We may assume that 0 is a deficient value of $F$. Taking a sequence $s_k$ tending to infinity such that the circles $|z| = s_k$ do not meet the $C^0$ set $C_3$, and such that

$$2s_k \leq s_{k+1} \leq 4s_k,$$

for each $k$, we may choose a path $\Gamma$ tending to infinity in one of the $P_{1,j}$ and avoiding $C_3$, such that

$$\log |F(z)| < -cT(|z|, F)$$

as $z$ tends to infinity on $\Gamma$, for some positive constant $c$, which may be chosen arbitrarily close to $\pi$. This path $\Gamma$ may be formed so as to consist of radial segments joining $|z| = s_k$ to $|z| = s_{k+1}$ and, if necessary, arcs of the circles $|z| = s_k$, each of length $o(s_k)$, joining these radial segments. The function $g$ maps $\gamma$ onto a path $g \Gamma$ which tends to infinity, by Lemma 6, and Lemma 2 implies that $g \Gamma$ must pass through annuli of the form

$$A_1 = \{ z : K^{-1}R \leq |z| \leq KR \},$$

with $K$ a large positive constant and $R$ arbitrarily large, on which

$$f(\zeta) = \alpha \zeta^n(1 + o(1)), \quad \zeta f'(\zeta)/f(\zeta) = n + o(1),$$

in which $\alpha \neq 0$ and $n$ is an integer, both possibly depending on the annulus $A_1$. We choose $z_5$ on $\Gamma$ such that $|g(z_5)| = R$. By Lemma 9 and the construction of $\Gamma$, we may now choose $z_6$ lying on $\Gamma$ such that $z_6 = (1 + o(1))z_5$ and $|g(z_6)| = R(1 + o(1))$, while the circle $|z| = s = |z_6|$ does not meet $C_3$. Because of the choice of $s$, we may assume that

$$\log |F(z)| < -(c/2)T(s, F), \quad z \in \Omega' = \{ u : u = z_6e^{it}, \quad -\sigma_1 < t < \sigma_1 \},$$

using $\sigma_j$ to denote positive constants not depending on $R$ or $s$. By Lemma 9 again, $g(z)$ lies in $A_1$ for $z$ lying on a subpath $\Gamma'$ of $\Gamma$, which contains $z_6$ and joins the circles $|z| = K^{-\sigma_2}s$ and $|z| = K^{\sigma_2}s$. Because

$$f(g(z)) = (1 + o(1))Q(z) = (1 + o(1))\beta z^d, \quad d \neq 0,$$

on $\Gamma$, we deduce that $n \neq 0$ in (3). Consequently, a branch $\psi$ of the inverse function $f^{-1}$ defined near $G(z_6)$ admits unrestricted analytic continuation in the annulus

$$A_2 = \{ w : 2K^{-1}|\alpha|R^n \leq |w| \leq (1/2)KR^n \},$$

taking values in $A_1$. Further, by (5), we have

$$|Q(z_6)| = (1 + o(1))|Q(z_5)| = (1 + o(1))|\alpha|R^n,$$

and the image of the annulus

$$A_3 = \{ z : s/4 \leq |z| \leq 4s \}$$

under $Q$ lies in $A_2$, provided $K$ was chosen large enough. In addition, with $\psi = f^{-1}$ the same branch of the inverse as above, $f^{-1}(Q(z))$ admits unrestricted analytic continuation in $A_3$, starting at $z_6$, such that

$$|f^{-1}(Q(z))| \leq KR = (1 + o(1))K|g(z_6)| \leq s^{\sigma_3}$$

there, using (44) of the paper. Moreover, the derivative $\psi' = (f^{-1})'$ may be defined on a simply-connected subdomain $D^*$ of the annulus $A_2$, this subdomain containing the images of $\Omega'$ under $G$ and $Q$, with

$$|w\psi'(w)/\psi(w)| = |\zeta f'(\zeta)/f(\zeta)|^{-1} \leq 2,$$

and

$$1$$
and so
\[ |\psi'(w)| \leq 2|\psi(w)/w| \leq 4K R K \|\alpha R^n\|^{-1} \leq s^{\sigma_d}, \quad w \in D^*, \]
by (7) and (9). We thus have, for \( z \in \Omega' \), using (4) and (10),
\[
|g(z) - f^{-1}(Q(z))| = |f^{-1}(G(z)) - f^{-1}(Q(z))| \leq \\
\leq \exp\left(-\frac{c}{4} T(s, F)\right).
\]
We define
\[ g_1(u) = g(z_0 u^2), \quad f_1(u) = f^{-1}(Q(z_0 u^2)) \]
on the simply-connected domain
\[ D_1 = \{u : |\log |u|| \leq \log 2, \quad |\arg u| < \pi\}. \]
Since
\[ \log |g_1(u)| \leq \log M(4s, g) = o(T(s, F)) \]
and since (9) ensures that
\[ \log |f_1(u)| \leq O(\log s) \]
on \( D_1 \), (11) and the same argument as used at the end of the proof of Lemma 6 show that
\[ g_1(u) - f_1(u) = o(1), \text{ and so } \log |g(z_0 u^2)| \leq O(\log s), \text{ for } |u| = 1, \quad |\arg u| \leq \pi/2. \]This gives
\[ \log M(s, g) = O(\log s) \]
and contradicts the hypothesis that \( g \) is transcendental.