

# How do we do proofs? (Part I)

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2007-2008

Last year, you saw quite a few proofs.

You probably saw several different types of proof.

### Question 1

How many different kinds of proof can you name or at least describe?

Consider the following question about odd numbers, and try to find a proof.

Later we will come back to the proof and look at the process we went through to find it.

### Question 2

Prove that, for every odd integer  $n$ ,  $n^4 - 1$  is divisible by 8.

Students sometimes feel that 'proofs are hard'.

Here are some questions about understanding lecturers' proofs.

- 1 Do you find it easy to follow the individual steps in proofs you see in lectures?
- 2 Do you find it easy to see the overview of what needs to be established during the proof?
- 3 Do you find that you understand the proof once you see it?

What about when you want to do proofs yourself?

- 1 Do you find that you can learn how to do proofs by reading and understanding lecturers' proofs?
- 2 Do you feel that you have no idea how the lecturer thought of which step to do when?
- 3 Do you feel that it is much harder to find your own proofs than to follow the proofs given by the lecturer?
- 4 Do you find it particularly hard to know how to **start** proofs?

## Questions about questions

When you come across a question which asks you to prove something, you may find it useful to ask the following ‘meta-questions’, i.e., questions about the question.

- 1 What does the question mean?
- 2 Are there several ways to ask the same question?
- 3 How do we use formal definitions?
- 4 How do we start a proof?
- 5 What are we allowed to assume during a proof?
- 6 Which type of proof is appropriate here?

Hopefully we have answered Question 2 above, which means that we have managed to prove the following statement.

For every odd integer  $n$ ,  $n^4 - 1$  is divisible by 8. (\*)

Statement (\*) has several equivalent formulations, some of which make it easy to see how to start a proof: here are a few possibilities.

- Let  $n$  be an odd integer. Then  $n^4 - 1$  is divisible by 8.
- If  $n$  is an odd integer, then  $n^4 - 1$  is divisible by 8.
- Let  $n$  be an integer. Then  
 $n$  is odd  $\Rightarrow n^4 - 1$  is divisible by 8.

The main thing is that, in the proof, we are allowed to assume that  $n$  is odd, and then we have to deduce some further facts about  $n$ .

This is a **very** common approach when you are asked to prove that 'every object of type  $X$  has property  $Y$ '.

### Question 3

Which other equivalent formulations can you think of for statement (\*) above?



Our next task is to revisit our proof to see what we did when. Investigate the following questions.

- 1 Which definitions did we assume, and when did we use them?
- 2 Which standard results did we quote, and when did we use them?
- 3 Is it obvious why we chose to do each step when we did?

You may already know this result about limits and inequalities, but **you are never allowed to use a result to prove itself!**

### Question 4

Prove that, for every real number  $M$ , and every **convergent** sequence of real numbers  $x_1, x_2, x_3, \dots$  such that all of the terms  $x_n$  are  $\leq M$ , we have

$$\lim_{n \rightarrow \infty} x_n \leq M.$$

- 1 Which meta-questions should you ask?
- 2 Do you feel that you know how to start a proof?
- 3 Do you think that you can write down a proof?
- 4 What do you call a question about meta-questions?