How do we do proofs? (Part I)

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In a direct proof, we argue directly from a given starting point, using formal definitions, standard facts (or previously proved results), and logical reasoning, to obtain the result we want.

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You are more likely to spot the ideas you need for proofs if you are not distracted by having to think about the routine parts of doing proofs.

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- You may, if you wish, quote the standard (binomial theorem) expansion of $(a + b)^4$.

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2. Do you find it easy to see the overview of what needs to be established during the proof?
3. Do you find that you understand the proof once you see it?

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4. Do you find it particularly hard to know how to **start** proofs?

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This is a **very** common approach when you are asked to prove that 'every object of type X has property Y'.

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4. How much choice did we have at each stage as to what we should try next?
5. Is it obvious why we chose to do each step when we did?

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1. Which terms are mentioned in the question? Can you write down the precise definitions of these terms?
2. Can you reformulate the statement in a way that makes it more obvious how to start your proof?