math.FA/9809073 14 Sep 1998

STRONG REGULARITY FOR UNIFORM ALGEBRAS

J. F. Feinstein and D. W. B. Somerset

Abstract. A survey is given of the work on strong regularity for uniform algebras over the last thirty years, and some new results are proved, including the following. Let A be a uniform algebra on a compact space X and let E be the set of all those points $x \in X$ such that A is not strongly regular at x. If E has no non-empty, perfect subsets then A is normal, and X is the character space of A. If X is either $[0,1]$ or the circle T and E is meagre with no non-empty, perfect subsets then A is trivial. These results extend Wilken's work from 1969. It is also shown that every separable Banach function algebra which has character space equal to either $[0,1]$ or T and has a countably-generated ideal lattice is uniformly dense in the algebra of all continuous functions.

The study of strong regularity for uniform algebras was initiated by Wilken in 1969 [Wi2]. He had two main results. The first (a lemma) was that if a uniform algebra A is strongly regular on a space X then X is necessarily the character space of A, and A is normal. The second result was that there is no strongly regular uniform algebra on the unit interval $[0,1]$ other than the trivial one $C[0,1]$ itself. In this paper we survey the subsequent developments of the study, and extend Wilken's results. Various pertinent examples are given.

We begin by introducing the terminology, definitions and notation which we shall need, along with some background results concerning strong regularity.

Terminology and notation. In our terminology a *compact space* is a compact, Hausdorff topological space. For any compact space X we denote the algebra of all continuous, complex-valued functions on X by $C(X)$.

Let A be a commutative, unital Banach algebra. We denote by Φ_A the character space of A.

Definition. Let X be a compact space. A *function algebra* on X is a subalgebra of $C(X)$ which contains the constant functions and separates the points of X. A function algebra A on X is *trivial* if $A = C(X)$. A *Banach function algebra* on X is a function algebra on X with a complete algebra norm. A *uniform algebra* on X is a Banach function algebra on X whose norm is the uniform norm on X.

1991 *Mathematics Subject Classification.* Primary 46J10, secondary 46J20.

Key words and phrases. Functional Analysis, uniform algebras, ideals, strong regularity.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Every commutative, unital, semisimple Banach algebra may be regarded, via the Gelfand transform, as a Banach function algebra on its character space. In this paper we give results for unital Banach algebras. However, our results carry through without much difficulty to the non-unital case (with appropriate definitions) simply by considering the standard unitisation of any algebra under consideration.

Notation. Let A be a Banach function algebra on a compact space X, and let $x \in X$. We denote the evaluation character at x by ϵ_x. We define the ideals J_x, M_x as follows:

$$M_x = \{f \in A : f(x) = 0\} = \ker \epsilon_x;$$

$$J_x = \{f \in A : f^{-1}(\{0\}) \text{ is a neighbourhood of } x\}.$$

Let K be a closed subset of X. Then we define the ideal $I(K)$ by

$$I(K) = \{f \in A : f(K) \subseteq \{0\}\}.$$

It is standard to identify the set of evaluation characters $\{\epsilon_x : x \in X\}$ with X.

Definition. Let A be a Banach function algebra on a compact space X. Then A is *regular on* X if, for each closed set F contained in X and each $x \in X \backslash F$, there is an $f \in A$ with $f(F) \subseteq \{0\}$ and $f(x) = 1$; A is *regular* if it is regular on Φ_A. The algebra A is *normal on* X if, for every pair of disjoint closed sets E, F contained in X, there is an $f \in A$ with $f(E) \subseteq \{0\}$ and $f(F) \subseteq \{1\}$; A is *normal* if it is normal on Φ_A. Now let $x \in X$. Then A is *strongly regular at* x if J_x is dense in M_x. The algebra A is *strongly regular on* X if it is strongly regular at every point of X; A is *strongly regular* if it is strongly regular on Φ_A. Finally, A *has spectral synthesis* if A is regular and if every closed ideal of A is an intersection of maximal ideals.

A simple compactness argument shows that every strongly regular Banach function algebra is regular. It is standard, see e.g. [St, 27.2], that every regular Banach function algebra is normal.

We now come to Wilken's results (mentioned earlier) about strong regularity for uniform algebras. The first is this.

Theorem 1 [Wi2]**.** *Let A be a uniform algebra on a compact space X. Suppose that A is strongly regular on X. Then*
(a) $\Phi_A = X$*;*
(b) A is normal.

The second matter addressed by Wilken was the question of whether there actually are any non-trivial, strongly regular uniform algebras. He was able to show that there are no non-trivial, strongly regular uniform algebras on $[0, 1]$. In 1973 his method was extended by Chalice [Ch] to show that there are no non-trivial, strongly regular uniform algebras on the circle T either. In 1974 some further results in this direction were obtained by Batikjan [Ba] for locally connected, compact spaces X, under additional technical conditions.

Also in [Ch], Chalice gave an example of a non-trivial uniform algebra which was strongly regular at all points of a dense subset of its character space. He raised the

question whether a uniform algebra could be strongly regular at a non-peak point, and this question was resolved affirmatively by Wang in 1975 [Wa]. The general question of the existence of non-trivial, strongly regular uniform algebras remained open, however.

Meanwhile in 1987 Mortini gave an elementary proof of Wilken's result [M]. Although not mentioned there, his method actually applies to all Banach function algebras.

Theorem 2. *Let A be a Banach function algebra on a compact space X, and let K be a closed subset of X such that $I(K)$ is the zero ideal. Suppose that A is strongly regular at every point of K. Then*
(a) $\Phi_A = X = K$;
(b) A is normal.

Proof. Suppose, for contradiction, that there exists a character ϕ in $\Phi_A \backslash K$. For every $x \in K$ we have that $\phi \neq \epsilon_x$, and so M_x is not a subset of $\ker \phi$. Since J_x is dense in M_x, there must be a function f_x in $J_x \setminus \ker \phi$. Set

$$N_x = \{y \in X : f_x(y) = 0\}.$$

Then, by the compactness of K, there are finitely many elements $x_1, x_2, \ldots x_n$ of K such that

$$K \subseteq \bigcup_{k=1}^{n} N_{x_k}.$$

Set $f = f_{x_1} f_{x_2} \cdots f_{x_n}$. Then $\phi(f) \neq 0$. But $f \in I(K)$, and so $f = 0$. This contradiction proves (a), and (b) is now immediate. □

In 1992 the first author answered Wilken's question affirmatively by exhibiting a non-trivial, strongly regular uniform algebra [F1], using Cole's systems of root extensions [Co]. The same method also produces examples of non-trivial, normal uniform algebras which fail the condition of strong regularity at exactly one point. An example where the character space is metrizable can be found in a remark on page 300 of [F1] (see also [F1, Theorem 5.3]). Taking finite direct sums, one can then obtain examples which fail strong regularity at any specified finite number of points. A similar method produces examples which fail strong regularity at countably many points.

Example 3. There exists a normal uniform algebra B on a compact, metrizable space Y such that the set of points of Y at which B is not strongly regular is countably infinite.

Proof. Let A be a normal uniform algebra on a compact metric space X such that there is exactly one point of X at which A is not strongly regular [F1]. Let B be the standard unitisation of the c_0 direct sum of countably many copies of A (so that B is the algebra of all those sequences $(f_n) \subseteq A$ which converge uniformly on X to some constant function). Let Y be the one-point compactification of $X \times \mathbf{N}$, and regard B as a uniform algebra on Y in the obvious way.

It is elementary to check that this example has the required properties. □

It is natural to wonder if there is a connection between strong regularity and Gleason parts. A curious example is given in [F2] of a strongly regular uniform algebra for which every Gleason part except one is a singleton, and the other part has exactly two points.

All these examples are normal, and this brings us to the first main question of the paper. Can Wilken's first result be extended to the case where strong regularity holds except on some small exceptional set? Notice that Mortini's argument does not help with this question, as the following example shows.

Example 4. Let $X = \{0, 1, 1/2, 1/3, \dots\}$, and let A be the restriction to X of the disc algebra. By the identity principle, the restriction map is an isomorphism, and so A is a Banach function algebra on X (where the norm is the uniform norm of the functions on the closed unit disc). It is easy to see that this Banach function algebra is strongly regular at every point of X except for the point 0. But, of course, $\Phi_A \neq X$, and A is not normal.

To extend Wilken's result therefore, we shall have return to his original method for uniform algebras. We shall need the following results. The first is due to Rudin from 1957.

Theorem 5 [R, Theorem 4]**.** *Let X be a compact space which has no non-empty, perfect subsets. Then there are no non-trivial uniform algebras on X.*

The next is more recent. We need a couple of definitions.

Definition. Let X be a compact space, and let A be a Banach function algebra on X. A point $x \in X$ is an *R-point* if $y \in X$ with $y \neq x$ implies that $M_y \not\supseteq J_x$. Evidently x is an R-point whenever A is strongly regular at x. A simple compactness argument shows that A is regular on X if and only if every point of X is an R-point.

The algebra A is *2-local on X* if the following condition holds: if $f \in C(X)$ is such that there are elements g_1 and g_2 in A so that every point of X has neighbourhood on which f agrees with either g_1 or g_2, then f is in fact in A. Every local Banach function algebra is, of course, 2-local on its character space. Thus every normal Banach function algebra is 2-local on its character space. It is shown in [Wi1, 2.3, 3.1] that a uniform algebra whose character space is equal to either $[0, 1]$ or T is 2-local on its character space.

Theorem 6 [FS2]**.** *Let A be a Banach function algebra on a compact space X, and let $y \in X$. The set $F = \{x \in X : M_y \supseteq J_x\}$ is closed, and is connected if A is 2-local on X. If A is a uniform algebra on X then F is either a singleton or contains a non-empty perfect subset.*

Since points of $F \backslash \{y\}$ are not R-points, it follows that the set of non-R-points has a non-empty, perfect subset if A is a uniform algebra which is not regular on X.

We also recall some standard facts about Jensen measures. Wilken used the existence of these in his proof of Theorem 1.

Definition. Let A be a uniform algebra on a compact space X, and let $\phi \in \Phi_A$. Then a *Jensen* measure for ϕ is a regular, Borel probability measure μ on X such that, for all $f \in A$,

$$\log|\phi(f)| \leq \int_X \log|f(x)| \mathrm{d}\mu(x) \tag{1.1}$$

(where $\log(0)$ is defined to be $-\infty$).

It is standard (see, for example, [G, p.33]) that every $\phi \in \Phi_A$ has a Jensen measure supported on X, and that each such measure *represents* ϕ, i.e., for all $f \in A$,

$$\phi(f) = \int_X f(x) \mathrm{d}\mu(x).$$

We are now ready for our first new result.

Theorem 7. *Let A be a uniform algebra on a compact space X. Set*

$$E = \{x \in X : \overline{J_x} \neq M_x\}.$$

Suppose that E has no non-empty, perfect subsets. Then
(a) $\Phi_A = X$;
(b) A is normal.

Proof. For (a), let $\phi \in \Phi_A$. Let μ be a Jensen measure for ϕ supported on X, and let F be the closed support of μ. Suppose first that $F \backslash E \neq \emptyset$. Let $x \in F \backslash E$. Then it follows from (1.1) that $J_x \subseteq M_\phi$. Since $\overline{J_x} = M_x$, we obtain $M_x \subseteq M_\phi$, and so $\phi = \epsilon_x$. Now suppose instead that $F \subseteq E$. Set $K = F \cup \{\phi\}$. Then K is a compact subset of Φ_A. Since $F \subseteq E$ it follows that F has no non-empty, perfect subsets. The same must also be true for K. Thus, by Theorem 5, $A|K$ is uniformly dense in $C(K)$. On the other hand, $\delta_\phi - \mu$ is a regular Borel measure on K which annihilates A. Thus we must have $\delta_\phi = \mu$. Since μ is supported on X, it follows that $\phi \in X$, as required.

For (b), observe that if A were not normal on X then by Theorem 6 there would be a non-empty, perfect subset of X consisting of points which were not R-points, and a fortiori not points of strong regularity. Thus E would have a non-empty, perfect subset. This contradiction establishes that A is normal. □

The following examples illustrate what can happen once a uniform algebra fails to be strongly regular on a large enough subset.

Examples 8. (a) Let $S = \bigcup_{n=1}^{\infty} \left\{ \left(1 - \frac{1}{n}\right) e^{\frac{2\pi k i}{n}} : 1 \leq k \leq n \right\}$, and let $X = S \cup T$. Let A be the restriction to X of the disc algebra. By the maximum modulus principle, the restriction map is an isometric isomorphism, so A is a uniform algebra on X. Every point of S is isolated, so A is strongly regular at every point of S, which is a dense open subset of X. But, of course, $\Phi_A \neq X$, and A is not normal.

(b) Let A be the uniform algebra of continuous functions on a solid cylinder which are analytic on the base of the cylinder (often called 'the tomato-can algebra').

Strong regularity holds everywhere except the base, hence on a dense open subset of Φ_A, but A is not normal.

(c) Let A be the uniform algebra obtained by restricting H^∞ (the algebra of bounded functions on the disc which are analytic on the open disc) to the fibre of its maximal ideal space associated with a point on the unit circle, see [H, p.187ff]. Then A is regular on its Shilov boundary, but A is not normal. This example shows that the requirement of strong regularity outside the exceptional set in Theorem 7 cannot be relaxed to regularity (i.e. simply requiring that each x outside the exceptional set should be an R-point). In this example the exceptional set of non-R-points is actually empty.

(d) O'Farrell [O] has given an example of a normal uniform algebra $R(X)$, consisting of the closure of the rational functions having poles off the Swiss cheese X, with the property that there are continuous point derivations on a set of positive measure. Since strong regularity fails for normal uniform algebras wherever there is a continuous point derivation, it follows that $R(X)$ fails strong regularity at uncountably many points.

We turn now to consider the second part of Wilken's work on strong regularity for uniform algebras, namely that every strongly regular uniform algebra on $[0,1]$ is trivial. The question of whether there are any non-trivial uniform algebras with character space equal to $[0,1]$ seems still to be open. With Theorem 6, however, we are able to push Wilken's method a little further.

Definition. A closed set $E \subseteq X$ is a *peak set* for A if there is a function f in A such that f is constantly 1 on E, but such that $|f(t)| < 1$ for all $t \in X \backslash E$. A *peak point* for A is a point $x \in X$ such that the set $\{x\}$ is a peak set for A.

Now let A be a uniform algebra on a compact space X. It is standard that when X is metrizable (which occurs if and only if A is separable) the set of peak points is a dense, G_δ in the Shilov boundary of A, see II.11.2 and II.12.10 of [G]. Furthermore countable intersections and finite unions of peak sets are again peak sets for A, see Section II.12 of [G]. In particular the union of two peak points is a peak set.

For the next lemma, let us say that an ideal I in a Banach algebra *factors* if, for all f in I, there are g, h in I such that $f = gh$. By Cohen's factorization theorem, see [P, 5.2.2], I factors whenever I has a bounded approximate identity. If x is a peak point for a uniform algebra then M_x has a bounded approximate identity, and hence factors.

Lemma 9. *Let A be a Banach function algebra on a compact space X. Let F be the set of all those x in X such that M_x factors and A is strongly regular at x. Suppose that $x \in F$, and that $y \in X$ with A strongly regular at y. Then $J_x \cap J_y$ is dense in $M_x \cap M_y$.*

Proof. Let $f \in M_x \cap M_y$. Because M_x factors we can write $f = gh$ where $g, h \in M_x$. Since $f \in M_y$, at least one (h say) of g and h belongs to M_y. Choose sequences $(g_n) \subseteq J_x$ and $(h_n) \subseteq J_y$ converging to g and h respectively. Then $(g_n h_n) \subseteq J_x \cap J_y$ and this sequence converges to f. The result follows. $\square$

Proposition 10. *Let X be $[0,1]$ or T, and let A be a uniform algebra on X. Suppose that A is 2-local on X and that X has a dense subset consisting of peak points at which A is strongly regular. Then A is trivial.*

Proof. The proofs for $[0,1]$ and for T are subtly different, due to the existence of end-points in $[0,1]$. Let S be the dense subset of X consisting of peak points of X at which A is strongly regular.

First suppose that $X = [0,1]$. Let x be in S. We proceed as in Wilken's original proof: choose $f \in A$ such that $f(x) = 1$ and such that $|f(t)| < 1$ for all other points of $[0,1]$. Choose a sequence g_n in J_x converging to $1 - f$ in A. Define $h_n(t)$ to be 1 on $[0,x]$ and $1 - g_n(t)$ for all other t. Then the functions h_n agree locally on X with either 1 or $1 - g_n$, so each h_n is in A, since A is 2-local on X.

Also, the functions h_n converge to a function h in A which is 1 on $[0,x]$, but with $|h(t)| < 1$ for all other t. Thus $[0,x]$ is a peak set for A, and, similarly, so too is $[x,1]$. Since S is dense, and countable intersections and finite unions of peak sets are peak sets, we see that $[x,y]$ is a peak set for all x, y in $[0,1]$, and thus that every closed subset of $[0,1]$ is a peak set. It follows, by [Br, 2.4.3], that for every closed subset E of $[0,1]$ the restriction of A to E is closed in $C(E)$. Thus $A = C([0,1])$ by Glicksberg's theorem, see [St, 13.5].

Now suppose instead that $X = T$. Let E be any closed arc in T with endpoints in S. We shall show that E is a peak set for A. The result will then follow, as before. Let x, y be the end-points of E. Then (by the earlier remarks) $\{x,y\}$ is a peak set for A. Choose $f \in A$ such that $f(x) = f(y) = 1$ and such that $|f(t)| < 1$ for all other points of T. By Lemma 9, we can find a sequence (g_n) in $J_x \cap J_y$ converging to $1 - f$ in A.

Define $h_n(t)$ to be 1 for $t \in E$, and set $h_n(t) = 1 - g_n(t)$ for all other t. As above, each h_n is in A. The functions h_n converge uniformly to a function h which is constantly 1 on E, but such that $|h(t)| < 1$ for all other t. Thus E is a peak set for A, as claimed. The rest of the proof is identical to the proof above for the interval, noting, of course, that T itself is trivially a peak set for A. □

Theorem 11. *Let X be $[0,1]$ or T, and let A be a uniform algebra on X. Set*

$$E = \{x \in X : \overline{J_x} \neq M_x\}.$$

Suppose that A is 2-local on X and that E is meagre. Then A is trivial.

Proof. Since A is 2-local on X, Theorem 6 shows that if A were not regular on X then there would be an interval consisting of non-R-points, hence points of E. This would contradict the hypothesis that E is meagre. Hence A is regular on X, so X is the Shilov boundary of A. This implies that the set of peak points is a dense G_δ in X, so its intersection with the complement of the meagre set E gives a dense subset of peak points at which strong regularity holds. Thus A is trivial by Proposition 10. □

If A is a uniform algebra with Φ_A equal to either $[0,1]$ or T then A is 2-local on $X = \Phi_A$ [Wi1, 2.3, 3.1]. Thus if the set E above is meagre then A is trivial.

Corollary 12. *Let A be a uniform algebra on X, where X is either $[0,1]$ or T. Set*

$$E = \{x \in X : \overline{J_x} \neq M_x\}.$$

Suppose that E is meagre and has no non-empty, perfect subsets. Then A is trivial.

Proof. Theorem 7 shows that X is the character space of A, and that A is normal on X, and hence 2-local. The result now follows from Theorem 11. □

In particular, if the exceptional set E is countable then A is trivial. The following example shows that things can go wrong if the exceptional set is permitted to have a non-empty perfect subset.

Example 13. Let A be the non-trivial uniform algebra on the Cantor set described in Theorem 9.3 of [We]. The character space of A is the whole of the Riemann sphere (see 9.2 and 9.3′ of [We]), so [St, 27.3] shows that A is not normal on the Cantor set. Let B be the algebra of continuous functions on $[0,1]$ whose restrictions to the Cantor set lie in A. Then B is a non-normal uniform algebra on $[0,1]$, strongly regular on a dense, open subset of $[0,1]$. The character space of B is not equal to $[0,1]$ since it contains a copy of the Riemann sphere. Theorem 11 shows that B is not 2-local on $[0,1]$.

For the last part of the paper we consider a condition on Banach algebras which has been of interest recently, and which has consequences for strong regularity. For a Banach algebra A, let $Id(A)$ be the lattice of closed, two-sided ideals of A. Then $Id(A)$ is *countably-generated* if there is a countable subset S of $Id(A)$ such that for all $I \in Id(A)$

$$I = \left(\bigcup\{J \in S : J \subseteq I\}\right)^{-}.$$

Examples of Banach algebras with countably-generated ideal lattices include separable C*-algebras, TAF-algebras, and separable Banach algebras with spectral synthesis, see [So]. For further results on this property, see [Be1] and [Be2]. We shall need the following result of Beckhoff's.

Lemma 14. [Be1] *Let A be a separable Banach algebra. The following are equivalent.*

(a) $Id(A)$ is countably-generated.

(b) There is a countable subset B of A such that for all $I \in Id(A)$, $I = \overline{I \cap B}$.

Proposition 15. *Let A be a separable Banach function algebra on a compact space X. Set*

$$E = \{x \in X : \overline{J_x} \neq M_x\}.$$

If $Id(A)$ is countably-generated then E is a meagre subset of X.

Proof. By Lemma 14 there is a countable subset B of A such that $I = \overline{I \cap B}$ for each $I \in Id(A)$. Note that for $b \in B$ and $x \in X$, a necessary condition for $b \in M_x \setminus \overline{J_x}$ is that x should belong to the boundary of the zero set of b (for otherwise either

$b \notin M_x$ or $b \in J_x$). The boundary of the zero set of b is a closed set without interior, hence meagre. Thus the set

$$F = \{x \in X : M_x \cap B \neq \overline{J_x} \cap B\}$$

is meagre, being a countable union of meagre sets. But for $x \in X$, $x \notin F$ if and only if $M_x \cap B = \overline{J_x} \cap B$, which holds if and only if $M_x = \overline{J_x}$. Thus $E = F$, so E is meagre. □

Examples 16. (a) For $\alpha \in (0,1)$, let $A = lip_\alpha[0,1]$ be the little Lipschitz algebra on the metric space $([0,1], d^\alpha)$, where $d^\alpha(x,y) = |x-y|^\alpha$, $(x, y \in [0,1])$. Then A is separable and $\Phi_A = [0,1]$. It was shown in [Sh, Corollary 4.3] that A has spectral synthesis. Hence $Id(A)$ is countably-generated, by the remarks above.

(b) Let A be the Banach function algebra consisting of those continuous functions f on $[0,1]$ which are differentiable at 0, with norm given by

$$\|f\| = \|f\|_\infty + \sup\left\{\left|\frac{f(t) - f(0)}{t}\right| : t \in (0,1]\right\},$$

where $\|\cdot\|_\infty$ denotes the uniform norm. Then A is separable and normal on $[0,1]$ and strong regularity fails only at the point 0. The ideal lattice of A is countably-generated, see [Be2, p.455].

(c) Let $A = C^1[0,1]$. Then A is separable and normal but there is no point in $[0,1]$ at which A is strongly regular. Hence $Id(A)$ is not countably-generated.

(d) Let A be the disc algebra. Then A is separable but there is no point of the disc at which A is strongly regular. Hence again $Id(A)$ is not countably-generated.

The next lemma shows that strong regularity is preserved under uniform closure.

Lemma 17. *Let B be a Banach function algebra on a compact space X and let $x \in X$. Let A be the uniform closure of B in $C(X)$. If B is strongly regular at x, then $J_x \cap B$ is uniformly dense in M_x (where M_x and J_x denote the ideals of A).*

Proof. Let $f \in A$ with $f(x) = 0$. Choose a sequence of functions $(f_n) \subseteq B$ converging uniformly to f on X. By subtracting the constant $f_n(x)$ from f_n if necessary, we may assume that $f_n(x) = 0$. Then, since B is strongly regular at x, there are functions g_n in B each of which vanishes on a neighbourhood of x, and such that $\|g_n - f_n\|_B < 1/n$. But the norm in B must dominate the uniform norm on X (which is at most equal to the spectral radius), so it follows that the sequence (g_n) converges to f uniformly on X. The result follows. □

Theorem 18. *Let B be a separable Banach function algebra on X, where X is either $[0,1]$ or T. Suppose either that B is 2-local on X, or that $\Phi_B = X$. If $Id(B)$ is countably-generated, then B is uniformly dense in $C(X)$.*

Proof. Let A be the uniform closure of B in $C(X)$. Suppose first that B is 2-local on X. As in Theorem 11, it follows from Theorem 6 and Proposition 15 that B must be regular on X. Hence A is also regular on X, so X is the Shilov boundary of A. Thus the set of peak points of A is a dense G_δ of X. Proposition 15 shows

that strong regularity holds for B on a dense G_δ of X, and strong regularity holds for A at every point of X where it held for B, by Lemma 17. Thus X has a dense subset S consisting of peak points of A where strong regularity holds for A. We now follow the proof of Proposition 10.

First suppose that $X = [0,1]$. Let x be in S. Choose $f \in A$ such that $f(x) = 1$ and such that $|f(t)| < 1$ for all other points of $[0,1]$. Lemma 17 shows that there is a sequence (g_n) of elements of B each vanishing in a neighbourhood of x, and converging in A to $1 - f$. Define $h_n(t)$ to be 1 on $[0,x]$ and $1 - g_n(t)$ for all other t. Then each h_n agrees locally on X with either 1 or $1 - g_n$, so each h_n is in B, since B is 2-local. The rest of the proof for $X = [0,1]$ now follows as in Proposition 10.

Now suppose that $X = T$. Let E be any closed arc in T with endpoints $x, y \in S$. Choose $f \in A$ such that $f(x) = f(y) = 1$ and such that $|f(t)| < 1$ for all other points of T. Then $(1 - f) \in M_x \cap M_y$ (where again we use M_x, J_x, etc. to denote ideals of A), so as in Lemma 9 there exist $p \in M_x$ and $q \in M_y$ such that $(1-f) = pq$. By Lemma 17 there are sequences (p_n) in $J_x \cap B$ and (q_n) in $J_y \cap B$ converging to p and q respectively in A. Hence if $g_n = p_n q_n$ then $(g_n) \subseteq J_x \cap J_y \cap B$ and (g_n) converges to $1 - f$ in A. Define $h_n(t)$ to be 1 for $t \in E$, and $h_n(t) = 1 - g_n(t)$ for all other t. As above, each h_n is in B. The proof is now concluded as in Proposition 10.

Finally, suppose that $\Phi_B = X$. Then $\Phi_A = X$ too, and again Proposition 15 and Lemma 17 shows that strong regularity holds for A on a dense G_δ of X. Hence A is trivial, by the remark after Theorem 11. $\square$

Of course, the conditions of Theorem 18 are by no means necessary for the conclusion, as Example 16(c) shows, for instance.

Let us conclude by mentioning a couple of open questions. It follows from Theorem 5.1 of [Wh] that whenever A is a normal uniform algebra on a compact space X and $x \in X$ is such that J_x has a bounded approximate identity, then A is strongly regular at x. It appears to be unknown, however, whether a normal uniform algebra must be strongly regular at each of its peak points (recall that if a point x is a peak point then M_x has a bounded approximate identity). If this should be the case, then the methods above will show immediately that every normal uniform algebra on $[0,1]$ or T is trivial. For some positive results in this direction, see [Ba].

It is also unknown whether or not there are any non-trivial uniform algebras that have spectral synthesis. In [FS1] there is an example of a strongly regular uniform algebra A such that every maximal ideal of A has a bounded approximate identity, but such that A does not have spectral synthesis. In the same paper it is shown that the method of taking systems of root extensions cannot produce a uniform algebra with spectral synthesis unless it is applied to such an algebra in the first place.

References

[Ba] B. T. Batikjan, *Regular subalgebras of $C(X)$*, Moscow Univ. Math. Bull. **29** (1974), 89–90.

[Be1] F. Beckhoff, *Topologies of compact families on the ideal space of a Banach algebra*, Studia Math. **118** (1996), 63–75.

[Be2] F. Beckhoff, *Countability properties of the ideal space a Banach algebra and D-algebras*, Rend. del Circ. Mat. di Palermo **46** (1997), 451–464.
[Br] A. Browder, *Introduction to Function Algebras*, W. A. Benjamin, Inc., New York, 1969.
[Ch] D. R. Chalice, *S-algebras on sets in C^n*, Proc. Amer. Math. Soc. **39** (1973), 300–304.
[Co] B. Cole, *One point parts and the peak point conjecture.*, Ph.D. Dissertation, Yale University (1968).
[F1] J. F. Feinstein, *A non-trivial, strongly regular uniform algebra*, J. London Math. Soc. **45** (1992), 288–300.
[F2] J. F. Feinstein, *Regularity conditions for Banach function algebras*, Function spaces (Edwardsville, Il, 1994), Lecture Notes in Pure and Appl. Math., vol. 172, Dekker, New York, 1995, pp. 117–122.
[FS1] J. F. Feinstein and D. W. B. Somerset, *A note on ideal spaces of Banach algebras*, Bull. London Math. Soc. (to appear).
[FS2] J. F. Feinstein and D. W. B. Somerset, *Non-regularity for Banach function algebras*, in preparation (1998).
[G] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1969.
[H] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.
[M] R. Mortini, *Closed and prime ideals in the algebra of bounded analytic functions*, Bull. Austral. Math. Soc. **35** (1987), 213–229.
[O] A. G. O'Farrell, *A regular uniform algebra with a continuous point derivation of infinite order*, Bull. London Math. Soc. **11** (1979), 41–44.
[P] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras*, C. U. P., Cambridge, 1994.
[R] W. Rudin, *Continuous functions on compact spaces without perfect subsets*, Proc. Amer. Math. Soc. **8** (1957), 39–42.
[Sh] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240–272.
[So] D. W. B. Somerset, *Spectral synthesis for Banach algebras*, Quart. J. Math. Oxford (to appear).
[St] E. L. Stout, *The Theory of Uniform Algebras*, Bogden and Quigley, Inc., New York, 1971.
[Wa] J. L. M. Wang, *Strong regularity at nonpeak points*, Proc. Amer. Math. Soc. **51** (1975), 141–142.
[We] J. Wermer, *Banach algebras and analytic functions*, Advances in Math. **1** (1961), 51–102.
[Wh] M. C. White, *Injective modules for uniform algebras*, Proc. London Math. Soc. **73** (1996), 155–184.
[Wi1] D. R. Wilken, *Approximate normality and function algebras on the interval and the circle*, Function Algebras (Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965), Scott-Foresman, Chicago, Ill., 1966, pp. 98–111.
[Wi2] D. R. Wilken, *A note on strongly regular uniform algebras*, Canad. J. Math. **21** (1969), 912–914.

School of Mathematical Sciences, University of Nottingham, NG7 2RD, U.K.
E-mail address: Joel.Feinstein@nottingham.ac.uk

Department of Mathematical Sciences, University of Aberdeen, AB24 3UE, U.K.
E-mail address: ds@maths.abdn.ac.uk