How do we do proofs (Part II)

2008 – 2009

3 handouts at front

- Copy of today's slides
- Convergence and divergence for some standard series.
- Student Feedback form.

Please complete both sides of the student feedback form and place it in the box when you leave.

Question 1

Strictly increasing sequence (n_k) of natural numbers,

$$n_1, n_2, n_3, \dots$$

Clearly $n_1 \in \mathbb{N}$, so $n_1 \geq 1$.

Then $n_2 > n_1$, so $n_2 \geq 2$

etc., giving $n_{200} \geq 200$

and $n_k \geq k$ for all $k \in \mathbb{N}$.

The sequence $n_k = k$ is strictly increasing, so $n_k = k$ can be achieved.

You can prove this more rigorously by induction (exercise).

Question 2 and its meta-questions.

[Typo: one occurrence of "Question 3" should be "Question 2". Now fixed on web.]

2A. An equivalent reformulation is "Let (n_k) be a strictly increasing sequence of natural numbers. Then $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ converges."

2B. Good idea to start a proof with "Let (n_k) be a strictly increasing sequence of natural numbers".

2C – We can quote standard fact that $\sum_{k=1}^{\infty} k^{-p}$ converges if $p>1$.

– We saw earlier that $n_k \geq k$ for all $k \in \mathbb{N}$.

– We could use some form of the comparison test.

e.g. if (a_n), (b_n) are sequences of real numbers, with

$0 \leq a_n \leq b_n$ for all n,

then if $\sum_{k=1}^{\infty} b_k$ converges then

$\sum_{k=1}^{\infty} a_k$ converges too.

Contrapositive:

If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$

also diverges.

Proof

Let (n_k) be a strictly increasing sequence of natural numbers. Then we know $n_k \geq k$ for all k. So we have

$$0 \leq \frac{1}{n_k^2} \leq \frac{1}{k^2}.$$

But $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (standard series)

and so $\sum_{k=1}^{\infty} \frac{1}{n_k^2}$ converges too,

by the comparison test. $\square$

Question 3.

The answer is no.

To give a satisfactory answer, you need to give a <u>specific</u> counterexample. We want an example of a strictly increasing sequence (n_k) such that $\sum_{k=1}^{\infty} 1/n_k$ diverges. We know that $\sum_{k=1}^{\infty} 1/k$ is a standard divergent series (the harmonic series). So we can take $n_k = k$ to give us our example.

(One counterexample is enough.)