How do we do proofs? (Part I)

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Last year, you saw quite a few proofs.

You probably saw several different types of proof.

As an exercise, later, you make wish to make a list of all the different kinds of proof that you can think of.

Many of the proofs in second-year Pure Mathematics modules can be described as **direct proofs**.

In a direct proof, we argue directly from a given starting point, using formal definitions, standard facts (or previously proved results), and logical reasoning, to obtain the result we want.

In these sessions on how to do proofs, we will focus mainly on direct proofs.

The aim is to show you that some parts of doing proofs are relatively systematic, and should become almost routine with practice.

Question sheet questions give you a great opportunity to practice writing proofs!

Of course, there is a creative element to doing proofs, and you may need to think of clever ideas. However . . .

You are more likely to spot the ideas you need for proofs if you are not distracted by having to think about the routine parts of doing proofs.

Consider the following question about odd numbers, and try to find a proof, using the hints below or otherwise.

Question 1

Prove that, for every odd integer n, $n^4 - 1$ is divisible by 8.

Hints

- Before you start, write down the definitions of the terms **odd** and **divisible by 8**.
- You should start your proof with the sentence **'Let *n* be an odd integer.'**
- You may, if you wish, quote the standard (binomial theorem) expansion of $(a + b)^4$.

Students sometimes feel that ‘proofs are hard’.

Here are some questions about understanding lecturers’ proofs.

1. Do you find it easy to follow the individual steps in proofs you see in lectures?
2. Do you find it easy to see the overview of what needs to be established during the proof?
3. Do you find that you understand the proof once you see it?

What about when you want to do proofs yourself?

1. Do you find that you can learn how to do proofs by reading and understanding lecturers' proofs?
2. Do you feel that you have no idea how the lecturer thought of which step to do when?
3. Do you feel that it is much harder to find your own proofs than to follow the proofs given by the lecturer?
4. Do you find it particularly hard to know how to **start** proofs?

Above, we proved the following statement.

For every odd integer n, $n^4 - 1$ is divisible by 8. $(*)$

Statement $(*)$ has several equivalent formulations: here are two.

- Let n be an odd integer. Then $n^4 - 1$ is divisible by 8.
- If n is an odd integer, then $n^4 - 1$ is divisible by 8.

The main thing is that, in the proof, we are allowed to assume that n is odd, and then we have to deduce some further facts about n.

This is a **very** common approach when you are asked to prove that 'every object of type X has property Y'.

I recommend that, later, you look back at our proof above, and decide whether you know the answers to the following questions.

1. Why did we start the proof the way we did?
2. Which definitions did we assume, and when did we use them?
3. Which standard results did we quote, and when did we use them?
4. How much choice did we have at each stage as to what we should try next?
5. Is it obvious why we chose to do each step when we did?

Knowing a fact from an early age can sometimes make it harder to write down a proof, rather than easier!

Question 2

Prove that, for every pair of odd integers m and n, $m + n$ is even.

You should give a direct proof, in a similar style to our earlier proof.

Before you start, you may wish to think about the following.

1. Which terms are mentioned in the question? Can you write down the precise definitions of these terms?
2. Can you reformulate the statement in a way that makes it more obvious how to start your proof?