How do we do proofs? (Part II)

Joel Feinstein

School of Mathematical Sciences
University of Nottingham

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In the module G12MAN, when we talk about a **sequence** (x_k), **by default** we assume that the sequence is infinite and that we start from $k = 1$. In other words, the sequence has the form $x_1, x_2, x_3, \ldots$.

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If we wish to work with sequences starting from a different first term, we will specify this. For example, a sequence of the form $x_0, x_1, x_2, \ldots$ can be denoted by $(x_k)_{k=0}^{\infty}$.

Question 1

What does it mean to say that a sequence of real numbers (x_k) is:
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Question 2

(i) Are these two properties equivalent? (That is, does each property imply the other?)

(ii) Do either of these properties imply the other?

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Question 3 (formal justification not required)

Let (n_k) be a strictly increasing sequence of natural numbers.

- What is the smallest possible value that n_2 could have? What about n_{200}?
- For a given positive integer k, what is the minimum possible value that n_k could have? (Give your answer in terms of k.)

I accept that, once you have decided which definition of $\mathbb{N}$ you are working with, the answers to Question 3 are **clear**.

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Question 4

How would you justify your answers to Question 3 formally if you were required to?

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Note that, **in the module G12MAN**, 0 will **not** count as a natural number: for us

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Question 5

Prove that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ is convergent.

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How could you start a formal proof here?

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Meta-question 5B

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Meta-question 5C

What are you allowed to assume, state or use during the proof?

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Meta-question 6C

Given that you are expected to justify your answer fully, what expectations are implicit in Question 6, i.e., what do you have to do to give a satisfactory answer to Question 6?