How do we do proofs? (Part II)

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Throughout this session, when we talk about a **sequence** (x_k), we will assume that the sequence is infinite and that we start from $k = 1$. In other words, the sequence has the form $x_1, x_2, x_3, \ldots$.

Of course, we can change the name of the index without changing the sequence. So we can also denote the above sequence by (x_n).

We recall the following definition concerning sequences of real numbers.

Definition

A sequence (x_k) of real numbers is said to be **strictly increasing** if we have

$$x_1 < x_2 < x_3 < \cdots .$$

Different authors disagree about whether or not 0 is a natural number.

For us, $\mathbb{N} = \{1, 2, 3, \dots\}$, so $0 \notin \mathbb{N}$.

Question 1 (formal justification not required)

Let (n_k) be a strictly increasing sequence of natural numbers.

- What is the smallest possible value that n_2 could have? What about n_{200}?
- For a given positive integer k, the minimum possible value that n_k could have clearly depends on k. In terms of k, what is this minimum possible value?
  Your answer should be a function of k.
- If you **did** have to justify your answers formally here, how would you do so?

Before attempting the next two questions, you should attempt the meta-questions (i.e. questions about questions) which follow them.

You will have an opportunity to think about the meta-questions and to give feedback, before being given more time to work on the question itself.

These questions will build on our discussion above, and also on your previous knowledge of results concerning the convergence and divergence for series.

Note again that, for us, 0 will **not** count as a natural number: for us $0 \notin \mathbb{N}$, and so

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

Question 2

Prove that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ is convergent.

Question 3

Is it true that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent?

Question 2

Prove that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ is convergent.

Meta-question 2A

Can you think of several equivalent, different ways of expressing the fact that you are asked to prove in Question 2? If so, which do you find most helpful?

Meta-question 2B

How could you start a formal proof here?

Meta-question 2C

What sort of things will you need to assume, state or use during the proof?

Question 3

Is it true that, for every strictly increasing sequence of natural numbers (n_k), the series $\sum_{k=1}^{\infty} \frac{1}{n_k}$ is convergent?

Meta-question 3A

Does it make the question harder when you are not told the answer?

Meta-question 3B

Do you have a guess as to what the answer might be?

Meta-question 3C

Given that you are expected to justify your answer fully, what expectations are implicit in Question 3, i.e., what do you have to do to give a satisfactory answer to Question 3?