The zeros of $ff'' - b$

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Abstract

We prove that if $f$ is meromorphic in the plane of order $L$, and with few poles, then the exponent of convergence of the zero sequence of $ff'' - b$ is $L$, for every non-zero constant $b$.

1 Introduction

In his influential paper [6] on differential polynomials in meromorphic functions, Hayman proved that if $f$ is transcendental and meromorphic in the plane and $n$ is an integer with $n \geq 4$ then $(f^n)'$ takes every finite non-zero value infinitely often in the plane, the same true for $n \geq 3$ if $f$ is entire. Hayman subsequently conjectured [8, Problem 1.19] that the same result holds for $n \geq 2$ and $f$ transcendental meromorphic and this question, following partial results by several authors [2, 10, 12], was settled by Bergweiler and Eremenko [1].

**Theorem 1.1** ([1]) Let $f$ be transcendental and meromorphic in the plane, let $n$ and $k$ be integers with $n > k \geq 1$, and let $b$ be a non-zero complex number. Then $(f^n)^{(k)} - b$ has infinitely many zeros.

We turn now to a related conjecture, advanced in [16].

**Conjecture 1.1** If $f$ is transcendental and meromorphic in the plane and $k$ is a positive integer and $b$ a non-zero complex number, then $ff^{(k)} - b$ has infinitely many zeros.

Obviously Theorem 1.1 shows that Conjecture 1.1 is true for $k = 1$. The examples $f(z) = R(z)e^{P(z)}$, with $R$ a rational function and $P$ a non-constant polynomial, show that
$\mathcal{F}^{(k)}$ may have finitely many zeros: for $k \geq 2$ these are the only functions meromorphic in the plane and with this property [4, 11]. Results related to Conjecture 1.1, concerning the value distribution of $f(f^{(k)})^n$ when $k \geq 1$ and $n \geq 2$, appear in [16, 18].

In the present paper we establish a strong version of Conjecture 1.1 for $k = 2$ and $f$ with few poles. Suppose first that $f$ is transcendental and meromorphic of order zero in the plane, with finitely many poles. Then it is elementary to show that so is $f f''$, and hence $f f'' - b$ has infinitely many zeros, for every finite $b$. For functions of positive order, we prove the following.

**Theorem 1.2** Let $f$ be meromorphic in the plane of positive order $L \leq \infty$, and assume that $N(r, f)$ has order less than $L$. Let $b$ be a non-zero complex number. Then the zero sequence of $f f'' - b$ has exponent of convergence $L$.

The papers [14] and [15] claim to prove Conjecture 1.1 for entire functions. However, the proofs offered appear to contain gaps: see, for example, the statement and application ([14, p.267] and [15, pp.42,45]) of results of Edrei and Fuchs [3]. In any case, Theorem 1.2 establishes a stronger conclusion for $k = 2$, namely that if the poles of $f$ are Borel exceptional then $f f''$ has no finite non-zero Borel exceptional value.

The main idea of the proof of Theorem 1.2 is as follows. Assume as a special case that $f$ is a transcendental entire function of finite order and

$$f(z) f''(z) = \frac{1}{2} + e^{P(z)},$$

with $P$ a polynomial. We prove the existence of a local solution $y(z)$ of the associated differential equation $yy'' = \frac{1}{2}$ such that $y(z) - f(z)$ is dominated by $e^{P(z)/2}$ on rays where $e^{P(z)}$ is large, but by $e^{P(z)}$ on rays where $e^{P(z)}$ is small. These asymptotics are then shown to be incompatible, using properties of harmonic measure. The proof depends fundamentally on the existence of solutions of $yy'' = \frac{1}{2}$ in a sufficiently large region, and it seems very difficult to extend the method to $k \geq 3$.

I would like to thank my wife for translating [14] and [15] for me, and to acknowledge valuable conversations with Walter Bergweiler and Rod Halburd.
2 Lemmas needed for Theorem 1.2

Lemma 2.1 Let
\[ v(u) = \int_0^u e^{t^2} \, dt, \quad \lambda = i \int_0^\infty e^{-t^2} \, dt = \frac{i\sqrt{\pi}}{2}. \] (1)

Then \( v \) is entire, with no critical values, and with two asymptotic values \( \pm \lambda \).

Lemma 2.2 Let \( z_1, w_1, C \) be complex numbers, with \( w_1 \neq 0, e^{-C} \), and define a branch of \( \phi(w) = (\log w + C)^{\frac{1}{2}} \) near \( w_1 \). Assume that, with \( v, \lambda \) as in (1),
\[ v(\phi(w_1)) = v((\log w_1 + C)^{\frac{1}{2}}) \neq \pm \lambda. \] (2)

Then there exists a solution \( Y(z) \) of the equation
\[ Y(z)Y''(z) = \frac{1}{2}, \] (3)

analytic near \( z_1 \), with
\[ Y(z_1) = w_1, \quad Y'(z_1) = \phi(w_1), \] (4)

and with the following properties. There exist complex numbers \( D, c_1, c_2 \), with
\[ \frac{(c_j - D)e^C}{2} = (-1)^j \lambda, \quad c_j \neq z_1, \] (5)

such that \( Y(z) \) admits unrestricted analytic continuation in \( \mathbb{C} \setminus \{c_1, c_2\} \), the continuations satisfying
\[ \frac{(z - D)e^C}{2} = v(Y'(z)). \] (6)

Proof. We choose \( D \) such that
\[ x_1 = \frac{(z_1 - D)e^C}{2} = v(\phi(w_1)). \] (7)

Let \( \psi \) be that branch of the inverse function \( v^{-1} \) which maps \( x_1 \) to \( \phi(w_1) \). For \( z \) near \( z_1 \), define \( Y(z) \) by
\[ Y(z) = e^{-C} \exp \left( \psi^2 \left( \frac{(z - D)e^C}{2} \right) \right). \] (8)

Then \( Y(z_1) = w_1 \), by (7) and the choice of \( \psi \). The analytic continuation of \( Y \) along any path in \( \mathbb{C} \setminus \{c_1, c_2\} \) is possible, since the only singularities of \( \psi \) in the finite plane are \( \pm \lambda \), by Lemma 2.1, and we have \( z_1 \neq c_1, c_2 \), by (2), (5) and (7). Near \( z_1 \) we have, from (8),
\[ \psi \left( \frac{(z - D)e^C}{2} \right) = (\log Y(z) + C)^{\frac{1}{2}} \]
for some choice of the logarithm and square root and so, since \( \psi(x_1) = \phi(w_1) \),
\[
\frac{(z - D)e^C}{2} = v(\phi(Y(z))) = v(\log Y(z) + C)^{\frac{1}{2}}. \tag{9}
\]

Thus, again near \( z_1 \),
\[
\frac{e^C}{2} = v'(\phi(Y))\phi'(Y)Y' = \frac{e^{\phi(Y)^2}Y'}{2\phi(Y)} = \frac{e^CY'}{2\phi(Y)}, \quad Y'(z) = \phi(Y(z)), \tag{10}
\]
which gives \( Y'(z_1) = \phi(w_1) \) and \( Y'(z)^2 = \log Y(z) + C \), from which (3) follows. Finally, (6) follows from (9) and (10).

The next lemma summarizes some facts from the Wiman-Valiron theory [9]: these are normally stated for a transcendental entire function \( F \), but obviously hold if \( F \) is a polynomial not vanishing identically, in which case the central index \( N \) is simply the degree.

**Lemma 2.3** Let \( F(z) = \sum_{n=0}^{\infty} a_n z^n \neq 0 \) be an entire function and let \( \delta_1 > 0 \). Then there exists a set \( E \) of finite logarithmic measure with the following properties. For \( r \in [1, \infty) \setminus E \), the central index \( N = \nu(r, F) \) is the largest \( n \) for which
\[
|a_n|r^n = \max\{|a_m|r^m : m \geq 0\},
\]
and \( N \) satisfies
\[
N \leq \log M(r, F)^{1+\delta_1}. \tag{11}
\]
Further, if \( |z_0| = r \) and \( |F(z_0)| \geq (1 - o(1))M(r, F) \) then for \( |\log z/z_0| \leq N^{-5/8} \) we have
\[
F'(z)/F(z) = (N/z)(1 + o(1)), \quad F''(z)/F(z) = (N/z)^2(1 + o(1)) \tag{12}
\]
and
\[
F(z) = F(z_0)(z/z_0)^N(1 + o(1)). \tag{13}
\]
Finally,
\[
M(s, F) = (s/r)^N M(r, F)(1 + o(1)) \quad \text{for} \quad |\log s/r| \leq N^{-5/8}. \tag{14}
\]

We shall, following standard terminology [9], refer to those \( r \) in \( [1, \infty) \setminus E \) as normal for \( F \) with respect to the Wiman-Valiron theory.
3 Proof of Theorem 1.2

Suppose that \( f \) is meromorphic in the plane, of positive order \( \rho(f) = L \leq \infty \), that \( N(r, f) \) has order less than \( L \), and that the zero sequence of \( ff'' - b \), for some finite, non-zero complex number \( b \), has exponent of convergence less than \( L \). There is no loss of generality in assuming that \( b = \frac{1}{2} \), and we may write

\[
f(z) = f_1(z)/f_2(z), \quad f''(z) = g_1(z)/g_2(z), \quad 2f(z)f''(z) = 1 + W(z)e^{-2g(z)},
\]

with the \( f_j, g_j \) and \( g \) entire, and \( W \) meromorphic in the plane, such that

\[
\max\{\rho(f_2), \rho(g_2), \rho(W)\} < \sigma < \tau < L.
\]  

(16)

Obviously \( f_1 \) has order \( L \), and we can assume that \( g \neq 0 \). The following lemma is standard.

**Lemma 3.1** Let \( p_j, j = 1, 2, \ldots \) be the zeros of \( f_2, g_2, W \) and \( 1/W \) in \( |z| > 1 \), with repetition according to multiplicity. Then \( \sum |p_j|^{-\sigma} < \infty \) and we have

\[
|\log |f_2(z)|| + |\log |g_2(z)|| + |\log |W(z)|| \leq |z|^{\sigma}
\]

(17)

provided \( |z| \) is large and \( z \) lies outside the union of the discs \( B(p_j, |p_j|^{-\sigma}) \). Further, there exist \( \theta_0 \in [0, 2\pi) \) and \( R_0 > 1 \) such that (17) holds for all \( z = re^{i\theta_0} \) with \( r \geq R_0 \).

**Lemma 3.2** Let \( \delta_1 \) be a small positive constant. There exist arbitrarily large positive \( r_0 \) normal for \( g \) and \( f_1 \) with respect to the Wiman-Valiron theory as in Lemma 2.3, and such that, with \( \tau \) as in (16), the following hold:

\[
\log M(r_0, f_1)^{1+\delta} > \nu(r_0, f_1) > r_0^{\tau(1+\delta_1)}; \]

(18)

\[
\log M(r_0, f_1) \leq M(r_0, g) + O(r_0^\alpha);
\]

(19)

\[
r_0^\tau = o(M(r_0, g)), \quad T(r_0, f) + T(r_0, f') + T(r_0, f'') \leq 14M(r_0, g).
\]

(20)

Further, \( g \) is transcendental if \( L = \infty \), while if \( L \) is finite then \( g \) is a polynomial and has positive degree \( L \).
Proof. We first note that by [5] there is a set $E_0$ of finite logarithmic measure such that

$$|f_2^{(j)}(z)/f_2(z)| \leq r_0^{(\sigma-1)}, \quad j = 1, 2.$$  \hfill (21)

holds for $|z| = r_0 \in E_1 = [1, \infty) \setminus E_0$. We may then choose $r_0 \in E_1$ so that (17) holds for $|z| = r_0$, and such that $r_0$ is normal for $g$ and $f_1$ with respect to the Wiman-Valiron theory, and further such that (18) holds, using (11) and (16), as well as

$$T(r_0, f') \leq 2T(r_0, f), \quad T(r_0, f'') \leq 4T(r_0, f).$$  \hfill (22)

Taking $u_0$ with $|u_0| = r_0$ and $|f_1(u_0)| = M(r_0, f_1)$, the formula

$$f_1''/f_1 = f''/f + 2((f'/f)(f_2'/f_2) + f_2''/f_2) = f''/f + 2(f_1'/f_1 - f_2''/f_2) + f_2''/f_2$$

and (12), (16), (18) and (21) give

$$|f''(u_0)/f(u_0)| = (1 + o(1))|f''_1(u_0)/f_1(u_0)| \geq r_0^{-2}.$$

Thus

$$|f_1(u_0)|^2 \leq \exp(r_0^2) \exp(2M(r_0, g))$$

from (15), (16) and (17), which gives (19). Now (20) follows, using (15), (16), (18), (19) and (22), and we obtain the final assertions of the lemma, since $\tau$ may be chosen arbitrarily close to $L$ in (18).

\[\square\]

**Lemma 3.3** Let $r_0$ be large and as in Lemma 3.2, and set $N = \nu(r_0, g)$. Let $q_j, j = 1, 2, \ldots$, be the zeros and poles of $f, f', f''$ and $W$ in $r_0^{\frac{1}{3}} \leq |z| \leq r_0 e^{-1/2N}$, repeated according to multiplicity, and define the discs $B_j$ by $B_j = B(q_j, 2M(r_0, g)^{-2})$. Then the $B_j$ have sum of radii $o(N^{-1})$, and with $d$ denoting positive constants independent of $r_0$ we have

$$|f'(z)/f(z)| + |f''(z)/f'(z)| \leq M(r_0, g)^d$$  \hfill (23)

and

$$|\log |W(z)|| + |\log |f_2(z)|| + |\log |g_2(z)|| \leq o(M(r_0, g)),$$  \hfill (24)

for all $z$ satisfying $r_0^{\frac{1}{3}} \leq |z| \leq r_0 e^{-1/N}$ and outside the union of the discs $B(q_j, M(r_0, g)^{-2})$.

Finally,

$$\log M(r_0 e^{-2/N}, f_j) + \log M(r_0 e^{-2/N}, g_j) \leq dM(r_0, g), \quad j = 1, 2.$$  \hfill (25)

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Proof. Using (16), (20) and the standard estimate
\[ N(R_2, h) \geq \int_{R_1}^{R_2} \frac{n(t, h) - n(0, h)}{t} \, dt, \quad 1 < R_1 < R_2, \]
the number of \( q_j \) is at most
\[ r_0^2 + dNM(r_0, g) \leq dNM(r_0, g). \]
Hence the sum of the radii of the \( B_j \) is \( O(NM(r_0, g)^{-1}) = o(N^{-1}) \), by (11). Using (11), (16) and (20) we then obtain (24) from the Poisson-Jensen formula, and (23) from its differentiated form \([7, \text{p.22}]\). It remains only to establish (25). For \( f_1, f_2 \) and \( g_2 \) there is nothing to prove, by (16), (19) and (20). To obtain (25) for \( g_1 \), we choose \( r_0^* \in [r_0 e^{-2/N}, r_0 e^{-1/N}] \) such that the circle \( |z| = r_0^* \) meets none of the discs \( B_j \), and apply (19), (23) and (24). \( \square \)

Lemma 3.4 Let \( \eta(r) \) be a positive function tending slowly to \( \infty \) as \( r \to \infty \), in particular so slowly that \( \eta(r) = o(\log r) \) and, if \( g \) is transcendental, \( \eta(r) = \nu(r, g)^{o(1)} \).

Let \( r_0 \) be large and as in Lemma 3.2, and set \( \eta = \eta(r_0) \) and \( N = \nu(r_0, g) \). Choose \( z_0 \) with
\[ |z_0| = r_0, \quad g(z_0) = |g(z_0)| > (1 - o(1))M(r_0, g). \tag{26} \]
Then (12) and (13) hold for \( |\log z/z_0| \leq N^{-2/3} \) and we may write
\[ g(z) = M(r_0, g)(z/z_0)^N(1 + o(1)), \quad e^{-2g(z)} = S(z)e^{-2M(r_0, g)(z/z_0)^N} \]  
for \( |\log z/z_0| \leq N^{-2/3} \), \( |z| \leq r_0 \). Here \( h(z) = S(z) \) for \( z \in H \) shall mean that
\[ \log |h(z)| \leq o(M(r_0, g)) \quad \forall z \in H. \]

Next, let \( r_1 \) satisfying \( r_0 e^{-4n/N} \leq r_1 \leq r_0 e^{-2n/N} \) be such that the circle \( |z| = r_1 \) meets none of the discs \( B_j \) of Lemma 3.3. Then for \( |z| = r_1 \) we have (23) and (24), as well as
\[ f(z) = S(z), \quad f'(z) = S(z), \quad f''(z) = S(z). \tag{28} \]

Proof. We can choose \( z_0 \) satisfying (26), by Lemma 2.3, and for \( |\log z/z_0| \leq N^{-2/3} \) we obtain at once (12) and (13), while if in addition \( |z| \leq r_0 \) we also get (27). Next, (23) and (24) hold for \( |z| = r_1 \), by Lemma 3.3. To prove (28) we note that
\[ M(r_1, g) = o(M(r_0, g)), \]

which follows from (14) if \( g \) is transcendental and is obvious if \( g \) is a non-constant polynomial, in both cases using the fact that \( \eta(r) \to \infty \). Thus writing
\[
(f'')^2 = (f'') (f'' / f)
\]
we see from (15), (23) and (24) that \( f''(z) = S(z) \) for \( |z| = r_1 \). Using (15) and (24) again, we obtain
\[
\log M(r_1, g_1) = o(M(r_0, g)).
\]
It now follows using Lemma 3.1 and (20) that \( f''(z) = S(z) \) for \( z = re^{i\theta_0} \) and for \( R_0 \leq r \leq r_1 \). Integration of \( f'' \) along the ray \( \arg z = \theta_0 \) and around the circle \( |z| = r_1 \) now gives (28).

Next, we estimate \( f \) on a radial line segment on which \( e^{-2M(r_0, g)(z/z_0)^N} \) is large.

**Lemma 3.5** Let \( \varepsilon \) be a small positive constant. Let \( r_0 \) and \( r_1 \) be large, chosen in accordance with Lemmas 3.2 and 3.4. Then there exists \( \delta \in (\varepsilon, 2\varepsilon) \) such that \( T^* = \{sz_0 \exp(\pm i\delta/N) : s > 0 \} \) meets none of the discs \( B_j \) of Lemma 3.3, and we have, for \( j = 0, 1, 2 \),
\[
f^{(j)}(z) = S(z)e^{-iM(r_0, g)(z/z_0)^N}, \quad z \in T_0 = \{z_0 se^{i\delta/N} : r_1 / r_0 \leq s \leq e^{-1/N} \}.
\]

**Proof.** We may choose \( \delta \) so that \( T^* \) meets none of the discs \( B_j \), since the \( B_j \) have sum of radii \( o(N^{-1}) \), by Lemma 3.3. On \( T_0 \) we have, by (23), (24) and (27),
\[
f''(z) / f(z) = S(z), \quad 1 + W(z)e^{-2g(z)} = 1 + S(z)e^{-2M(r_0, g)(z/z_0)^N}
\]
and \( \Re(-2i(z/z_0)^N) > 0 \) so that \( |e^{-2iM(r_0, g)(z/z_0)^N}| > 1 \). Using (15) and (29) this gives
\[
f''(z) = S(z)e^{-iM(r_0, g)(z/z_0)^N} = S(z)e^{(z/z_0)^N M(r_0, g) \sin \delta}
\]
for \( z \) in \( T_0 \). Since we have (28) on \( |z| = r_1 \), we get (30) by integration.

The next lemma gives estimates for \( f \) on regions in which \( e^{-2M(r_0, g)(z/z_0)^N} \) is small.

**Lemma 3.6** Fix large \( r_0 \) and \( r_1 \), chosen in accordance with Lemmas 3.2 and 3.4, take \( \delta \) as in Lemma 3.5, and define the logarithmic rectangle \( J_0 \) by
\[
J_0 = \{z_0 e^\zeta : \log r_1 / r_0 \leq \Re(\zeta) \leq -1/N, \quad |\Im(\zeta) + \delta/N| \leq 8M(r_0, g)^{-1}\}.
\]
Then

\[ |e^{-2iM(r_0,g)(z/z_0)}| < 1, \quad f'(z)/f(z) = S(z), \quad f''(z)/f'(z) = S(z) \quad \text{for} \quad z \in J_0, \quad (32) \]

and there exist a constant \( C \) and a branch of the logarithm such that

\[ f'(z)^2 = \log f(z) + C + S(z)e^{-2iM(r_0,g)(z/z_0)} \quad \text{for} \quad z \in J_0. \quad (33) \]

Further,

\[ f(z) = S(z), \quad f'(z) = S(z), \quad f''(z) = S(z) \quad \text{for} \quad z \in J_0. \quad (34) \]

**Proof.** We have (32) by (23), since the choice of \( \delta \) in Lemma 3.5 implies that \( J_0 \) does not meet any of the discs \( B(q_j, M(r_0, g)^{-2}) \) of Lemma 3.3. Further, multiplying (15) by \( f'/f \) and using (24), (27) and (32) gives

\[ 2f''(t)f'(t) = f'(t)/f(t) + S(t)e^{-2iM(r_0,g)(t/z_0)} \quad \text{for} \quad t \in J_0. \quad (35) \]

We may choose \( z^* \in J_0 \) with the property that for each \( z \in J_0 \) there exists a path in \( J_0 \) of length \( O(r_0) \) from \( z^* \) to \( z \) on which

\[ |e^{-2iM(r_0,g)(t/z_0)}| \leq |e^{-2iM(r_0,g)(z/z_0)}|, \]

and integrating (35) along these paths from \( z^* \) we obtain (33), with some constant \( C \). Now (28), (32) and (33) together give

\[ f'(z) = S(z), \quad \log f(z) + C = S(z) \]

for \( z \) in \( J_0 \) with \( |z| = r_1 \). Integrating \( f'/f \) and using (32) again we get \( \log f(z) + C = S(z) \) on \( J_0 \). Thus (33) now gives \( f'(z) = S(z) \) on \( J_0 \), so that integration of \( f' \) and use of (28) and (32) give (34). \( \square \)

The next step is to refine the estimate (33) on a smaller logarithmic rectangle.

**Lemma 3.7** Let

\[ J_1 = \{z_0e^{i\delta} : -2048/N \leq \text{Re}(\zeta) \leq -1/N, \quad |\text{Im}(\zeta) + \delta/N| \leq 8M(r_0, g)^{-4}\}. \quad (36) \]
Then $J_1 \subseteq J_0$ and on $J_1$ we have
\[
|e^{-2g(z)}| = S(z) |e^{-2iM(r_0,g)(z/z_0)^N}| \leq e^{-cM(r_0,g)\sin(\delta/2)} \tag{37}
\]
and
\[
f'(z) = (\log f(z) + C) \frac{1}{2}(1 + \alpha(z)), \quad \alpha(z) = S(z)e^{-2iM(r_0,g)(z/z_0)^N} = o(1). \tag{38}
\]
In (36) and henceforth we use $c$ to denote positive constants independent of $r_0$ and $\varepsilon$.

**Proof.** The estimate (37) follows at once from (27). Next, we note that for $z \in J_1$ we have $f(z) = S(z)$, by (34), and $2f(z)f''(z) = 1 + o(1)$, by (15), (24) and (37). This gives $1/f''(z) = S(z)$ and so $1/f'(z) = S(z)$, using (32). Hence (33) and (37) yield
\[
\frac{\log f(z) + C}{f(z)^2} = 1 + o(1), \quad (\log f(z) + C)^{-1} = S(z)
\]
for $z \in J_1$, which leads to (38). \qed

We now approximate $f$ by a solution of (3).

**Lemma 3.8** Let
\[
J_2 = \{z_0e^\zeta : -256/N \leq \text{Re}(\zeta) \leq -32/N, \quad |\text{Im}(\zeta) + \delta/N| \leq M(r_0,g)^{-1}\}. \tag{39}
\]
Then there exists a solution $y(z)$ of the equation (3), analytic in $J_2$, such that
\[
f^{(j)}(z) - y^{(j)}(z) = S(z)e^{-2iM(r_0,g)(z/z_0)^N}, \quad y^{(j)}(z) = S(z), \quad z \in J_2, \quad j = 0, 1, 2. \tag{40}
\]
Further, the solution $y(z)$ admits unrestricted analytic continuation in $\mathbb{C} \setminus \{c_1, c_2\}$, with $c_j$ not in $J_2$, and the continuations satisfy (6), with $v$ as in (1) and $D$ a constant.

**Proof.** Choose $z_1 = z_0e^\zeta$ with
\[
-4/N \leq \text{Re}(\zeta) \leq -2/N, \quad \text{Im}(\zeta) = -i\delta/N - i4M(r_0,g)^{-1},
\]
\[
f(z_1) \neq e^{-C}, \quad v((\log f(z_1) + C)^{1/2}) \neq \pm \lambda, \tag{41}
\]

with the same $C$ and the same branch of the logarithm and square root as in (38), and with $v$ and $\lambda$ as in (1). Let

$$J_3 = \{z_0 e^\zeta : -1024/N \leq \text{Re}(\zeta) \leq \log |z_1/z_0|, \ |\text{Im}(\zeta) + \delta/N| \leq 4M(r_0, g)^{-1}\}. \quad (42)$$

Then $J_3 \subseteq J_1$ and $z_1$ lies on the boundary of $J_3$ and there exists, for each $z \in J_3$, a path from $z_1$ to $z$ in $J_3$, of length $O(r_0)$, on which

$$|e^{-2iM(r_0, g)(z/z_0)^N}| \leq |e^{-2iM(r_0, g)(z/z_0)^N}|. \quad (43)$$

On $J_3$ we define

$$\beta(z) = z_1 + \int_{z_1}^z (1 + \alpha(t)) dt = z + S(z)e^{-2iM(r_0, g)(z/z_0)^N} = z + o(1), \quad (44)$$

using (37), (38) and (43). Further, $\beta$ is univalent on an open logarithmic rectangle $J_4$ containing $J_3$, and with $\beta(J_4) \subseteq J_1$, using (38), (44) and the fact that any two points $u_1, u_2$ in $J_4$ can be joined by a path in $J_3$ of length $O(|u_1 - u_2|)$. We write

$$f(z) = y(\beta(z)), \quad z \in J_4. \quad (45)$$

Then

$$y(z_1) = y(\beta(z_1)) = f(z_1). \quad (46)$$

Further, for $z$ in $J_4$, (38) gives

$$y'(\beta(z)) = (\log y(\beta(z)) + C)^{\frac{1}{2}},$$

from which it follows that

$$y'(z_1) = (\log y(z_1) + C)^{\frac{1}{2}} = (\log f(z_1) + C)^{\frac{1}{2}}, \quad (47)$$

and further that

$$y'(u) = (\log y(u) + C)^{\frac{1}{2}}, \quad 2y(u)y''(u) = 1, \quad u \in \beta(J_4). \quad (48)$$

Let

$$J_5 = \{z_0 e^\zeta : -512/N \leq \text{Re}(\zeta) \leq -16/N, \ |\text{Im}(\zeta) + \delta/N| \leq 2M(r_0, g)^{-1}\}. \quad (49)$$
Then for $z$ in $J_5$ the line segment from $z$ to $\beta(z)$ lies in $\beta(J_3)$, by (11), (37), (42) and (44). From (32), (38), (44) and (45) we get $y'(z)/y(z) = S(z)$ for $z \in \beta(J_3)$ and so

$$\log \frac{y(z)}{f(z)} = - \log \frac{y(\beta(z))}{y(z)} = - \int_z^{\beta(z)} \frac{y'(t)}{y(t)} dt = S(z)[|\beta(z) - z| = S(z)e^{-2M(r_0,g)(z/z_0)^{\gamma}}$$

for $z$ in $J_5$. Recalling (11) and (34) and applying Cauchy’s estimate for derivatives now yields (40) for $z$ in $J_2$.

It remains only to establish that $y(z)$ can be analytically continued as asserted. By (32), (41), (46) and (47), we may apply Lemma 2.2 with $w_1 = f(z_1)$, to obtain a solution $Y$ of (3) analytic near $z_1$, admitting unrestricted analytic continuation in $\mathbb{C} \setminus \{c_1, c_2\}$ for some $c_1, c_2$, and satisfying (4) and (6). But then (46), (47), (48) and the existence-uniqueness theorem give $Y \equiv y$ near $z_1$. Since $y$ is analytic on $J_2$, we get $c_j \notin J_2$. \hfill $\Box$

**Lemma 3.9** Choose $s_1$ with

$$-128/N \leq s_1 \leq -64/N$$

and such that (i) the circle $|z| = r_0e^{s_1}$ meets none of the exceptional discs $B_j$ of Lemma 3.3, and (ii) the singularities $c_1, c_2$ of $y(z)$ and the constant $D$ of (6) do not lie in the annulus $r_0e^{s_1-1/N} \leq |z| \leq r_0e^{s_1+1/N}$. Set

$$J = \{z_0e^{\zeta} : s_1 - \delta^{1/2}N^{-1} \leq \text{Re}(\zeta) \leq s_1 + \delta^{1/2}N^{-1}, \quad |\text{Im}(\zeta)| \leq \delta N^{-1}\}, \quad z_2 = z_0e^{s_1}.$$  \hfill (51)

Then $y(z)$ extends to an analytic function satisfying

$$y^{(j)}(z) = S(z) \quad \text{for} \quad z \in J \quad \text{and} \quad j = 0, 1, 2.$$  \hfill (52)

**Proof.** It is clearly possible to choose $s_1$ satisfying conditions (i) and (ii) of Lemma 3.9, since the discs $B_j$ of Lemma 3.3 have sum of radii $o(N^{-1})$. Obviously we have

$$J \subseteq F_0 = \{z_0e^{\zeta} : s_1 - 2\delta^{1/2}N^{-1} \leq \text{Re}(\zeta) \leq s_1 + 2\delta^{1/2}N^{-1}, \quad |\text{Im}(\zeta)| \leq 2\delta N^{-1}\}$$

and, by (39) and (50), $F_0 \cap J_2$ is non-empty. Thus the function $y$ extends to be analytic on $F_0$, using condition (ii) and Lemma 3.8. By (ii) and (51) we have

$$|z_2 - D| \geq r_0e^{s_1} \min\{1 - e^{-1/N}, e^{1/N} - 1\} \geq cr_0e^{s_1}N^{-1},$$

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whereas
\[ |z - z_2| \leq c\delta^4 r_0 e^{\alpha_1} N^{-1} \quad \text{for} \quad z \in F_0, \]
still denoting by \( c \) constants independent of \( r_0 \) and \( \delta \). Since \( \delta \) is assumed small, it follows
that the variation of \( \arg(z - D) \) on \( F_0 \) is at most \( \frac{\pi}{4} \). Using (6), we see that the image of \( F_0 \)
under \( v(y'(z)) \) fails to meet either the non-positive, or the non-negative, real axis. Thus so
does the image of \( F_0 \) under \( y'(z) \), since \( v \) is an odd entire function, positive on the positive
real axis. Hence Bloch’s theorem applied to \( \log y'(z) \) gives
\[
\left| \frac{d \log y'}{d \log z} \right| \leq \frac{cN}{\delta}, \quad z \in J. \tag{53}
\]
But \( J \cap J_2 \) is non-empty. Hence, using (40), integration now gives \( y'(z) = S(z) \) and \( y(z) = S(z) \) on \( F_1 \). Applying (53) again we obtain (52) for \( z \) in \( J \).

\[ \square \]

4  Completion of the proof

We apply the two-constants theorem for subharmonic functions \[13\] to the function
\[
G(z) = \log |f_2(z)(f(z) - y(z))e^{iM(r_0,g)(z/z_0)^N}| - \log |(f_1(z) - f_2(z)y(z))e^{iM(r_0,g)(z/z_0)^N}|
\]
on the region \( J \) defined in (51). On the boundary of \( J \) we have
\[
G(z) \leq cM(r_0,g),
\]
using (25), (51), (52) and denoting by \( c \) constants independent of \( r_0 \) and \( \delta \). On the intersection
of \( \partial J \) with the ray \( z = z_0 e^{i\zeta}, \text{Im}(\zeta) = i\delta N^{-1} \), we have \( \text{Im}((z/z_0)^N) > 0 \) and
\[
G(z) \leq o(M(r_0,g)),
\]
using (25), (30), and (52). On the other hand, on the intersection \( F \) of \( \partial J \) with the ray
\( z = z_0 e^{i\zeta}, \text{Im}(\zeta) = -i\delta N^{-1} \) we have, by (40) and (50),
\[
|(f(z) - y(z))e^{iM(r_0,g)(z/z_0)^N}| = |S(z)e^{-iM(r_0,g)(z/z_0)^N}| = S(z)e^{-cM(r_0,g)\sin \delta},
\]
and so, using (25),
\[
G(z) \leq -cM(r_0,g) \sin \delta.
\]

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By Carleman’s estimate for harmonic measure [17, Lemma E], each circular arc $T_j$ of $\partial J$ satisfies
\[
\omega(z_2, \Omega, T_j) \leq \exp(-c\delta^{-\frac{1}{2}}),
\]
in which $z_2$ is given by (51) and $\Omega$ is the interior of $J$. Hence by the symmetry of $\Omega$ we have $\omega(z_2, \Omega, F) > \frac{1}{4}$ if $\delta$ is small enough. Thus
\[
G(z_2) \leq cM(r_0, g) \exp(-c\delta^{-\frac{1}{2}}) + o(M(r_0, g)) - cM(r_0, g) \sin \delta.
\]
Since $\delta$ may be chosen arbitrarily small we obtain, using (24) and the choice of $s_1$,
\[
|f(z_2) - y(z_2)| \leq \exp(-cM(r_0, g) \sin \delta). \tag{54}
\]
In the same way we get
\[
|f''(z_2) - y''(z_2)| \leq \exp(-cM(r_0, g) \sin \delta),
\]
this time using the function
\[
G_1(z) = \log |g_2(z)(f''(z) - y''(z))e^{iM(r_0, g)(z/z_0)N}| = \log |(g_1(z) - g_2(z)y''(z))e^{iM(r_0, g)(z/z_0)N}|.
\]
Since
\[
\frac{1}{2} We^{-2ig} = f f'' - \frac{1}{2} = f f'' - y y'' = f(f'' - y'') + y''(f - y),
\]
by (15) and (48), this gives, using (52) and (54) again,
\[
|W(z_2)^{-1} e^{2ig(z_2)}| \geq \exp(cM(r_0, g) \sin \delta).
\]
But (24), (27), (50), (51) and the choice of $s_1$ give
\[
\text{Im}(g(z_2)) = o(M(r_0, g)), \quad e^{2ig(z_2)} = S(z_2), \quad W(z_2)^{-1} = S(z_2),
\]
and we have a contradiction. Theorem 1.2 is proved.

References


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