How do we do proofs? (Part I)

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Last year, you saw quite a few proofs.

You probably saw several different types of proof.

Question 1

How many different kinds of proof can you name or at least describe?

Types of Proof

- Proof by induction
- Proof by "contrapositive", or contraposition.
- Proof by contradiction

Both of these are done by showing that the opposite (negation) of the result you want is impossible.

- Direct proof (Start from information given and make deductions).
- Etc.!

Consider the following question about odd numbers, and try to find a proof.

Later we will come back to the proof and look at the process we went through to find it.

Question 2

Prove that, for every odd integer n, $n^4 - 1$ is divisible by 8.

Proof (Several possible methods)

- Dangerous to start by assuming the result you want to prove: you <u>can</u> deduce true statements from false statements.

- Use definition of "odd".

n odd $\Rightarrow n = 2k+1$ some integer k.

Then $n^4 - 1 = (2k+1)^4 - 1$

= (binomial theorem)

$((2k)^4 + 4(2k)^3 + 6(2k)^2 + 4(2k) + 1) - 1$

$= 16k^4 + 32k^3 + 24k^2 + 8k$

$= 8(2k^4 + 4k^3 + 3k^2 + k)$,

and this is clearly divisible by 8. □

Comments

Start proof by assuming that n is an odd integer.

← Must <u>use</u> definition of "odd".

← Use a standard theorem to help.

Make deductions, and arrive at the result:

$(n^4 - 1)$ divisible by 8.

Another method:

write

$$n^4-1 = (n^2+1)(n^2-1)$$
$$= (n^2+1)(n+1)(n-1).$$

Since n is odd, so is n^2, so

n^2+1, $n+1$, $n-1$ are all even,

i.e. each bracket is divisible by 2.

Thus the product of the 3 brackets is

divisible by $2^3 = 8$. $\square$

Students sometimes feel that ‘proofs are hard’.

Here are some questions about understanding lecturers’ proofs.

1. Do you find it easy to follow the individual steps in proofs you see in lectures?
2. Do you find it easy to see the overview of what needs to be established during the proof?
3. Do you find that you understand the proof once you see it?

What about when you want to do proofs yourself?

1. Do you find that you can learn how to do proofs by reading and understanding lecturers' proofs?
2. Do you feel that you have no idea how the lecturer thought of which step to do when?
3. Do you feel that it is much harder to find your own proofs than to follow the proofs given by the lecturer?
4. Do you find it particularly hard to know how to **start** proofs?

Suggestion for discussion on Module Newsgroup.

With the above questions to help you, can you say how you feel about doing proofs.
Are proofs "hard"?
What do you find hardest about doing proofs?

Questions about questions

When you come across a question which asks you to prove something, you may find it useful to ask the following 'meta-questions', i.e., questions about the question.

1. What does the question mean?
2. Are there several ways to ask the same question?
3. How do we use formal definitions?
4. How do we start a proof?
5. What are we allowed to assume during a proof?
6. Which type of proof is appropriate here?

Hopefully we have answered Question 2 above, which means that we have managed to prove the following statement.

For every odd integer n, $n^4 - 1$ is divisible by 8. $(*)$

Statement $(*)$ has several equivalent formulations, some of which make it easy to see how to start a proof: here are a few possibilities.

- Let n be an odd integer. Then $n^4 - 1$ is divisible by 8.
- If n is an odd integer, then $n^4 - 1$ is divisible by 8.
- Let n be an integer. Then

$$n \text{ is odd} \Rightarrow n^4 - 1 \text{ is divisible by } 8.$$

The main thing is that, in the proof, we are allowed to assume that n is odd, and then we have to deduce some further facts about n.

This is a **very** common approach when you are asked to prove that 'every object of type X has property Y'.

Question 3

Which other equivalent formulations can you think of for statement $(*)$ above?

Equivalent formulations

Exercise: see how many of these you can think of.

Our next task is to revisit our proof to see what we did when.
Investigate the following questions.

1. Which definitions did we assume, and when did we use them?
2. Which standard results did we quote, and when did we use them?
3. Is it obvious why we chose to do each step when we did?

You may already know this result about limits and inequalities, but **you are never allowed to use a result to prove itself**!

Question 4

Prove that, for every real number M, and every **convergent** sequence of real numbers $x_1, x_2, x_3, \ldots$ such that all of the terms x_n are $\leq M$, we have

$$\lim_{n \to \infty} x_n \leq M \,.$$

1. Which meta-questions should you ask?
2. Do you feel that you know how to start a proof?
3. Do you think that you can write down a proof?
4. What do you call a question about meta-questions?

Exercise.

Think about the meta-questions above, and try to reformulate the statement above in a "friendlier" form which makes it clearer how a proof should start.