More practice with definitions, proofs and examples

This optional sheet gives you a chance to practise constructing proofs. It also offers you a chance to practice thinking up specific examples with a variety of properties.

You should not hand in your answers to this sheet. However, feel free to ask Dr Feinstein (at any time) or one of the G12MAN Workshop Helpers if you wish to discuss your attempted solutions to these questions.

Many of these questions have no direct connection with the material in the G12MAN lecture notes.

Where we discuss matrices below, you may assume that these matrices have entries which are real numbers if you wish (though the results are also valid when the entries are complex numbers). We will only discuss square matrices.

Very quick proofs using definitions

The first few proofs follow very quickly from the basic definitions. You should practise on these until you are fluent (and possibly bored!) with such proofs.

1. Prove that, for every pair of odd integers $m$ and $n$, we have that their sum $m + n$ is even, and that their product $mn$ is odd.

2. (Revision of matrix definitions.)
   (i) Prove that the only square matrices which are both upper triangular and lower triangular are the diagonal matrices.
   (ii) Prove that the only square matrices which are both lower triangular and also strictly upper triangular are the square zero matrices.

3. Prove that the constant function $0$ (the zero function) from $\mathbb{R}$ to $\mathbb{R}$ is the only function $f$ from $\mathbb{R}$ to $\mathbb{R}$ such that $f$ is both an odd function and an even function.

4. Prove that the constant functions from $\mathbb{R}$ to $\mathbb{R}$ are the only functions from $\mathbb{R}$ to $\mathbb{R}$ which are both monotone increasing and monotone decreasing.

5. Working in $\mathbb{R}$, give a careful proof that $1$ is not an interior point of the set $[0, 1]$.

6. Prove directly from the definition of interior given in the notes that every point of $\mathbb{R}$ is an interior point of $\mathbb{R}$.

7. Prove that, whenever $A$ and $B$ are bounded subsets of $\mathbb{R}^d$, then $A \cap B$ and $A \cup B$ are also bounded.

8. Let $A$ and $B$ be subsets of $\mathbb{R}$, and consider the Cartesian product $A \times B \subseteq \mathbb{R}^2$. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

9. Let $A$, $B$, $C$ and $D$ be subsets of $\mathbb{R}$. Prove that
   
   $$(A \setminus C) \times (B \setminus D) \subseteq (A \times B) \setminus (C \times D).$$

10. Here we use the definition of absorption given in lectures (or see Dr Feinstein’s blog).

    Let $A$ and $B$ be disjoint subsets of $\mathbb{R}$ (i.e., $A \cap B = \emptyset$), and let $(x_n)$ be a sequence of real numbers. Prove that it is impossible for both $A$ and $B$ to absorb the sequence $(x_n)$.

11. Prove that, for every pair of real numbers $x$ and $y$ with $x \neq y$, there are open intervals $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. 

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More practice

The next few proofs still follow fairly quickly from the definitions, but may need a little more work
and/or thought. You are also asked to find some examples and do some calculations. (Note that
examples also require justification.)

We use the following standard definitions concerning square matrices.

**Definition:** Let $A$ be a square matrix. Then we say that $A$ is **idempotent** if $A^2 = A$; the matrix $A$
is nilpotent if there exists a natural number $n$ such that $A^n = 0$ (the square zero matrix of the
appropriate size).

The **trace** of the matrix $A$ is the sum of the entries on the leading (main) diagonal of $A$.

The **characteristic polynomial** of $A$, $\chi_A$, is defined by

$$
\chi_A(t) = \det(tI - A)
$$

(note that some authors use $\det(A - tI)$, where $I$ is the identity matrix of the appropriate size.

1. Let $z \in [0, 10)$. Prove that there exist $x \in [0, 3)$ and $y \in [0, 7)$ with $x + y = z$.

2. Prove that the only square matrices which are both idempotent and nilpotent are the square
zero matrices.

3. Give an example of a non-zero $2 \times 2$ matrix which is nilpotent.

4. Let $A$ be a (square) diagonal matrix.
   (i) Prove that $A$ is idempotent if and only if all of the entries on the diagonal are either 0 or 1.
   (ii) Prove that $A$ is nilpotent if and only if $A = 0$.
   [Don’t forget what you found out above! You MUST use the fact that $A$ is a diagonal matrix
   here.]

5. Let $A$ be a nilpotent square matrix. Prove that $A$ can not have any non-zero eigenvalues.

6. Let $A$ be an idempotent square matrix and suppose that $\lambda$ is an eigenvalue for $A$. Prove that
   $\lambda \in \{0, 1\}$.

7. **Throughout this question**, let $A$ be a $2 \times 2$ square matrix.
   (i) By direct calculation, determine formulae for the coefficients of the characteristic polynomial
   $\chi_A(t)$ in terms of the entries of $A$.
   (ii) What is the relationship between the characteristic polynomial of $A$, the trace of $A$ and the
determinant of $A$?
   (iii) By direct calculation, verify the Cayley-Hamilton Theorem for $A$.

8. Give an example of a sequence of **finite** subsets $A_n$ of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a finite
subset of $\mathbb{R}$.

9. Give an example of a sequence of **bounded** subsets $A_n$ of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a
bounded subset of $\mathbb{R}$.

10. Give an example of a sequence of **closed** subsets $A_n$ of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is **not** a closed
subset of $\mathbb{R}$.

11. Let $(x_n) \subseteq \mathbb{R}$. Suppose that $(y_n)$ is a subsequence of $(x_n)$, and that $(z_n)$ is a subsequence of
$(y_n)$. Prove that $(z_n)$ is also a subsequence of $(x_n)$.
   [One way to express this result is to say that every sub-subsequence of a sequence is also a
subsequence of that sequence.]
Even more practice

The remaining questions are of variable difficulty, perhaps depending on the method you choose or whether you spot a helpful idea. You may wish to use some of the results mentioned above to help.

Don’t forget that you can also get plenty of practice with proofs and examples by doing the standard question sheet questions, and perhaps trying the questions on the Challenging problems sheet.

Sums of subsets of $\mathbb{R}$.

Let $A$ and $B$ be subsets of $\mathbb{R}$, then we define $A + B \subseteq \mathbb{R}$ by

$$A + B = \{x + y \mid x \in A \text{ and } y \in B\} = \{z \in \mathbb{R} \mid \text{there are } x \in A \text{ and } y \in B \text{ with } x + y = z\}$$

i.e., $A + B$ is the set of all real numbers of the form $x + y$ where $x \in A$ and $y \in B$.

1. Prove that, for all integers $n$, $n^3 - n$ is divisible by three.
2. (i) Prove that there are no positive integers $n$ for which $1 + n + n^2 + n^3$ is prime.
   (ii) For which integers $n$, if any, is $-(1 + n + n^2 + n^3)$ prime?
3. Prove that the only invertible, square matrices which are idempotent are the square identity matrices.
4. Let $A$ be a singular $2 \times 2$ matrix.
   (i) Prove that, if the trace of $A$ is 1, then $A$ is idempotent.
   (ii) Is the converse to the statement in (i) true?
5. Determine all $2 \times 2$ idempotent matrices.
6. Prove that the only nilpotent, diagonalizable square matrices are the square zero matrices.
7. Are there any invertible, nilpotent square matrices?
8. Starting from a suitable definition of convergence for sequences of real numbers, prove that limits of convergent sequences are unique: no sequence of real numbers $(x_n)$ can converge to two different real numbers.
   [Note here that you must not assume that $\lim_{n \to \infty} x_n$ is unique in your proof!]
9. Prove the following set equality involving sums of sets, as defined above:

$$[0, 10) = [0, 3) + [0, 7).$$
10. Let $R$ and $S$ be positive real numbers. Prove that

$$[0, R] + [0, S] = [0, R + S]$$

and that

$$[0, R) + [0, S) = [0, R + S).$$
11. Give an example of a pair of subsets $A$ and $B$ of $\mathbb{R}$ such that $A + B \neq A \cup B$.
12. Give an example of a pair of subsets $A$ and $B$ of $\mathbb{R}$ such that $A + B = A \cup B$.
13. Find infinitely many different pairs of subsets $A$ and $B$ of $\mathbb{R}$ such that $A + B = A \cup B$.
14. Which of the three sets $\mathbb{Q}$, $\mathbb{Q}^c$ or $\mathbb{R}$ is $\mathbb{Q} + \mathbb{Q}^c$ equal to?