be a *-semisimple Banach *-algebra. The 'usual definition of spectral synthesis' is that A has spectral synthesis if each closed subset of $Prim_*(A)$ is the hull of a unique closed ideal of A. We begin by showing that for a large class of Banach *-algebras, which probably contains all the relevant examples, this 'usual definition' is equivalent to our definition. Our method requires A to be hermitian and *-regular, but spectral synthesis (in either sense) is a very strong property, and it seems unlikely that a Banach *-algebra could have spectral synthesis but fail to be hermitian and *-regular.

Theorem 2.1 *Let A be a hermitian, *-regular, *-semisimple Banach *-algebra. Then A has spectral synthesis in the sense of this paper if and only if A has spectral synthesis in the usual sense for *-semisimple Banach *-algebras.*

Proof. As we noted above, we have $Prim(A) \subseteq Prim_*(A)$, since A is hermitian. We also have that $Prim_*(A) \subseteq Prim^s(A)$ since every $P \in Prim_*(A)$ is prime and semisimple.

Now if A has spectral synthesis in our sense then, by condition (iii), the elements of $Prim(A)$ separate the closed ideals of A. Since $Prim(A) \subseteq Prim_*(A)$, it follows that the elements of $Prim_*(A)$ separate the closed ideals of A, and thus that A has spectral synthesis in the usual sense. Conversely, if A has spectral synthesis in the usual sense then the elements of $Prim_*(A)$ separate the closed ideals of A. Since each element of $Prim_*(A)$ is semisimple, it follows that the primitive ideals of A separate the closed ideals of A, and thus condition (iii) of our version of spectral synthesis holds. It remains to show that conditions (i) and (ii) of our definition of spectral synthesis are automatically satisfied.

Since A is *-regular, $Prim_*(A)$ is locally compact, which implies that $Prim^s(A)$ is locally compact, by Remark (a) after [13; Definition 1.2]. Thus property (i) of spectral synthesis holds.

For property (ii), first observe that if the normality property fails on $Prim(A)$ for an element $a \in A$, then it also fails for a^*a. For suppose that (P_α) is a net in $Prim(A)$ converging to $P \in Prim(A)$, and that $a \notin P$, but that $\lim_\alpha \|a + P_\alpha\| = 0$. Then $a^*a \notin P$, because P is the kernel of a topologically irreducible $*$-representation of A on Hilbert space, but $\lim_\alpha \|a^*a + P_\alpha\| = 0$. Thus it is enough to establish that the normality property holds for self-adjoint elements. Let $a \in A_{sa}$ and let $P \in Prim_*(A)$. Then the quotient norm of $a + P$ in A/P dominates the spectral radius of $a + P$ in A/P. Let C be the completion of A/P in the C*-norm from the corresponding topologically irreducible *-representation of A/P. Then the spectral radius of the canonical image of $a+P$ in C is less than or equal to the spectral radius of $a+P$ in A/P. But the C*-norm of $a+P$ is equal to the spectral radius in C. Thus we have shown that, for self-adjoint elements, the quotient norm dominates the corresponding C*-norm, for each $P \in Prim_*(A)$. On the other hand, the norm functions are lower semicontinuous on $Prim_*(A)$ for the C*-norms, since $Prim_*(A) \cong Prim(A)$, and it is straightforward to show from this that the normality property holds for the quotient norms on $Prim_*(A)$. This establishes (ii). Q.E.D.

If G is a locally compact group G, then $A = L^1(G)$ is a *-semisimple Banach *-algebra, and $C^*(A)$ is the (full) group C*-algebra $C^*(G)$ of G. The class of locally compact groups for which $L^1(G)$ is hermitian and *-regular includes all nilpotent groups and all connected groups of polynomial growth [27], [9]. It also includes all groups in $[\mathrm{FC}]^-$ [17], where a group belongs to $[\mathrm{FC}]^-$ provided that each conjugacy class has compact closure.

For a locally compact abelian group G, $L^1(G)$ has spectral synthesis if and only if G is compact. This classical theorem is chiefly due to Malliavin. In the next section we exhibit a non-compact, non-abelian group G for which $L^1(G)$ has spectral synthesis.

It is not difficult to see that an algebra A in the class of hermitian, *-regular, *-semisimple Banach *-algebras has spectral synthesis if and only if every closed ideal of A is semisimple. Thus from [13; 1.15] we have the following useful extension result.

Proposition 2.2 *Let A be a hermitian, *-regular, *-semisimple Banach *-algebra, and let $J \in Id(A)$. If both J and A/J have spectral synthesis then A has spectral synthesis.*

Proof. It follows from the remarks after [13; 1.15] that every closed ideal of A is semisimple. Hence A has spectral synthesis, as we have just observed. Q.E.D.

We are interested in showing that spectral synthesis is equivalent to the Hausdorffness of τ_∞ on $Id(A)$. The next result establishes one direction of this, for a particular class of Banach *-algebras.

Theorem 2.3 *Let A be a *-semisimple Banach *-algebra, and suppose that $Prim(C^*(A))$ is Hausdorff. If τ_∞ is Hausdorff on $Id(A)$ then A has spectral synthesis (in the sense of this paper).*

Proof. By [13; 1.11] it is enough to show that every proper, closed, prime ideal of A belongs to $Prim_*(A) \subseteq Prim^s(A)$. Thus let P be a proper, closed, prime ideal of A. Since $\bigcap\{Q : Q \in Prim_*(A)\} = \{0\}$, $Prim_*(A)$ is τ_w-dense in $Prime(A)$. Thus there is a net (P_α) in $Prim_*(A)$ converging to P (τ_w). Let (q_α) be the corresponding net of quotient norms in the compact space $S_1(A)$. By passing to a subnet, if necessary, we may assume that (q_α) converges to some seminorm q, say, in $S_1(A)$. Set $Q = \ker q$. Then $P_\alpha \to Q$ (τ_∞), and the normality property for τ_w [33; 2.5] implies that $Q \subseteq P$. Set $B = C^*(A)$. For each α, let $\widetilde{P}_\alpha \in Prim(B)$ such that $\widetilde{P}_\alpha \cap B = P_\alpha$. By the τ_∞-compactness of $Id(B)$ we may assume that the net $(\widetilde{P}_\alpha)$ is τ_∞-convergent in $Id(B)$, with limit $\widetilde{R}$ say. Since $Prim(B)$ is Hausdorff, the set $Prim(B) \cup \{B\}$ is τ_∞-closed in $Id(B)$, by [6; Proposition 8] and [2; 3.3(b), 4.3(b)]. Hence either $\widetilde{R} = B$, or $\widetilde{R} \in Prim(B)$. But since $\widetilde{P}_\alpha \cap A = P_\alpha \to \widetilde{R} \cap A$ (τ_∞) (the restrictions to A of the quotient C*-seminorms converge), we have that $\widetilde{R} \cap A = Q$, by the Hausdorffness of τ_∞ on $Id(A)$. Hence $\widetilde{R} \neq B$ so $\widetilde{R} \in Prim(B)$. Thus $P \in Prim_*(A)$, as required. Q.E.D.

It was shown in [20] that if G is an $[FC]^-$-group then $Prim(C^*(G))$ is Hausdorff.

Now we work in the other direction, trying to show that spectral synthesis implies that τ_∞ is Hausdorff. The following definition is taken from [13].

Let A be a Banach algebra, and suppose that there is a continuous norm γ on A such that B, the γ-completion of A, is a C*-algebra (in this paper A will be a Banach *-algebra with $B = C^*(A)$, so γ will be the maximal C*-seminorm on A). Extending the definition from the Haagerup tensor product of C*-algebras [1; §6], we refer to those closed ideals in A of the form $J \cap A$ ($J \in Id(B)$) as *upper ideals*. The set of upper ideals is denoted $Id^u(A)$. Note that if I is an upper ideal of A then in fact $I = J \cap A$ where J is the closure of I in B.

Definition [13; 2.4] Let A be a Banach algebra. We shall say that A has *property (P)* if A satisfies the following conditions:

(a) there is a continuous norm γ on A such that B, the γ-completion of A, is a C*-algebra;

(b) every primitive ideal of A is an upper ideal, i.e. $Prim(A) \subseteq Id^u(A)$;

(c) there is a subset R of $A \cap B_{sa}$ such that each $a \in R$ is contained in a completely regular, commutative Banach *-subalgebra A_a of A (where the norm and the involution on A_a are those induced by B), and such that if $I \in Id^u(A)$ and $J \in Id(A)$ with $J \not\subseteq I$ then there exists $a \in R$ such that $A_a \cap J \not\subseteq I$.

Recall that a *-semisimple Banach *-algebra A is said to be *locally regular* if there is a dense subset R of A_{sa} such that each $a \in R$ generates a completely regular commutative Banach *-subalgebra A_a [5; §4]. A locally regular, *-semisimple Banach *-algebra is automatically *-regular [5; Theorem 4.3]. It is shown in [5; Theorem 4.1] that if a locally compact group G has polynomial growth then $L^1(G)$ is locally regular. Furthermore, if ω is a polynomial weight and either G is compactly generated and of polynomial growth [5], or $G \in$[SIN] [32], then the Beurling algebra $L^1(G, \omega)$ is also locally regular.

Theorem 2.4 *Let A be a hermitian, locally regular, *-semisimple Banach *-algebra. Then A has property (P). Hence every upper ideal of $Id(A)$ is τ_∞-closed in $Id(A)$, and if A has spectral synthesis then τ_∞ is Hausdorff on $Id(A)$. If $Prim(C^*(A))$ is Hausdorff then $Id(A)$ is τ_∞-Hausdorff if and only if A has spectral synthesis.*

Proof. First we show that A has property (P). Condition (a) is immediate (see [10; p.223] if need be). Condition (b) follows from the fact that A is hermitian, see [25; pp.50-51]. It is shown in [17; Lemma 1.2] that if A is a hermitian, locally regular, *-semisimple Banach *-algebra then for each closed subset E of $Prim(A)$ there is a smallest ideal $J(E)$ of A with hull equal to E. Furthermore, $J(E)$ is generated by elements from the set $\bigcup\{A_a : a \in R\}$. Let I be an upper ideal of A and let J be a closed ideal of A not contained in I. Set $E = \{P \in Prim(A) : P \supseteq I\}$ and $F = \{P \in Prim(A) : P \supseteq J\}$. Then $E \not\subseteq F$, so [17;

Lemma 1.2] shows that $J(F) \not\subseteq I$. Hence there exists $a \in R$ such that $A_a \cap J(F) \not\subseteq I$. Since $J(F) \subseteq J$, we have established that condition (c) holds.

It now follows from [13; Proposition 2.6] that every upper ideal of $Id(A)$ is τ_∞-closed in $Id(A)$, and if A has spectral synthesis then τ_∞ is Hausdorff on $Id(A)$. If, furthermore, $Prim(C^*(A))$ is Hausdorff then it follows from [13; Proposition 2.6] and Theorem 2.3. that $Id(A)$ is τ_∞-Hausdorff if and only if A has spectral synthesis. Q.E.D.

Corollary 2.5 *Let G be an [FC]$^-$-group. Then τ_∞ is Hausdorff on $Id(L^1(G))$ if and only if $L^1(G)$ has spectral synthesis.*

Proof. $L^1(G)$ is hermitian and locally regular, and $Prim(C^*(G))$ is Hausdorff [20], so the result follows from Theorem 2.4. Q.E.D.

In the same way it also follows from Theorem 2.4 that if G is a connected group of polynomial growth then every upper ideal of $L^1(G)$ is τ_∞-closed in $Id(L^1(G))$.

It is unknown, for the class of [FC]$^-$-groups, whether spectral synthesis for $L^1(G)$ is equivalent to the compactness of G. Indeed very little is known about spectral synthesis for [FC]$^-$-groups. It is not even known whether singletons in the primitive ideal space are sets of synthesis.

3. $\mathbf{L^1}$-group algebras and Fourier algebras

In this section we employ group-theoretic techniques, concentrating mainly on L^1-group algebras. We begin with an example of a non-compact, non-abelian group G for which $L^1(G)$ has spectral synthesis. Next we show that if G is a finite extension of an abelian group then the topology τ_r is Hausdorff on $Id(L^1(G))$ if and only if $L^1(G)$ has spectral synthesis, which occurs if and only if G is compact. General results allow one to apply this to the classes of nilpotent groups, [FD]$^-$-groups, and Moore groups (where [FD]$^-$ denotes the class of locally compact groups for which the commutator subgroup has compact closure, and a locally compact group G is a Moore group if every irreducible unitary representation of G is finite dimensional). Finally we show that if G is a non-discrete group then τ_r fails to be Hausdorff on the ideal space of the Fourier algebra $A(G)$.

Example 3.1 *A non-compact group with spectral synthesis.* Let p be a prime and let N be the field of p-adic numbers. Let K be the subset of elements of N of valuation 1. Then K is a compact subgroup under multiplication. Let $G = K \propto N$, where K acts on the additive group N by multiplication. The group G is often referred to as Fell's non-compact group with countable dual.

The dual group $\widehat{N}$ is canonically isomorphic to N. More precisely, there is a character χ of N such that $\chi(x) = 1$ if and only if x is a p-adic integer, and then the mapping $y \mapsto \chi_y$, where $\chi_y(x) = \chi(xy)$, is a topological isomorphism between N and $\widehat{N}$. The irreducible representations of G are easy to determine using Mackey's theory. For $y \in N_j$, the K-orbit of χ_y is equal to $\{\chi_t : t \in N_j\}$. Let $\pi_j = \mathrm{ind}_N^G \chi_y$, for $y \in N_j$. Then $\widehat{G} = \widehat{K} \cup \{\pi_j : j \in \mathbf{Z}\}$. The topology of $\hat{G}$ has been described in [4; 4.6]. Both $\widehat{K}$ and $\widehat{G}\backslash\widehat{K}$ are discrete, and a sequence $(\pi_{j_k})_k$ converges to some (and hence all) $\sigma \in \widehat{K}$ if and only if $j_k \to -\infty$.

Now let E be a closed subset of $\widehat{G}$, and let $J_E = \{j \in \mathbf{Z} : \pi_j \in E\}$ and

$$F = \{0\} \cup \bigcup_{j \in J_E} \{\chi_y : y \in N_j\} \subseteq \widehat{N}.$$

Then F is closed and G-invariant, and since the N_j are open and closed in $\widehat{N}$, the boundary of F is contained in the singleton $\{0\}$. Thus F is a spectral set for $L^1(N)$. The projection theorem for spectral sets [18] shows (in the notation of [18]) that

$$h(e_N(k(F))) = \widehat{K} \cup E$$

is a spectral set for $L^1(G)$. Since $\widehat{K}$ is discrete, E is open (and closed) in $\widehat{K} \cup E$, so since $L^1(G)$ has the Wiener property [27], it follows that E is a spectral set for $L^1(G)$, see [17; Remark 1.3].

This example is hermitian by [26] and *-regular, so the two possible versions of spectral synthesis coincide, by Theorem 2.1. The group G also has polynomial growth, so $L^1(G)$ is locally regular. Hence the topologies τ_∞ and τ_r are Hausdorff on $Id(L^1(G))$, by Theorem 2.4 and [34; 3.1.1].

Another strategy for showing that $L^1(G)$ above has spectral synthesis might be to proceed as follows. Let J be the ideal of $A = L^1(G)$ given by $J = \ker \widehat{K}$. Then J is a semisimple Banach algebra with discrete primitive ideal space. If it could be shown directly that J has spectral synthesis then since $A/J \cong L^1(K)$ also has spectral synthesis (because K is compact), it would follow from Proposition 2.2 that A has spectral synthesis.

We now go on to show that the failure of spectral synthesis implies that the topology τ_r is non-Hausdorff, at least for a large class of groups. We are grateful to Colin Graham for informing us about the following proposition, which he can prove by tensor methods. Here we provide an alternative proof, in keeping with the methods of this paper.

Proposition 3.2 *Let G be a non-compact locally compact abelian group. Then every non-empty open subset of $\widehat{G}$ contains a closed subset which is non-spectral.*

Proof. Notice first that for $\alpha \in \widehat{G}$ and $I \in Id(L^1(G))$, the mapping $f \mapsto \alpha f$ is an automorphism of $L^1(G)$ and $h(\alpha I) = \alpha^{-1}h(I)$. It is enough therefore to show that every open neighbourhood V of 1 contains a closed non-spectral set.

Since the topology on $\widehat{G}$ is the topology of uniform convergence on compact subsets of G, we can assume that V is of the form

$$V = \{\alpha \in \widehat{G} : |\alpha(x) - 1| < \epsilon \text{ for all } x \in C\},$$

where $\epsilon > 0$ and C is a compact subset of G. Let H be any compactly-generated (open) subgroup of G containing C, and let $\phi : \widehat{G} \to \widehat{H}$ denote the restriction map $\alpha \mapsto \alpha|_H$. Then

$$V = \phi^{-1}\left(\{\gamma \in \widehat{H} : |\gamma(x) - 1| < 1 \text{ for all } x \in C\}\right).$$

In particular, when H is compact, $\widehat{G/H}$ is open in $\widehat{G}$ and $\widehat{G/H} \subseteq V$. Since, in this case, G/H is non-compact, $\widehat{G/H}$ has a closed subset which is non-spectral for $L^1(G/H)$, and hence is non-spectral for $L^1(G)$ by the injection theorem for spectral sets. Thus we can henceforth assume that H is non-compact.

By the projection theorem for spectral sets, for any closed subset F of $\widehat{H}$, F is a spectral set for $L^1(H)$ if and only if $\phi^{-1}(F)$ is a spectral set for $L^1(G)$. Therefore it suffices to treat the case when G is non-compact and compactly generated. By the structure theorem,

$$G = \mathbf{R}^m \times \mathbf{Z}^n \times K,$$

where K is a compact group and $m+n \geq 1$. When $m \geq 1$, pass to $H = G/\mathbf{R}^{m-1}\times\mathbf{Z}^n\times K = \mathbf{R}$, and when $m = 0$ pass to $H = G/\mathbf{Z}^{n-1} \times K = \mathbf{Z}$. Suppose that we know that every neighbourhood of 1 in $\widehat{H}$ contains a closed non-spectral set. Then applying the injection theorem for spectral sets once more, the same follows for $\widehat{G}$. Thus we are reduced to the two cases $G = \mathbf{R}$ and $G = \mathbf{Z}$.

Consider $G = \mathbf{R}$ first. It is well-known that $\widehat{\mathbf{R}} = \mathbf{R}$ contains a compact non-spectral set, although we have not been able to find a specific reference for this fact (in fact such a set can be formed as a finite union of translates by integers of a suitable compact subset E of $[0, 1]$, where E is such that the set $\{\exp(2\pi is) : s \in E\}$ is a non-spectral set for $L^1(\mathbf{Z})$). Thus it is enough to show that if $E \subseteq \mathbf{R}$ is non-spectral then so is sE for every $s > 0$. Now it is easily verified that the mapping $\theta : L^1(\mathbf{R}) \to L^1(\mathbf{R})$ given by

$$(\theta f)(x) = \frac{1}{s} f\left(\frac{x}{s}\right), \quad x \in \mathbf{R},$$

is an (isometric) isomorphism of $L^1(\mathbf{R})$ and satisfies

$$\widehat{\theta f}(y) = \widehat{f}(sy), \quad y \in \mathbf{R}.$$

It follows that sE is non-spectral whenever E is.

Finally, let $G = \mathbf{Z}$. Identifying $\widehat{\mathbf{R}}$ with $\mathbf{R}$ and $\widehat{\mathbf{Z}}$ with $\mathbf{T}$, the restriction map $p : \widehat{\mathbf{R}} \to \widehat{\mathbf{Z}}$ is given by $p(y) = e^{2\pi i y}$. Now if V is any neighbourhood of 1 in $\mathbf{T}$, choose $0 < \delta < \frac{1}{2}$ such that $p((-\delta, \delta)) \subseteq V$. Let E be a closed subset of $\mathbf{R}$ such that E is non-spectral for $L^1(\mathbf{R})$ and $E \subseteq (-\delta, \delta)$, and let $F = p(E) \subseteq V$. Then E is open (and closed) in $p^{-1}(F) = \bigcup_{m \in \mathbf{Z}} (m + E)$. Since a clopen subset of a spectral set is itself spectral, it follows that $p^{-1}(F)$ is not spectral. Hence the projection theorem implies that F is non-spectral. Q.E.D.

In order to exploit the existence of non-spectral sets, we need some information about quotient norms. The following definition is useful.

Let A be a completely regular, natural Banach function algebra on its maximal ideal space $Max(A)$. Recall that a Gelfand compact subset X of $Max(A)$ is a *Helson set* if $A|_X = C(X)$ (where $C(X)$ is the algebra of continuous complex functions on X). Letting I be the closed ideal consisting of elements of A which vanish on X, the least constant k such that

$$k \sup\{|f(x)| : x \in X\} \geq \|f + I\| \quad \text{for all } f \in A$$

is called the *Helson constant* of X. We say that a Banach function algebra A has the *Helson property* (with constant K) if there is a constant K such that whenever U is a non-empty Gelfand open subset of $Max(A)$ there is an increasing net $(F_\alpha)_\alpha$ of Helson sets of constant bounded by K contained in U such that $\bigcup_\alpha F_\alpha$ is Gelfand dense in U.

A subset E of an abelian group G is said to be an *independent set* [31; 5.1.1] if it has the following property (following [31] we use additive notation): for every choice of distinct points $x_1, \ldots, x_k$ of E and integers $n_1, \ldots, n_k$, either

$$n_1 x_1 = n_2 x_2 = \ldots = n_k x_k = 0$$

or

$$n_1 x_1 + n_2 x_2 + \ldots + n_k x_k \neq 0.$$

The importance of this definition is that if G is a locally compact abelian group and $F \subseteq \widehat{G}$ is a finite independent set then F is a Helson set with Helson constant bounded by 2 [31; 5.6.7].

A locally compact abelian group is an *I-group* [31; 2.5.5] if every neighbourhood of the identity contains an element of infinite order. A simple argument shows that this implies that every non-empty open subset contains an element of infinite order—indeed the set of elements of finite order is meagre. The next lemma is similar to [31; 5.2.3].

Lemma 3.3 *Let N be a locally compact, second countable abelian group and suppose either that N is an I-group or that N is an uncountable group in which every non-trivial* *element has order q with q prime. Let U be a non-empty open subset of N. Then there exists an increasing net (F_α) of finite independent sets contained in U, such that $\bigcup_\alpha F_\alpha$ is dense in U.*

Proof. Suppose first that N is an I-group. Let U be a non-empty open subset of N and let (X_i) be a base of non-empty open sets for the topology on U. Choose $x_1 \in X_1$ of infinite order. Suppose, for an inductive hypothesis, that we have chosen an independent set $\{x_1, \ldots, x_k\}$ with each $x_i \in X_i$ and of infinite order. Let S be the countable set $S = \{m_1x_1 + \ldots + m_kx_k : m_i \in \mathbf{Z}\}$. Let $s \in S$ and let $m \in \mathbf{Z} \setminus \{0\}$. Let $sol_m(s) = \{x \in X_{k+1} : mx = s\}$. Then either $sol_m(s)$ is empty (hence meagre) or else there exists $y \in sol_m(s)$. In this case

$$\begin{aligned} sol_m(s) &= \{z \in X_{k+1} : m(y-z) = 0\} \\ &= \{x \in N : y + x \in X_{k+1},\ mx = 0\}. \end{aligned}$$

But $\{x \in N : mx = 0\}$ is a closed set, with no interior since N is an I-group, so once again $sol_m(s)$ is meagre. Thus $\bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus \{0\}\}$ is also meagre, so

$$\bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus \{0\}\} \cap X_{k+1}$$

is meagre in X_{k+1}. Since the set of elements of finite order is also meagre, there exists $x_{k+1} \in X_{k+1} \setminus \bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus \{0\}\}$ with x_{k+1} of infinite order. It is straightforward to check that $\{x_1, \ldots, x_k, x_{k+1}\}$ satisfies the inductive hypothesis. Evidently $\bigcup_{i=1}^{\infty}\{x_i\}$ is dense in U.

Now suppose that N is uncountable and that every non-trivial element of N has order q with q prime. As before, let U be an open subset of N and let $(X_i)_{i\geq 1}$ be a base of non-empty open sets for the topology on U. Choose $0 \neq x_1 \in X_1$. Suppose, for an inductive hypothesis, that we have chosen an independent set $\{x_1, \ldots, x_k\}$ with each $0 \neq x_i \in X_i$. Let S be the countable set $S = \{m_1x_1 + \ldots + m_kx_k : m_i \in \mathbf{Z}\}$. Let $s \in S$ and let $m \in \mathbf{Z} \setminus q\mathbf{Z}$. Let $sol_m(s) = \{x \in X_{k+1} : mx = s\}$. Then either $sol_m(s)$ is empty or else there exists $y \in sol_m(s)$. In this case

$$sol_m(s) = \{z \in X_{k+1} : m(y-z) = 0\} = \{y\}$$

since q does not divide m. Thus $\bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus q\mathbf{Z}\}$ is finite or countably infinite, so

$$X_{k+1} \setminus \bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus q\mathbf{Z}\}$$

is uncountable. Let $0 \neq x_{k+1} \in X_{k+1} \setminus \bigcup\{sol_m(s) : s \in S, m \in \mathbf{Z} \setminus q\mathbf{Z}\}$. It is straightforward to check that $\{x_1, \ldots, x_k, x_{k+1}\}$ satisfies the inductive hypothesis. Evidently $\bigcup_{i=1}^{\infty}\{x_i\}$ is dense in U. Q.E.D.

It follows from Lemma 3.3 that if N is a locally compact abelian group such that $\widehat{N}$ is either a second countable I-group or a second countable, uncountable group in which every non-trivial element has order q with q prime then $L^1(N)$ has the Helson property (with constant 2). In the case of the second countable I-groups, the independent set F constructed in Lemma 3.3 consists of elements of infinite order. This implies that F is actually a Helson set with Helson constant 1 [31; 5.1.3 Corollary, and 5.5.2].

For the next lemma, let D_q be the compact group obtained as the direct product of countably many copies of the cyclic group of order q, where q is an integer, $q \geq 2$.

Lemma 3.4 *Let G be a locally compact group with an abelian, non-compact, closed, normal subgroup N of finite index m. Then G has a quotient G' which has a non-compact closed normal subgroup N' of finite index, such that $\widehat{N'}$ is either a second countable I-group or a second countable, uncountable group in which every non-trivial element has order q where q is a prime.*

Proof. We begin by reducing to the case where G is discrete. Since G has an abelian subgroup of finite index, G is a projective limit of Lie groups. Thus, passing to a quotient modulo some compact normal subgroup, we may assume that G is a Lie group. Let N_0 denote the connected component of N containing the identity. Then N_0 is open in G.

Suppose first that N/N_0 is finite. Then, by the structure theory of locally compact, abelian groups, N is isomorphic to $\mathbf{R}^n \times M$ where M is a compact group and $n \geq 1$ (since N is non-compact). Then M is normal in G, and $\widehat{N/M}$ is isomorphic to $\mathbf{R}^n$, which is a second countable I-group. Thus we are left with the case where N/N_0 is infinite, and so, passing to G/N_0 we may assume that G is discrete.

By [31; 2.5.5] there is a closed subgroup H of N such that either $\widehat{N/H}$ is a second countable I-group or such that $\widehat{N/H}$ is isomorphic to D_q for some prime q. However H may not be normal in G, so set $\widetilde{H} = \bigcap_{x \in G} x^{-1}Hx$. Then $\widetilde{H}$ is normal in G. Choose coset representatives $x_1, \ldots, x_m$ for G/N, and for $1 \leq i \leq m$ set $H_i = x_i^{-1}Hx_i$. Then the homomorphism

$$\widehat{N/H_1} \times \cdots \times \widehat{N/H_m} \to \widehat{N/\widetilde{H}}$$

from the product of the subgroups $\widehat{N/H_i}$ of $\widehat{N/\widetilde{H}}$ given by $(\alpha_1, \ldots, \alpha_m) \mapsto \alpha_1 \ldots \alpha_m$ is continuous and has dense range in $\widehat{N/\widetilde{H}}$, since this range separates the points in $N/\widetilde{H}$. Since N is discrete, all of the $\widehat{N/H_i}$ are compact and hence the above homomorphism is surjective. Thus $\widehat{N/\widetilde{H}}$ is a quotient of the product $\widehat{N/H_1} \times \cdots \times \widehat{N/H_m}$. Note also that $\widehat{N/\widetilde{H}}$ is compact and infinite, so is certainly uncountable.

Now if $\widehat{N/H}$ is isomorphic to D_q, then every $\widehat{N/H_i}$ is isomorphic to D_q and this implies that $\widehat{N/\widetilde{H}}$ is second countable, and that every non-trivial element of $\widehat{N/\widetilde{H}}$ has order q.

Finally, if $\widehat{N/H}$ is a second countable I-group, then $\widehat{N/H_1} \times \cdots \times \widehat{N/H_m}$ is also a second countable I-group. Clearly, then, $\widehat{N/\widetilde{H}}$ is second countable. Since the homomorphism above is continuous and each $\widehat{N/H_i}$ is an I-group which is a subgroup of $\widehat{N/\widetilde{H}}$, the latter is an I-group as well. Q.E.D.

We now introduce some further notation. For a locally compact group G and a closed subset X of $\widehat{G}$, let $K(X)$ denote the kernel of X in $L^1(G)$. Then $K(X)$ is the largest ideal of $L^1(G)$ with hull equal to X. Ludwig has shown that if G has polynomial growth and $L^1(G)$ is hermitian then there is a smallest closed ideal $j(X)$ whose hull is equal to X [28].

Now let G be a locally compact group and suppose that G has an abelian, closed, normal subgroup N of finite index. Let G act on $\widehat{N}$ in the usual way by $(x, \alpha) \mapsto x \cdot \alpha$ $(x \in G, \alpha \in \widehat{N})$, where $x \cdot \alpha(n) = \alpha(x^{-1}nx)$ $(n \in N)$. Let $\widehat{N}/G$ be the set of G-orbits in $\widehat{N}$, with the quotient topology, and let $q : \widehat{N} \to \widehat{N}/G$ be the quotient map. Let $\phi : \widehat{G} \to \widehat{N}/G$ be the map defined $\phi(\pi) = G \cdot \alpha$, where $\alpha \in \widehat{N}$ and $\ker(\alpha) \supseteq \ker(\pi|_N)$. Then q and ϕ are both continuous and both open. For a set $C \subseteq \widehat{N}$, let $\widetilde{C} = \phi^{-1}(q(W))$.

Lemma 3.5 *Let G be a locally compact group and suppose that G has an abelian, closed, normal subgroup N of finite index. Suppose that X is a G-invariant compact subset of $\widehat{N}$ and that Y is a G-invariant closed subset of $\widehat{N}$ such that X is contained in the interior of Y and such that the complement ot Y in $\widehat{N}$ is relatively compact. Then $K(\widetilde{Y}) \subseteq j(\widetilde{X})$.*

Proof. Let $f \in K(\widetilde{Y})$. By [28] it is enough to find $g \in K(\widetilde{X})$ such that $gf = f$. Let $h \in L^1(N)$ such that $h \in K(X)$ and $\hat{h}$ takes the constant value 1 on the closure of the complement of Y. Such a function exists because $L^1(N)$ is completely regular and the complement of Y is relatively compact. Since N is an open subset of G we may extend h to an element $g \in L^1(G)$ by setting $h(x) = 0$ for $x \in G \backslash N$. The element g has the required property. Thus $f \in j(\widetilde{X})$. Q.E.D.

Part of the argument of the next theorem adapts a method used in [12; 1.2].

Theorem 3.6 *Let G be a locally compact group and suppose that G has an abelian, non-compact, closed, normal subgroup N of finite index m. Then $L^1(G)$ does not have spectral synthesis, and τ_r is not Hausdorff on $Id(L^1(G))$.*

Proof. The properties of spectral synthesis and of τ_r being Hausdorff both pass to quotients [34; 2.9]. Thus by Lemma 3.4 we may suppose that the dual $\widehat{N}$ of the abelian subgroup N is either a second countable I-group or a second countable, uncountable group in which every non-trivial element has order q, with q prime. (This reduction step is not required for the proof that $L^1(G)$ does not have spectral synthesis).

For $\alpha \in \widehat{N}$, let G_α denote its stabilizer in G. The lengths of orbits of elements of $\widehat{N}$ are bounded by m. Let d be the maximal orbit length, and set $\widehat{N}_d = \{\alpha \in \widehat{N} : |G : G_\alpha| = d\}$. Then $\widehat{N}_d$ is non-empty and open. Let $\mathcal{H}$ be the set of subgroups of G of index d. Then $\widehat{N}_d = \bigcup_{H\in\mathcal{H}} \widehat{N}_H$, where the union is disjoint, and $\widehat{N}_H := \{\alpha \in \widehat{N} : G_\alpha = H\}$.

Fix $H \in \mathcal{H}$. Then there exists a non-empty open set $W \subseteq \widehat{N}_H$ such that the sets $x \cdot W$ $(x \in G/N)$ are pairwise disjoint. By Proposition 3.2, W contains a non-spectral set F for $L^1(N)$. Set $X = G \cdot F = \{x \cdot F : x \in G/N\}$. Then X is G-invariant, and is also non-spectral since it has F as a non-spectral non-empty clopen subset. Hence $\widetilde{X}$ is also non-spectral by the projection theorem [18; 2.6], which we may use because G has polynomial growth and $L^1(G)$ is hermitian. This shows that $L^1(G)$ does not have spectral synthesis.

Now let $(V_\alpha)_\alpha$ be a net of G-invariant, decreasing, open neighbourhoods of X, each having compact complement in $\widehat{N}$, such that $\bigcap_\alpha N_\alpha = X$ (where for each α, N_α is the closure of V_α). Then $(\widetilde{V}_\alpha)_\alpha$ is a net of decreasing open subsets of $\widehat{G}$, and $\bigcap_\alpha \widetilde{N}_\alpha = \widetilde{X}$. Hence $(K(\widetilde{N}_\alpha))_\alpha$ is an increasing net in $Id(L^1(G))$, and $K(\widetilde{N}_\alpha) \subseteq j(\widetilde{X})$ for all α by Lemma 3.5. Hence

$$I := \overline{\bigcup_\alpha K(\widetilde{N}_\alpha)} \subseteq j(\widetilde{X}).$$

By Lemma 3.3 there is for each α an increasing net $(H_{\beta(\alpha)})_{\beta(\alpha)}$ of Helson sets in V_α, each of Helson constant bounded by 2, such that $\bigcup_{\beta(\alpha)} H_{\beta(\alpha)}$ is dense in V_α. Then $F_{\beta(\alpha)} := G \cdot H_{\beta(\alpha)}$ is a G-invariant Helson set, with a bound K dependent on m, but independent of α, and of course $\bigcup_{\beta(\alpha)} F_{\beta(\alpha)}$ is also dense in V_α. Hence $(K(\widetilde{F_{\beta(\alpha)}}))_{\beta(\alpha)}$ is a decreasing net in $Id(L^1(G))$, and $\bigcap_{\beta(\alpha)} K(\widetilde{F_{\beta(\alpha)}}) = K(\widetilde{N}_\alpha)$. Hence $K(\widetilde{F_{\beta(\alpha)}}) \xrightarrow{\beta(\alpha)} K(\widetilde{N}_\alpha)$ (τ_r) by Lemma 1.1. But $K(\widetilde{N}_\alpha) \to I$ (τ_r), also by Lemma 1.1, so if $(K(\widetilde{F_\gamma}))_\gamma$ denotes the 'diagonal' net, see [24; §2, Theorem 4], then $K(\widetilde{F_\gamma}) \to I$ (τ_r).

Now we show that $K(\widetilde{F}_\gamma) \to K(\widetilde{X})$ (τ_r). First we establish convergence for τ_n. Suppose that $f \notin K(\widetilde{X})$. Then there exists $P \in \widetilde{X}$ such that $f \notin P$. For each $Q \in Prim_*(L^1(G))$, let Q_* be the primitive ideal of $C^*(G)$ whose intersection with $L^1(G)$ is equal to Q. Set $\epsilon = \frac{1}{2}\|f + P_*\|_*$, where $\|\cdot\|_*$ denotes the C*-norm on $C^*(G)$. By the lower semicontinuity of norm functions for C*-algebras, there is a hull-kernel neighbourhood M_* of P_* in $Prim(C^*(G))$ such that $\|f + Q_*\|_* > \epsilon$ for all $Q_* \in M_*$. Let $M = \{Q \in Prim_*(L^1(G)) : Q_* \in M_*\}$. Then M is an open neighbourhood of P in $Prim_*(L^1(G))$, because G is *-regular, so eventually there is for each γ and element $Q_\gamma \in \widetilde{F}_\gamma \cap M$. Hence eventually

$$\|f + K(\widetilde{F}_\gamma)\| \geq \|f + Q_\gamma\| \geq \|f + (Q_\gamma)_*\|_* > \epsilon.$$

This shows that $K(\widetilde{F}_\gamma) \to K(\widetilde{X})$ (τ_n).

Finally we show that $K(\widetilde{F}_\gamma) \to K(\widetilde{X})$ (τ_u). By [34; 2.1] it is enough to show that for each $f \in K(\widetilde{X})$ and $\epsilon > 0$ there is a neighbourhood $\widetilde{M}$ of $\widetilde{X}$ in $\widehat{G}$ such that $\|f+K(\widetilde{F}_\gamma)\| < \epsilon$

whenever $\widetilde{F_\gamma} \subseteq \widetilde{M}$. Since N is an open subgroup of G there is a projection $P : L^1(G) \to L^1(N)$ given by $P(g) = g|_N$ $(g \in L^1(G))$. Then for each coset representative $x \in G/N$, $P(L_x f) \in L^1(N)$, and in fact $P(L_x f) \in K(F)$. Thus there exists a neighbourhood M_x of F in $\widehat{N}$ such that $|P(L_x f)(s)| < \epsilon/(Km)$ for each $s \in M_x$. Hence by the Helson property, $\|P(L_x f) + K(F_\gamma)\| < \epsilon/m$ whenever $F_\gamma \subseteq M_x$. Set $M = \bigcap_{x\in G/N} M_x$. Then whenever $F_\gamma \subseteq M$, there is, for each coset representative $x \in G/N$, an $h_x \in K(F_\gamma)$ such that $\|P(L_x f) - h_x\| < \epsilon/m$. Define $h \in K(\widetilde{F_\gamma})$ by $h(xn) = h_x(n)$ $(n \in N)$. Then

$$
\begin{aligned}
\|f - h\| &= \left\| \sum_{x\in G/N} f1_{xN} - h1_{xN} \right\| \\
&= \sum_{x\in G/N} \|f1_{xN} - h1_{xN}\| \\
&= \sum_{x\in G/N} \|P(L_x f) - h_x\| < \epsilon.
\end{aligned}
$$

Hence $K(\widetilde{F_\gamma}) \to K(\widetilde{X})$ (τ_u), so

$$K(\widetilde{F_\gamma}) \to K(\widetilde{X}) \quad (\tau_r).$$

Since $I \subseteq j(\widetilde{X})$ and $j(\widetilde{X})$ is a strict subset of $K(\widetilde{X})$, we have $I \neq K(\widetilde{X})$, so τ_r is not Hausdorff. Q.E.D.

Note, for the next result, that if a group G has a compact normal subgroup K such that G/K is a finite extension of a nilpotent group then $L^1(G)$ is hermitian and G has polynomial growth [29], and hence $L^1(G)$ is *-regular [5]. Thus Theorem 2.1 applies.

Theorem 3.7 *Let G be a locally compact group. Suppose that G has a compact normal subgroup K such that G/K is a finite extension of a nilpotent group. Then the following are equivalent:*

(i) τ_r is Hausdorff on $Id(L^1(G))$,
(ii) τ_∞ is Hausdorff on $Id(L^1(G))$,
(iii) τ_r and τ_∞ coincide on $Id(L^1(G))$,
(iv) G is compact,
(v) $L^1(G)$ has spectral synthesis.

Proof. (iv)⇒(ii) is proved in [7; p.72]. The equivalence of (ii) and (iii), and hence the implication (ii)⇒(i), is proved in [34; 3.1.1]. The implication (iv)⇒(v) is well-known. It remains to prove that (i)⇒(iv) and that (v)⇒(iv). First we show that (i)⇒(iv).

Suppose that τ_r is Hausdorff on $Id(L^1(G))$, and suppose too, to begin with, that G is a finite extension of an $[\mathrm{FD}]^-$ group, so that G has a closed normal subgroup N of finite index with $N \in$[FD]$^-$. Let K be the closure of the commutator subgroup of N. Then K is compact, and K is normal in G. Hence G/K is a finite extension of the abelian group N/K, and τ_r is Hausdorff on $Id(L^1(G/K))$. Thus G/K is compact by Theorem 3.6, so G is compact.

Now we consider the general case when G has a compact normal subgroup K such that G/K is a finite extension of a nilpotent group. It is enough to show that G/K is compact, and hence we may suppose that G itself has a nilpotent closed normal subgroup N of finite index. We proceed by induction on the length $l(N)$ of nilpotency of N. If $l(N) = 1$ then N is abelian, so it follows from Theorem 3.6 that G is compact. Now suppose that we have established the result for $l(N) = k \geq 1$, and that $l(N) = k + 1$. Let $Z(N)$ be the centre of N. Then $Z(N)$ is normal in G, so $G/Z(N)$ is a finite extension of $N/Z(N)$, and $l(N/Z(N)) = k$, so $G/Z(N)$ is compact. Hence $N/Z(N)$ is compact, so $N \in$[Z], i.e. N is a central group. But [Z]$\subseteq$[FD]$^-$ [16], so we have that G is a finite extension of an [FD]$^-$-group. Hence G is compact by the previous paragraph.

Finally we show that (v)$\Rightarrow$(iv). This follows by exactly the same argument used for (i)$\Rightarrow$(iv), replacing the hypothesis that τ_r is Hausdorff by the hypothesis that $L^1(G)$ has spectral synthesis. This property also passes to quotients, and the appeals to Theorem 3.6 are still valid. Q.E.D.

Theorem 3.7, of course, covers the classes of nilpotent groups and [FD]$^-$-groups. Every Moore group is a finite extension of an [FD]$^-$-group [30], so Theorem 3.7 also covers Moore groups.

For the final result of the paper, we apply Theorem 3.5, in the abelian case, to the situation of Fourier algebras.

Theorem 3.8 *Let G be a non-discrete locally compact group with Fourier algebra $A(G)$. Then τ_r is not Hausdorff on $Id(A(G))$.*

Proof. Suppose that τ_r is Hausdorff on $Id(A(G))$. Then τ_r is Hausdorff on $Id(A(H))$ for every closed subgroup H of G, because $A(H)$ is isomorphic to $A(G)/I(H)$, where $I(H)$ is the ideal consisting of all functions in $A(G)$ which vanish on H [14; Lemma 3.8].

Suppose first that G is totally disconnected. Then because G is not discrete, G has an infinite, compact open subgroup K. By [35; Theorem 2], K has an infinite closed abelian subgroup H. Then τ_r is Hausdorff on $Id(A(H))$, and $A(H) = L^1(\widehat{H})$, so $\widehat{H}$ is compact by Theorem 3.6. Hence H is discrete, contradicting the fact that it is infinite and compact.

It remains therefore to show that if τ_r is Hausdorff on $Id(A(G))$ then G must be totally disconnected. Suppose, for a contradiction, that G_0, the connected component of the identity, is non-trivial. A connected group is a projective limit of Lie groups, so there is a compact normal subgroup K of G_0 such that G_0/K is a non-trivial Lie group. But $A(G_0/K)$ is a quotient of $A(G_0)$. Indeed the mapping $u \mapsto \dot{u}$, where

$$\dot{u}(xK) = \int_K u(xt)dt$$

(with normalized Haar measure on K) is a continuous homomorphism of $A(G_0)$ onto $A(G_0/K)$ [11]. Thus τ_r is Hausdorff on $Id(A(G_0/K))$. On the other hand, G_0/K is a connected Lie group, so it is generated by its one parameter subgroups, i.e. images of analytic homomorphisms of $\mathbf{R}$ into G_0/K. Hence G_0/K contains numerous non-discrete closed abelian subgroups. As in the previous paragraph, this contradicts the fact that τ_r is Hausdorff on $Id(A(F))$ for each closed abelian subgroup F of G_0/K, which forces each such F to be discrete. Q.E.D.

It was shown in [22] that if G is a discrete group then $A(G)$ has spectral synthesis (and hence τ_∞ and τ_r are Hausdorff) provided that $A(G)$ satisfies an additional weak condition. This additional condition is satisfied whenever G is an amenable discrete group, and is probably satisfied for all discrete groups.

We close this section by observing that the Hausdorffness of τ_r is *not* equivalent to spectral synthesis for general Banach *-algebras. The Banach algebra $C^1[0,1]$ is a hermitian, (locally) regular, *-semisimple Banach *-algebra without spectral synthesis, but τ_r is Hausdorff on $Id(C^1[0,1])$ [12].

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