Critical Points of Certain Discrete Potentials*

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We prove some results on critical points of potentials arising from discrete distributions of charge in the plane and in space, and on zeros of certain associated meromorphic functions.

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1. INTRODUCTION

A number of recent papers [2,3,10] have concerned the following conjecture [2].

CONJECTURE 1.1 [2] Let \( a_1, a_2, \ldots \) be positive real constants and let

\[
    f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}, \quad z \in \mathbb{C}, \quad z_k \to \infty, \quad \sum_{z_k \neq 0} \frac{|a_k|}{|z_k|} < \infty. \tag{1}
\]

Then \( f \) has infinitely many zeros.

Conjecture 1.1 has the following physical interpretation. If an infinite wire carrying a charge density \( a_k \) is placed perpendicular to the complex plane at each \( z_k \), then the resulting electrostatic field [8, p. 10] is given by the vector \( 2(\text{Re}(f(z)), -\text{Im}(f(z))) \). The conjecture then asserts that there must be infinitely many equilibrium points in the plane, such points being critical points of the associated subharmonic potential

\[
    u(z) = \sum_{k=1}^{\infty} a_k \log |1 - z/z_k|. \tag{2}
\]

* Dedicated to the memory of Matts Essén, a selfless mathematician and dear friend.

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It is known that Conjecture 1.1 is true when \( \sum_{|z_k| \leq r} a_k = o(\sqrt{r}) \) as \( r \to \infty \) [2, Theorem 2.10] (see also [4, p. 327]), and when \( \inf \{a_k\} > 0 \) [3] (see also [10]).

For the analogous problem involving positive charges \( a_k \) at the points \( x_k \in \mathbb{R}^3 \) the potential is given by

\[
u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|}, \quad a_k \geq 1, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty.
\]

and the equilibrium points correspond to critical points of \( u \), i.e. zeros of \( \nabla u \). The following theorem is proved in [2, Theorem 2.11].

**Theorem 1.1** [2] Let \( x_k \in \mathbb{R}^3 \), with \( \lim_{k \to \infty} |x_k| = \infty \), and let

\[
u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|}, \quad a_k \geq 1, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty.
\]

Then \( u \) has infinitely many critical points i.e. the equation \( \nabla u = 0 \) has infinitely many solutions in \( \mathbb{R}^3 \).

The higher dimensional analogue of Theorem 1.1 will be discussed in Section 3. We shall prove the following theorem, the main feature of which is the elimination of the requirement that all the \( a_k \) be at least 1.

**Theorem 1.2** Let \( x_k \in \mathbb{R}^m \), \( m \geq 3 \), with \( \lim_{k \to \infty} |x_k| = \infty \), and let

\[
u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, \quad a_k > 0, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} < \infty. \quad (3)
\]

Suppose further that \( \phi(t) \) is continuous, positive and non-decreasing on \( [1, \infty) \), with

\[
\lim_{t \to \infty} \phi(t) = \infty, \quad \int_1^{\infty} \frac{1}{t \phi(t)} \, dt = \infty, \quad (4)
\]

and that the number \( \pi(r) \) of \( x_k \) lying in \( |x| \leq r \) satisfies, as \( r \to \infty \),

\[
\log \pi(r) = O(\phi(r)) \quad (m = 3), \quad \left( \frac{\pi(r)^{m-3}}{r} \right)^{m-3} = O(\phi(r)) \quad (m \geq 4). \quad (5)
\]

Then \( u \) has infinitely many critical points.

The proof of Theorem 1.2 combines the method of [2] with the Cartan lemma [6, p. 366]. For \( m = 3 \) the choice \( \phi(r) = 1 + \log r \) in (4) corresponds to the sequence \( (x_k) \) having finite exponent of convergence. Since this follows automatically from (3) if \( \inf \{a_k\} > 0 \) it is clear that Theorem 1.2 for \( m = 3 \) represents a strong improvement of Theorem 1.1. We make some remarks about the case \( m \geq 4 \) in Section 3.

We return now to meromorphic functions of the form (1). A natural generalization of Conjecture 1.1 concerns the existence of zeros of the function \( f \) in (1), when the \( a_k \) are allowed to be complex rather than real and positive. In particular the result
from [4, p. 327] cited above shows that if \( f \) is given by (1) and has finite lower order, and if

\[ a_k \in \mathbb{C} \setminus \{0\}, \quad \sum |a_k| < \infty, \quad \sum a_k \neq 0, \]

then \( f \) has infinitely many zeros. Examples (see e.g. [10]) show that the hypothesis \( \sum a_k \neq 0 \) is not redundant here. A number of results for complex \( a_k \) were proved in [10]. In particular it was shown [10, Theorem 1.3] that if \( f \) has finite order and is given by (1) with all but finitely many of the \( a_k \) real and positive, and if

\[ \sum |a_k| = \infty, \quad \liminf_{r \to \infty} \frac{T(r,f)}{r} < \infty, \]

then \( f \) has infinitely many zeros. This was achieved by considering zeros of \( f(z) - S(z) \), with \( S \) a rational function, rather than of \( f \). Using this device, a finite number of terms with \( a_k \) non-positive may be subsumed into \( S \), and a refinement of the method of [3] may be applied. Further results for complex \( a_k \) were proved in [10], using a Fourier series method from [12], quasiconformal surgery [13], and Baernstein’s spread theorem [1], but with strong assumptions on the order of \( f \) and the arguments of the non-positive \( a_k \). We will prove the following theorem, in which the notation is that of [5].

**Theorem 1.3** Let

\[
f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k} + \sum_{k=1}^{\infty} \frac{b_k}{z - w_k},
\]

in which each of the sequences \((z_k),(w_k)\) tends to infinity without repetition, and

\[ a_k > 0, \quad b_k \in \mathbb{C}, \quad \sum_{k=1}^{\infty} \frac{a_k}{|z_k|} + \sum_{k=1}^{\infty} |b_k| |w_k|^N < \infty, \]

for some positive integer \( N \). Assume that

\[ \rho(f) = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} < \frac{2N}{3}, \quad \liminf_{r \to \infty} \frac{T(r,f)}{r} < \infty, \]

and that

\[ \sum_{k=1}^{\infty} a_k = \infty. \]

Let \( S \) be a rational function. Then \( f(z) - S(z) \) has infinitely many zeros.

It seems plausible that the convergence condition on the \( b_k \) in (7) could be weakened to \( \sum |b_k| < \infty \). This is certainly true if

\[ \liminf_{r \to \infty} \frac{T(r,f)}{\sqrt{r}} = 0, \]
because in this case if \( f - S \) has finitely many zeros the cos \( \pi \lambda \) theorem \([6]\) applied to \(1/(f - S)\) and the residue theorem give a sequence \( r_k \to \infty \) such that

\[\sum_{|z| < r_k} a_k + \sum_{|w_k| < r_k} b_k = O(1),\]

contradicting (9).

2. PROOF OF THEOREM 1.2

For \( x_0 \) in \( \mathbb{R}^m \) or \( \mathbb{C} \) and for \( r > 0 \) we denote by \( D(x_0, r) \) and \( S(x_0, r) \) the open ball/disc and sphere/circle respectively, of centre \( x_0 \) and radius \( r \). We proceed as in \([2]\), and first establish the following lemma.

**Lemma 2.1** With the hypotheses of Theorem 1.2, we have

\[\liminf_{r \to \infty} M(r) = 0, \quad M(r) = \max\{u(x): |x| = r\}.\]  

**Proof** The proof of Lemma 2.1 is based on the well known Cartan lemma, in the form proved in \([6, \text{pp. 366–367}]\). For \( x \in \mathbb{R}^m \) and positive \( r, t \) let

\[\mu(x, r, t) = \sum_{|x_k| \leq r, |x - x_k| \leq t} a_k, \quad n(r) = \sum_{|x_k| \leq r} a_k = \mu(0, r, r).\]  

Let \( r \) be large and positive, and apply \([6, \text{pp. 366–367}]\) with

\[M = n(r), \quad h = \frac{r}{192}, \quad A = 6.\]

Then \([6, (6.5.17), \text{p. 367}]\) gives a union \( E \) of balls, having sum of radii at most

\[Ah = \frac{r}{32},\]

such that if \( x \) lies outside \( E \) then

\[\mu(x, r, t) \leq \frac{Mt}{eh} = \frac{192n(r)t}{er}, \quad 0 < t < \infty.\]

Further, let \( F \) be the union of open balls defined by

\[F = \bigcup_{|x_k| \leq r} D\left(x_k, \frac{r}{32N}\right), \quad N = n(r).\]
Then the balls of $E \cup F$ have sum of diameters at most $r/8$, and we choose $r' \in [r/2, 3r/4]$ such that the sphere $|x| = r'$ does not meet $E \cup F$.

For $|x| = r'$ the last condition of (3) gives

$$\sum_{|x_k| > r} \frac{a_k}{|x - x_k|^{m-2}} \leq \sum_{|x_k| > r} \frac{4^{m-2}a_k}{|x_k|^{m-2}} = o(1)$$

as $r \to \infty$. Further, for $|x| = r'$ we have

$$\sum_{|x_k| \leq r} \frac{a_k}{|x - x_k|^{m-2}} \leq \int_{r/64N}^{2r} t^{2-m} d\mu(x, r, t) \leq \frac{n(r)}{(2r)^{m-2}} + (m - 2)I(x, r),$$

in which, using (12),

$$I(x, r) = \int_{r/64N}^{2r} \frac{\mu(x, r, t)}{t^{m-1}} dt \leq \frac{192n(r)}{er} \int_{r/64N}^{2r} t^{2-m} dt = \frac{192n(r)}{er} J(r).$$

Clearly

$$J(r) = \log(128N) \quad (m = 3), \quad J(r) \leq \left(\frac{64N}{r}\right)^{m-3} \quad (m \geq 4).$$

Since (3) and (11) give

$$\infty > \sum_{k=1}^{\infty} \frac{a_k}{|x_k|} = \int_0^\infty t^{-1}dn(t) \geq \int_1^\infty \frac{n(t)}{t^2} dt,$$

it follows using (4) that there exists a sequence $r_j \to \infty$ with

$$n(r_j) = o\left(\frac{r_j}{\phi(r_j)}\right),$$

and (5), (13), (14), (15) and (16) now give, with $r'_j$ defined in the obvious way,

$$M(r'_j) \leq o(1) + o\left(\frac{J(r_j)}{\phi(r_j)}\right) = o(1).$$

To prove Theorem 1.2, it obviously suffices to prove the following lemma, which is based on the method of [2].

**Lemma 2.2** Let $\varepsilon > 0$, and let $u$ satisfy the hypotheses of Theorem 1.2. Then there exists a critical point $x_0$ of $u$ with $0 < u(x_0) < \varepsilon$.  


Proof. Using Lemma 2.1, choose \( r, s, t \) satisfying the following:

(i) we have, with \( m(R) = \min\{u(x) : |x| = R\}, \)

\[
0 < r < s < t, \quad M(r) < \varepsilon, \quad M(t) < m(s);
\]  

(ii) there exist \( \alpha, \beta \in \mathbb{N} \) such that

\[
|x_{\alpha}| < r < |x_{\beta}| < s.
\]

Since \( u(x_{\alpha}) = u(x_{\beta}) = \infty \), the points \( x_{\alpha}, x_{\beta} \) obviously belong to distinct components, say \( C_{\alpha} \) and \( C_{\beta} \), of the set \( \{x : u(x) > M(r)\} \). Since \( u \) is superharmonic we have

\[
M(r) \geq m(r) > m(s) > M(t),
\]

and there is a component \( D \) of the set \( \{x : u(x) > m(s)\} \) which contains the ball \( D(0,s) \) and whose closure \( \overline{D} \) lies in \( D(0,t) \), and we have \( C_{\alpha} \cup C_{\beta} \subseteq D \).

Assume that \( u \) has no critical point \( x \) with \( m(s) \leq u(x) \leq M(r) \). Let \( H \) be the set of all \( h \in [m(s), M(r)] \) with the property that \( x_{\alpha}, x_{\beta} \) lie in distinct components \( U_{\alpha}(h), U_{\beta}(h) \) of the set \( \{x : u(x) > h\} \) and let \( h^* = \inf H \).

We assert that \( h^* \in H \). If not then \( h^* < M(r) \) and \( x_{\alpha}, x_{\beta} \) lie in the same component \( C^* \) of the set \( \{x : u(x) > h^*\} \) and may therefore be joined by a path on which \( u(x) \geq h^{**} > h^* \). This contradicts the fact that there must be elements of \( H \) in \( (h^*, h^{**}) \).

Hence \( h^* \in H \). In particular, since \( D(0,s) \subseteq D \), we have \( m(s) \notin H \) and so \( h^* > m(s) \).

Let \( 0 < \delta < h^* - m(s) \). Since there are no critical points with \( u(x) = h^* \) we have

\[
4\sigma = \text{dist} \{U_{\alpha}(h^*), U_{\beta}(h^*)\} > 0,
\]

and for each \( v_1 \in \partial U_{\alpha}(h^*) \) there is a neighbourhood \( V_1 \) of \( v_1 \) on which \( u(x) > h^* - \delta \), and of diameter at most \( \sigma \), and such that \( \{x \in V_1 : u(x) > h^*\} \) is connected (to see this, denote coordinates in \( \mathbb{R}^m \) by \( t_1, \ldots, t_m \), assume without loss of generality that \( \partial u/\partial t_1 \neq 0 \) at \( v_1 \), apply the inverse function theorem to the \( C^1 \) function \( L = (u, t_2, \ldots, t_m) \), and let \( V_1 \) be the pre-image of a ball of small radius centred at \( L(v_1) \)). Hence

\[
\{x \in V_1 : u(x) \geq h^*\} = V_1 \cap \overline{U_{\alpha}(h^*)}.
\]

Cover the compact set \( \partial U_{\alpha}(h^*) \) by finitely many such neighbourhoods \( V_1, \ldots, V_\rho \), and set

\[
V = U_{\alpha}(h^*) \cup V_1 \cup \ldots \cup V_\rho.
\]

Then \( u(x) > h^* - \delta > m(s) \) on \( V \), so that \( V \subseteq D \), and \( \partial V \cap U_{\alpha}(h^*) = \emptyset \), and hence \( \max\{u(x) : x \in \partial V\} < h^* \). Doing the same for \( U_{\beta}(h^*) \) we obtain \( h \in (m(s), h^*) \) and distinct components \( U_{\alpha}(h), U_{\beta}(h) \subseteq D \) containing \( x_{\alpha}, x_{\beta} \) respectively, and this is obviously a contradiction.
3. THE HIGHER DIMENSIONAL ANALOGUE OF THEOREM 1.1

The following theorem is stated in [2, Theorem 2.11]. The case $m = 3$ is Theorem 1.1.

**THEOREM 3.1** [2]

Let $x_k \in \mathbb{R}^m$, $m \geq 3$, with $\lim_{k \to \infty} |x_k| = \infty$, and let

$$u(x) = \sum_{k=1}^{\infty} \frac{a_k}{|x - x_k|^{m-2}}, \quad a_k \geq 1, \quad \sum_{k=1}^{\infty} \frac{a_k}{|x_k|^{m-2}} < \infty.$$  \hfill (18)

Then $u$ has infinitely many critical points in $\mathbb{R}^m$.

For $m \geq 4$ our convergence criterion for the sum in (3) is stronger than that in (18), although our Theorem 1.2 does not require $a_k \geq 1$. The proof of Theorem 3.1 in [2] is based on the observation that the positive superharmonic function $u$ in (18) satisfies

$$u(x) \leq \sum_{k=1}^{\infty} \frac{a_k}{|x| - |x_k|^{m-2}},$$

and on the following lemma [2, Lemma 8.1].

**LEMMA 3.1** [2]

Suppose that $m \geq 3$ is an integer, that $0 < x_1 < x_2 < \ldots$, and that

$$\sum_{k=1}^{\infty} a_k x_k^{2-m} < \infty, \quad a_k \geq 1 \ \forall k.$$ \hfill (19)

Then

$$\liminf_{x \to +\infty} \Psi(x) = 0, \quad \Psi(x) = \sum_{k=1}^{\infty} a_k |x - x_k|^{2-m}.$$ \hfill (20)

Lemma 3.1 is, however, false for $m \geq 4$, as the example $a_k = 1, x_k = k$ shows. Here (19) is satisfied, but we have $\Psi(x) \geq 1$ for $x > 0$, so that (20) fails. Indeed, for $m \geq 4$ and $u(x)$ as in (18), with $a_k = 1, x_k = (k, 0, \ldots, 0)$, it is clear that

$$\lim_{r \to \infty} m(r) = 0, \quad \liminf_{r \to \infty} M(r) \geq 1.$$ 

Hence the proof of Theorem 3.1 in [2] is only complete for $m = 3$, and the case $m \geq 4$ would seem to require methods different to those used in [2] and in the present paper.

4. LEMMAS REQUIRED FOR THEOREM 1.3

We need a number of lemmas. The main content of the first lemma is a slightly more precise version of [11, Lemma 2].
Lemma 4.1 Let $G$ be transcendental and meromorphic in the plane, and let $\alpha \in (1, \infty)$. For $w \in \mathbb{C}$ and positive $r$ and $R$, let $L(r, w, R, G)$ denote the length of the level curves $|G(z)| = R$ lying in $D(w, r)$, and set $L(r, R, G) = L(r, 0, R, G)$. Let $\psi(t)$ be continuous, positive and non-decreasing on $[1, \infty)$ such that
\[ \int_1^\infty \frac{1}{t\psi(t)} \, dt < \frac{\log \alpha}{4}. \] (21)

Then if the positive constant $S$ is large enough there exist uncountably many $R \in (S, 2S)$ such that
\[ L(r, R, G)^2 \leq cr^2 \psi(\alpha r)(T(\alpha r, G) + \log S), \quad r \geq 1, \] (22)
in which $c$ is a positive constant depending only on $\alpha$.

If in addition the sequence $(w_k)$ satisfies, for some $\rho \in (0, \infty)$,
\[ |w_k| > 2, \quad \sum |w_k|^{-\rho} < 1, \] (23)
then there exist uncountably many $R \in (S, 2S)$ such that (22) holds and
\[ \int_{|w_k|^{-5\rho}}^{1/2} \frac{dL(t, w_k, R, G)}{t} \leq C_1(T(2|w_k|, G) + \log S)^{1/2}|w_k|^\rho \log |w_k| \] (24)
for each $k$, in which $C_1$ is a positive constant depending only on $\rho$.

Proof We use the length-area inequality as in [7, Theorem 2.1, p. 29] (see also [15, p. 44]). Let $\Delta$ be an open disc in $\mathbb{C}$ of area $A$. Then
\[ \int_S^{2S} \frac{L(\Delta, R, G)^2}{p(\Delta, R, G)R} \, dR \leq 2\pi A, \] (25)
in which $L(\Delta, R, G)$ is the length of the curves $|G(z)| = R$ in $\Delta$ and
\[ p(\Delta, R, G) = \frac{1}{2\pi} \int_0^{2\pi} n(\Delta, Re^{i\phi}, G) \, d\phi, \] (26)
where $n(\Delta, a, G)$ is the number of roots of $G(z) = a$ in $\Delta$, counting multiplicity.

Denote by $c_j$ positive constants depending only on $\alpha$. Set $\beta = \sqrt{\alpha}$ and $r_q = \beta^q, q = 0, 1, 2, \ldots$. Then (21) gives
\[ \sum_{q=1}^{\infty} \frac{1}{\psi(r_q)} \leq \frac{1}{\log \beta} \int_1^\infty \frac{1}{t\psi(t)} \, dt < \frac{1}{2}. \] (27)
Assume that $S$ is large. Then for $\phi$ real and for $S \leq R \leq 2S$ we have $\infty \geq |G(0) - Re^{i\phi}| \geq 1$. This gives, for $r \geq 1$,

$$
n(D(0, r), Re^{i\phi}, G) \leq n(r, Re^{i\phi}, G) \leq c_0 N(\beta r, 1/(G - Re^{i\phi}))$$

$$
\leq c_0 T(\beta r, G - Re^{i\phi}) + C$$

$$
\leq c_0 T(\beta r, G) + c_0 \log R + c_0 \log 2 + C$$

$$
\leq c_0 T(\beta r, G) + c_1 \log S.$$

Here the constant $C$ only arises if $G(0) = \infty$. Substituting this estimate into (25) and (26) gives, for $r \geq 1$,

$$
\int_{S}^{2S} \frac{L(r, R, G)^2}{R} dR \leq c_2 r^2 (T(\beta r, G) + \log S).
$$

Hence if the positive constant $c_3$ is chosen large enough then for each $q \in \mathbb{N}$ there exists a subset $E_q$ of $(S, 2S)$ with

$$
\int_{E_q} \frac{dR}{R} < \log \frac{2}{\psi(r_q)}
$$

such that for all $R \in (S, 2S) \setminus E_q$ and for $r_{q-1} < r \leq r_q$ we have

$$
L(r, R, G)^2 \leq c_3 r_q^2 \psi(r_q) (T(\beta r_q, G) + \log S)
$$

$$
\leq c_4 r^2 \psi(\alpha r_q) (T(\alpha r, G) + \log S).
$$

Since (27) and (28) give

$$
\int_{U_{\epsilon=1}E_q} \frac{dR}{R} < \log \frac{2}{\psi(r_q)}
$$

(22) follows.

Suppose now that $\rho$ and the sequence $(w_k)$ satisfy (23), and denote by $C_j$ positive constants depending only on $\rho$. Then

$$
\int_{|w_k|^{1-p}}^{1} \frac{dL(t, w_k, R, G)}{t} \leq \frac{2L(|w_k|/2, w_k, R, G)}{|w_k|} + \int_{|w_k|^{1-p}}^{1} \frac{L(t, w_k, R, G)}{t^2} dt
$$

for $S < R < 2S$. From the Cauchy-Schwarz inequality and (25) we obtain

$$
\left( \int_{S}^{2S} \frac{L(\Delta, R, G)}{R} dR \right)^2 \leq 2\pi A \int_{S}^{2S} \frac{\rho(\Delta, R, G)}{R} dR.
$$

(31)
Provided \( S \) is chosen large enough we may write
\[
p(D(w_k, t), R, G) \leq \frac{1}{2\pi} \int_0^{2\pi} n(3|w_k|/2, Re^{i\phi}, G)d\phi \leq C_2(T(2|w_k|, G) + \log S)
\]
for each \( k \) and for \( 0 < t \leq |w_k|/2 \), so that (31) yields
\[
\int_S^{2S} L(t, w_k, R, G) \frac{dR}{R} \leq C_3 t(T(2|w_k|, G) + \log S)^{1/2}
\]
for \( 0 < t \leq |w_k|/2 \). Substituting this estimate into (30) gives
\[
\int_S^{2S} \int_{|w_k|^{-5\rho}}^{2S} \frac{dL(t, w_k, R, G) dR}{t R} \leq \left( C_4 + C_5 \int_{|w_k|^{-5\rho}}^{2S} \frac{dt}{t} \right)(T(2|w_k|, G) + \log S)^{1/2}.
\]

It follows that, for all \( R \in (S, 2S) \) outside a set \( F_k \) with
\[
\int_{F_k} \frac{dR}{R} < \frac{\log 2}{2|w_k|^2},
\]
we have (24), and the result follows using (23) and (29).

The next lemma is a standard application of estimates for harmonic measure [14, p. 116].

**Lemma 4.2**  Let \( u \) be subharmonic and non-constant in the plane, let \( U \) be a domain such that \( u \equiv 0 \) on \( \partial U \), and let \( w_1 \in U \) be such that \( u(w_1) > 0 \). For \( t > 0 \) let \( \theta_U^*(t) \) be the angular measure of the intersection of \( U \) with the circle \( |z| = t \), except that \( \theta_U^*(t) = \infty \) if the whole circle \( |z| = t \) lies in \( U \). Then
\[
\int_{2|w_1|}^{r} \frac{\pi dt}{|\theta_U^*(t)|} \leq \log B(2r, u) + O(1), \quad r \to \infty,
\]
in which \( B(2r, u) = \sup\{u(z) : |z| = 2r\} \).

We need next a refinement of [9, Lemma 2.4].

**Lemma 4.3**  Let \((w_j)\) be a complex sequence such that \( w_j \to \infty \) without repetition, and with \(|w_j| > 2\), and let \( N_1 > 0 \) be such that
\[
\sum |w_j|^{-N_1} < \infty. \tag{32}
\]

Let \( G \) be transcendental and meromorphic in the plane, of order less than \( \rho < \infty \), and with finitely many poles. Let
\[
N_2 \geq 3N_1 + 2\rho. \tag{33}
\]
For \( m = 1, 2 \), let \( H_m \) be the union of the closures of the \( D(w_j, |w_j|^{-N_m}) \), and let \( E \) be the set of \( t \in [1, \infty) \) such that the circle \( S(0, t) \) meets \( H_1 \).

Next, let \( R_1 > 4 \) be so large that

\[
G^{-1}((\infty)) \subseteq D\left(0, \frac{R_1}{2}\right), \quad M(R_1, G) = \max\{|G(z)| : |z| = R_1\} > e,
\]

and

\[
\log |G(z)| \leq \frac{|z|^\rho}{2} \quad \text{for} \quad |z| \geq R_1,
\]

and

\[
\left(\frac{R_1}{2}\right)^{N_2 - N_1} > 4, \quad \sum_{|w_j| \geq \frac{1}{2}R_1} 26|w_j|^{-N_1} < 1.
\]

Let \( v_0 \) lie outside \( H_1 \), with

\[
|v_0| > R_1, \quad |G(v_0)| > M_1^2, \quad M_1 > M(R_1, G)^2,
\]

and let \( A \) be the component of the set \( \{z \in \mathbb{C} \setminus H_2 : |G(z)| > M_1\} \) in which \( v_0 \) lies. Then \( A \) is unbounded and we have, as \( R \to \infty \) with \( R \not\in E \),

\[
\int_{[2|v_0|, R/2] \setminus E} \frac{\pi dt}{t^{\theta_A}(t)} \leq \log \log M(R, G) + O(1).
\]

**Proof** Let \( R \) be large and positive, not in \( E \). Let \( U \) be the component of \( A \cap D(0, R) \) in which \( v_0 \) lies.

Set \( U = U_0 \), and form domains as follows. If \( U_q \) has been formed, choose (if possible) a \( w_j \) such that \( S(w_j, |w_j|^{-N_2}) \) meets the boundary \( \partial U_q \) and \( S(w_j, \lambda_q) \subseteq U_q \) for some \( \lambda_q \) with \( |w_j|^{-N_2} \leq \lambda_q \leq |w_j|^{-N_1} \). If such a \( w_j \) exists, we set \( U_{q+1} = U_q \cup D(w_j, \lambda_q) \), while if no such \( w_j \) exists, we halt the process. Since \( w_j \) tends to infinity with \( j \), the process terminates after the formation of some \( U_n \), and we set \( W = U_n \).

It follows easily that \( U_q \subseteq D(0, R) \) and \( \partial U_{q+1} \subseteq \partial U_q \) for each \( q \). Thus \( \partial W \) is contained in a union of arcs of \( S(0, R) \), and arcs of level curves \( |G(z)| = M_1 \), as well as arcs of circles \( S(w_j, |w_j|^{-N_2}) \). Further, \( W \subseteq U \cup H_1 \subseteq A \cup H_1 \).

Suppose that \( S_j = S(w_j, |w_j|^{-N_2}) \) meets \( \partial W \). Then \( |w_j| > \frac{1}{2}R_1 \) and \( S(w_j, t) \) meets the complement of \( W \) for all \( t \) with \( |w_j|^{-N_2} \leq t \leq |w_j|^{-N_1} \). Thus a standard estimate for harmonic measure [14, p. 116], coupled with a change of variables (see also [9, Lemma 2.3]), gives

\[
\omega(v_0, S_j, W) \leq 26|w_j|^{2(N_1 - N_2)},
\]

using (36). Since (35) gives

\[
\log |G(z)| \leq |w_j|^\rho \quad \text{for} \quad |z - w_j| \leq 1, \quad |z| \geq R_1,
\]
the two-constants theorem, [14, p. 116] and (33), (37) and (39) lead to

\[
\log M_1 \leq \log |G(v_0)| - \log M_1 \\
\leq \sum_{|w_j| > \frac{1}{2}R_1} 26|w_j|^\rho + d \log M(R, G) \exp\left( - \int_{2|v_0|}^{R/2} \frac{\pi dt}{t\theta_W(t)} \right) \\
\leq \sum_{|w_j| > \frac{1}{2}R_1} 26|w_j|^{-N_1} + d \log M(R, G) \exp\left( - \int_{2|v_0|}^{R/2} \frac{\pi dt}{t\theta_W(t)} \right),
\]

denoting positive absolute constants by \(d\). Using (34), (36) and (37) again this shows at once that the closure of \(W\) meets \(S(0, R)\), and \(A\) is unbounded. Moreover

\[
\int_{2|v_0|}^{R/2} \frac{\pi dt}{t\theta_W(t)} \leq \log \log M(R, G) + d, \quad R \to \infty, \quad R \notin E.
\]

Since \(W \subseteq A \cup H_1\) the result follows. \(\blacksquare\)

5. PROOF OF THEOREM 1.3

To prove Theorem 1.3 assume that \(f\) and \(S\) satisfy the hypotheses, but that \(f - S\) has finitely many zeros. Choose \(\rho\) such that

\[
\rho(f) < \rho < \frac{2N}{3}. \tag{40}
\]

By (7) and (9) there is no loss of generality in assuming that if \(z_j = w_k\) then \(a_j \neq -b_k\). Since all \(w_k\) with \(b_k \neq 0\) are then poles of \(f\), we may assume that (23) holds, by otherwise incorporating finitely many terms \(b_k/(z-w_k)\) into \(S\). By [10, Lemma 3.5], we have \(S(\infty) = 0\).

Let the positive integer \(P\) and the function \(G\) satisfy

\[
P > 1 + \rho/2, \quad f(z) - S(z) = \frac{1}{z^{P}G(z)}. \tag{41}
\]

Then \(G\) is transcendental and meromorphic in the plane, of order less than \(\rho\), and with finitely many poles.

We apply Lemma 4.3 to \(G\), and by (23) we may evidently take

\[
N_1 = \rho, \quad N_2 = 5\rho. \tag{42}
\]

Let \(R_1\) be large and positive, in particular so large that the first condition of (34) is satisfied, as well as (35) and (36). Thus \(\log |G(z)|\) is subharmonic in \(|z| > R_1\), and we may assume further that \(R_1\) is so large that the second condition of (34) holds, and that all poles of \(S\) lie in \(|z| < R_1/2\).
LEMMA 5.1  If $S_0$ is sufficiently large and positive there exist a large positive $T$ and
uncountably many $M_1 \in (S_0, 2S_0)$ such that, with the notation of Lemma 4.1,
\begin{equation}
M_1 > M(R_1, G)^2, \quad L(r, M_1, G) \leq r^{1+\rho/2} \quad \forall r \geq T,
\end{equation}
and such that, for some $C > 0$,
\begin{equation}
\int_{|w_k|^{-\rho} t}^{w_k/2} dL(t, w_k, M_1, G) \leq C(T(2|w_k|, G) + \log S_0)^{1/2}|w_k|^\rho \log |w_k|, \quad \forall k.
\end{equation}

**Proof**  Take a small positive $\delta$ and apply Lemma 4.1 with $\alpha = e^\delta$ and $\psi(t) = \delta^{-1}r^{\delta}$ in (21).

We choose $M_1$ as in Lemma 5.1 and may assume that $G$ has no multiple points with $|G(z)| = M_1$, which together with the fact that $G$ has finite order makes the next lemma obvious.

LEMMA 5.2  The set \{ $z : |G(z)| > M_1$ \} has finitely many unbounded components $V_j$. If $U$ is one of the $V_j$ then $U$ lies in $|z| > R_1$. Further, the finite boundary $\partial U$ of $U$ is the union of countably many pairwise disjoint level curves of $G$, each either simple and going to infinity in both directions, or simple closed.

LEMMA 5.3  Choose $v_0$ and $A$ as in (37). Then $A \subseteq U$, in which $U$ is one of the finitely many unbounded components $V_j$ of Lemma 5.2. Next, let
\begin{equation}
h(z) = \sum_{k=1}^{\infty} \frac{b_k w_k^N}{z^N(z - w_k)},
\end{equation}
in which the positive integer $N$ is as in (7) and (8). Further, for some $\theta \in [0, 2\pi)$ let $X_0$ be the intersection of $A$ with the ray $\arg z = \theta$, and for some $R \geq R_1$ let $Y_R = S(0, R) \cap A$. Let $B = \partial A \cup X_0 \cup Y_R$.

Then denoting by $c_j$ positive constants independent of $\theta$ and $R$ we have
\begin{equation}
I_1 = \int_{\partial U \cup B} |f(z) - S(z)| \, |dz| \leq c_1, \quad I_2 = \int_{B} |h(z)| \, |dz| \leq c_2.
\end{equation}

**Proof**  Points $v_0$ as in (37) do indeed exist, by (32) and the fact that $G$ is transcendental. It is clear that $A$ is a subset of one of the $V_j$, and we evidently have
\begin{equation}
\partial A \subseteq \partial U \cup K, \quad K = U \cap \bigcup_{j=1}^{\infty} S(w_j, |w_j|^{-N_z}).
\end{equation}

Let
\begin{equation}
J_0 = \{ z \in \partial U \cup B : |z| < T \},
\end{equation}
in which \( T \) is as in (43), and for positive integer \( m \) let

\[
J_m = \{ z \in \partial U \cup B : 2^{m-1}T \leq |z| < 2^mT \}.
\]

Then for \( m = 0, 1, 2, \ldots \), the set \( J_m \) has length at most \( 3T^{1+\rho/2}2^{m(1+\rho/2)} \), by (33), (36) and (43). Here the factor 3 arises due to the contribution of \( X_\theta \) and \( Y_R \), and we use the fact that \( T \) is large. Since (41) gives

\[
|f(z) - S(z)| \leq M_1^{-1}|z|^{-\rho} \leq M_1^{-1} \leq 1, \quad z \in U \cup \partial U,
\]

we get

\[
I_1 \leq 3T^{1+\rho/2} \left( 1 + \sum_{m=1}^{\infty} 2^{m(1+\rho/2)}2^{-\rho(m-1)} \right) \leq c_3,
\]

using the fact that \( P > 1 + \rho/2 \) by (41).

The estimation of \( I_2 \) is slightly more complicated. Fix \( k \in \mathbb{N} \), and let \( B_k = B \cap D(w_k, |w_k|/2) \). Then for all \( z \) in \( B \setminus B_k \) we have

\[
|z^N(z - w_k)| \geq \frac{1}{4} |z|^{N+1}
\]

and proceeding as in (48) leads to

\[
\int_{B \setminus B_k} \left| \frac{w_k^N}{z^N(z - w_k)} \right| |dz| \leq c_4 |w_k|^N,
\]

since \( N + 1 > 1 + \rho/2 \) by (40).

Next, we estimate

\[
I_3 = \int_{B_k} \left| \frac{w_k^N}{z^N(z - w_k)} \right| |dz|,
\]

and we begin by noting that on \( B_k \) we have

\[
|z/w_k| \geq \frac{1}{2}, \quad |z - w_k| \geq |w_k|^{-5\rho} = |w_k|^{-N_2}.
\]

Hence the contribution to \( I_3 \) from each circle \( S(w_j, |w_j|^{-N_2}) \) which meets \( D(w_k, |w_k|/2) \) is at most \( c_5 \), and for large \( k \) there are at most \( |w_k|^{\rho} \) such circles. Moreover, the contribution to \( I_3 \) from \( X_\theta \) and \( Y_R \) is at most \( c_6 \log |w_k| \). Finally, the contribution to \( I_3 \) from the level curves \( |G(z)| = M_1 \) is by (44) at most

\[
2^N \int_{|w_k|^{-5\rho}}^{|w_k|/2} dL(t, w_k, M_1, G) \leq c_7 (T|w_k|, G) + \log M_1)^{1/2}|w_k|^{\rho} \log |w_k|.
\]
Hence
\[ I_3 \leq c_8 |w_k|^{5/2} \log |w_k| \leq c_9 |w_k|^N, \]

by (8) and (40). Now combine this estimate with (49), multiply by $|b_k|$ and sum over $k$. Using (7) gives the second estimate of (46).

**Lemma 5.4** Let $U$ be as in Lemma 5.3. Then $D(0, R_1)$ lies in an unbounded component of $\mathbb{C} \setminus U$.

**Proof** Assume that $D(0, R_1)$ lies in a bounded component of the complement of $U$. Then there exists a simple closed curve $\gamma_1$, a component of $\partial U$, such that $D(0, R_1)$ lies in the interior $U_1$ of $\gamma_1$. Since the $z_k$ are poles of $f$ we get, using (7) and (46),
\[ c_1 \geq \int_{\gamma_1} |f(z) - S(z)| |dz| \geq 2\pi \sum_{|z_k| < R_1} a_k - O(1), \]

which gives a contradiction provided $R_1$ was chosen large enough, using (9).

**Lemma 5.5** Let $U$ be as in Lemma 5.3. Let $C_1$ be the unbounded component of $\partial U$ which separates $U$ from $D(0, R_1)$. Let $D$ be the component of $\mathbb{C} \setminus C_1$ which contains $U$. Fix $w^* \in U$ and define a single valued branch of $\log z$, continuous on the closure of $D$, with $|\arg w^*| \leq \pi$. Then we have $\log z = O(\log |z|)$ as $z \to \infty$ in the closure of $D$.

**Proof** Define a subharmonic function $v(z)$ on $\mathbb{C}$ as follows:
\[ v(z) = \log |G(z)| - \log M_1, \quad z \in U; \quad v(z) = 0, \quad z \notin U. \quad (50) \]

Then $v$ has finite order, since $G$ has. The boundary of $D$ is the curve $C_1$, and $v(z) = 0$ there. Thus the Ahlfors spiral theorem [6, pp. 600–608] gives $\arg z = O(\log |z|)$ as $z$ tends to infinity on $\partial D$. Since $\arg z$ is monotone on arcs of circles centred at the origin, the result follows.

**Lemma 5.6** Let $u$ be defined by (2). Then $u(z) = O(\log |z|)$ as $z \to \infty$ in the closure of $A$.

**Proof** Let
\[ g(z) = \sum_{k=1}^{\infty} \frac{b_k}{z - w_k}. \quad (51) \]

Writing
\[ \frac{1}{z - w} = \frac{1}{z(1 - w/z)} = \frac{1}{z} + \frac{w}{z^2} + \cdots + \frac{w^{N-1}}{z^N} + \frac{w^N}{z^N(z - w)} \]

we see from (7) that there is a rational function $S_1$, with $S_1(\infty) = 0$, such that we may write
\[ S(z) - g(z) = S_1(z) - h(z), \quad (52) \]

in which $h$ is defined by (45).
Let $D$ and the branch of $\log z$ be as in Lemma 5.5. Since $S_1(\infty) = 0$ and $S_1$ is analytic in $|z| > R_1/2$ there exist a constant $c_4$ and a function $S_2$, analytic and bounded in $|z| > R_1$, such that

$$S_1(z) = \frac{d}{dz} (c_4 \log z + S_2(z))$$

(53)

for $z$ in the closure of $D$. By (2), (6), (51) and (52) we have

$$u_x - iu_y = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k} = f(z) - g(z) = f(z) - S(z) + S_1(z) - h(z).$$

(54)

For each $w$ in $A$, we join $w$ to the point $v_0$ chosen in Lemma 5.3 by a path $\sigma(w)$ in the closure of $A$ consisting of part of the ray $\arg z = \arg v_0$, part of the circle $|z| = |w|$, and part of the boundary of $A$. By Lemma 5.3 there is a constant $c_5$ independent of $w$ such that

$$\int_{\sigma(w)} |f(z) - S(z)| + |h(z)| |dz| < c_5.$$

This gives, using (53) and (54),

$$\left| \int_{\sigma(w)} (u_x - iu_y)(dx + idy) \right| \leq |c_4 \log w| + O(1)$$

and the result follows using Lemma 5.5.

**Lemma 5.7**  There exist positive constants $k_1, k_2$ with the following property. With $u$ as in (2), define $u_1$ by

$$u_1(z) = \max\{u(z) - k_1 \log |z| - k_2, 0\}, \quad |z| > R_1,$$

with $u_1(z) = 0$ for $|z| \leq R_1$. Then $u_1$ is non-constant and subharmonic in the plane, with $u_1(z) = 0$ on $A$, and $T(r, u_1) = o(r)$ as $r \to \infty$.

**Proof**  Choose $k_1$ and $k_2$ using Lemma 5.6, so that $u(z) \leq k_1 \log |z| + k_2$ on $|z| = R_1$ and on $A$. Thus $u_1$ is subharmonic, with $T(r, u_1) = o(r)$ by (7). Finally, $u_1$ is non-constant by (9).

We may now complete the proof of Theorem 1.3. Let $W$ be a component of the set $\{z : u_1(z) > 0\}$. Since $u_1$ vanishes on the closure of $A$, the Cauchy-Schwarz inequality gives

$$\frac{1}{\theta^*_A(t)} + \frac{1}{\theta^*_W(t)} \geq \frac{2}{\pi}, \quad t > 0.$$  

(55)
Let $E$ be as defined in Lemma 4.3. Since $E$ has finite linear measure, by (32), it clearly has finite logarithmic measure. Using (8) and (41), choose a sequence $R_p \to \infty$, $R_p \notin E$, with

$$\log M(R_p, G) = O(R_p).$$

Using Lemma 4.2, (38), (55) and the fact that $T(r, u_1) = o(r)$ this gives

$$2 \log R_p - O(1) \leq \int_{[r_0, R_p/2] \setminus E} \left( \frac{\pi}{t \theta^*_w(t)} + \frac{\pi}{t \theta^*_f(t)} \right) dt$$

$$\leq 2 \log R_p + \log o(1) + O(1)$$

for some large $r_0$, which is plainly impossible.

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**References**