How do we do proofs? (Part II)

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In the module G12MAN, when we talk about a **sequence** (x_k), **by default** we assume that the sequence is infinite and that we start from $k = 1$. In other words, the sequence has the form $x_1, x_2, x_3, \ldots$.

We may also denote such a sequence by, for example, $(x_k)_{k=1}^{\infty}$, if there is danger of ambiguity.

Of course, we can change the name of the index without changing the sequence. So we can also denote the above sequence by (x_n), or by $(x_n)_{n=1}^{\infty}$.

If we wish to work with sequences starting from a different first term, we will specify this. For example, a sequence of the form $x_0, x_1, x_2, \ldots$ can be denoted by $(x_k)_{k=0}^{\infty}$.

Question 1

What does it mean to say that a sequence of real numbers (x_k) is:
Decreasing? Nondecreasing? **Increasing**? **Strictly increasing**?

Suppose that (x_n) is a sequence of real numbers. Consider the following properties that (x_n) might have:

(a) (x_n) is a nondecreasing sequence;

(b) (x_n) is not a decreasing sequence.

Question 2

(i) Are these two properties equivalent? (That is, does each property imply the other?)

(ii) Do either of these properties imply the other?

Suggestions?

What does it mean to say (x_n) is decreasing?

Some authors use this to mean

$x_1 > x_2 > x_3 > \dots$

but others (including Dr. F.) say it means $x_1 \geq x_2 \geq x_3 \geq \dots$

(x_n) is nondecreasing if $x_1 \leq x_2 \leq x_3 \leq \dots$ (all authors agree)

Increasing: two opinions again.

For Dr. Feinstein, "increasing" means "nondecreasing".

(x_n) is **increasing** if

$x_1 \le x_2 \le x_3 \le \cdots$

(x_n) is **strictly increasing** if

$x_1 < x_2 < x_3 < \cdots$

Most sequences do not satisfy **any** of these increasing/decreasing conditions.

e.g. $1, 0, 1, 2, 3, \cdots$ does not satisfy **any** of the above conditions (because it is not a "monotone" sequence).

The sequence 1, 0, 1, 2, 3, ... is <u>not</u> a decreasing sequence (by our definition). But nor is it a nondecreasing sequence (by our definition).

<u>Constant</u> sequences qualify as <u>both</u> decreasing and nondecreasing by our definitions! For example, consider 1, 1, 1, 1, ... We have $1 \le 1 \le 1 \le \ldots$, so the sequence is nondecreasing, and $1 \ge 1 \ge 1 \ge 1 \ldots$, so the sequence <u>is</u> "decreasing". (but <u>not</u> strictly decreasing)

Different authors disagree about whether or not 0 is a natural number.

Some authors define $\mathbb{N} = \{1, 2, 3, \ldots\}$.
Others define $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

The answer to the next question depends on which definition of $\mathbb{N}$ you are working with: there are two possible answers for each part.

Question 3 (formal justification not required)

Let (n_k) be a strictly increasing sequence of natural numbers.

- What is the smallest possible value that n_2 could have? What about n_{200}?
- For a given positive integer k, what is the minimum possible value that n_k could have? (Give your answer in terms of k.)

If $\mathbb{N} = \{0, 1, 2, \ldots\}$

then: $n_2 \geq 1$

$n_{200} \geq 199$

$n_k \geq k-1$

If (as in GIRMAN) $\mathbb{N} = \{1, 2, 3, \ldots\}$

then: $n_2 \geq 2$

$n_{200} \geq 200$

$n_k \geq k$.

I accept that, once you have decided which definition of $\mathbb{N}$ you are working with, the answers to Question 3 are **clear**.

They can, and should, be stated without proof in this module by saying **‘clearly, . . . ’**.

Question 4

How would you justify your answers to Question 3 formally if you were required to?

Before attempting the next two questions, you should attempt the meta-questions (i.e. questions about questions) which follow them.

These questions are not particularly hard in themselves, though they require some understanding of convergence and divergence for series. It is the illustrative meta-questions which follow which are more important here.

Note that, **in the module G12MAN**, 0 will **not** count as a natural number: for us

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Question 5

Prove that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ is convergent.

Question 6

Is it true that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent?

Question 5

Prove that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ is convergent.

Meta-question 5A

Can you think of several equivalent, different ways of expressing Question 5? If so, which do you find most helpful?

Meta-question 5B

How could you start a formal proof here?

Meta-question 5C

What are you allowed to assume, state or use during the proof?

A formal proof could start with "Let (n_k) be a strictly increasing sequence of natural numbers" or "suppose that ....", etc.

We start from this assumption on (n_k).

We then deduce the result using facts/results that are either <u>standard</u> or "clear".

The result must then hold for <u>all</u> (n_k) of this type.

<u>Exercise.</u> Write down a full, formal proof.

Question 6

Is it true that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent?

Meta-question 6A

Does it make the question harder when you are not told the answer?

Meta-question 6B

Do you have a guess as to what the answer might be?

Meta-question 6C

Given that you are expected to justify your answer fully, what expectations are implicit in Question 6, i.e., what do you have to do to give a satisfactory answer to Question 6?