Integer-valued analytic functions in a half-plane

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Abstract

A classical theorem of Pólya states that if \( f \) is an entire function taking integer values at the non-negative integers and satisfying \( f(z) = O\left(|z|^{M_2}\right) \) as \( z \to \infty \), for some \( M > 0 \), then there exist polynomials \( P_1, P_2 \) with \( f(z) \equiv P_1(z)z^2 + P_2(z) \). It is shown that the same result holds for functions analytic in a half-plane \( \Re z \geq A \).

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1 Introduction

The following classical theorem of Pólya [12] (see also [19, p.55]) shows in particular that \( 2^z \) is the slowest growing transcendental entire function which is integer-valued, that is, takes integer values at the non-negative integers.

**Theorem 1.1 ([12])** Let \( f \) be an entire function which satisfies

\[
f(n) \in \mathbb{Z} \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{1}
\]

and

\[
M(r, f) = \max\{|f(z)| : |z| = r\} = O(r^{M_2 r}) \tag{2}
\]

as \( r \to \infty \), for some \( M > 0 \). Then there exist polynomials \( P_1, P_2 \) with \( f(z) \equiv P_1(z)z^2 + P_2(z) \).

In particular if (2) is replaced by

\[
\limsup_{r \to \infty} \frac{M(r, f)}{2^r} < 1 \tag{3}
\]

then \( f \) is a polynomial.

It was shown by Selberg [14] that if the entire function \( f \) satisfies (1) and

\[
\tau(f) = \limsup_{r \to \infty} \frac{\log M(r, f)}{r} \leq \log 2 + \frac{1}{1500} = 0.6938138 \ldots \tag{4}
\]

then again there exist polynomials \( P_1, P_2 \) with \( f(z) \equiv P_1(z)z^2 + P_2(z) \), and subsequently by Pisot [1, 11, 13] that (4) may be weakened to

\[
\tau(f) < |\log(3/2 + \sqrt{3}i/2)| = 0.758876 \ldots
\]

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Further results on integer-valued entire functions may be found in [1, 3, 9, 10, 13, 15, 16, 17, 18].

The present note is concerned with analogous results for functions analytic in a half-plane. In this direction it was proved in [6] that if \( f \) is analytic of polynomial growth in \( \text{Re} \, z \geq 0 \) and \( f \) satisfies (1) then \( f \) is a polynomial, a result which has several applications in value distribution theory and differential equations. These include: determining meromorphic functions \( f \) and \( g \) of finite order when \( f \) and \( g \) have the same zeros and poles and the same is true of \( f' \) and \( g' \) [7]; determining meromorphic functions \( f \) such that \( f \) and \( f'' + Bf \) have no zeros, where \( B \) is a rational function [6]; determining Bank-Laine functions of finite order from their zero-sequences [8].

It is then natural to consider functions \( f \) analytic in the right half-plane satisfying (1) and of exponential growth comparable to (2). The methods of [1, 11, 13] appear difficult to transfer to a half-plane, being based on the Borel-Laplace transform of an entire function of exponential type, and for technical reasons the approach of [14] also seems difficult to adapt. On the other hand the method of [12] may be modified using different contours and integral estimates, and even admits slight simplification in parts.

**Theorem 1.2** Let \( A \in \mathbb{R} \) and \( M > 0 \) and let \( f \) be analytic in the half-plane \( H \) given by \( \text{Re} \, z \geq A \), such that

\[
\tag{5} f(n) \in \mathbb{Z} \quad \text{for all } n \in \mathbb{Z} \text{ with } n \geq A.
\]

Assume that

\[
|f(z)| = O(\left|z\right|^{M2^{|z|}}) \quad \text{as } \left|z\right| \to \infty, z \in H.
\]  

\[
\tag{6}

Then there exist polynomials \( P_1, P_2 \) with \( f(z) \equiv P_1(z)2^z + P_2(z) \).

If \( A = 0 \) and (6) is replaced by

\[
\tag{7}
\limsup_{|z| \to \infty, z \in H} |f(z)|2^{-|z|} < 1,
\]

then \( f \) is a polynomial.

No analogous result is obtainable in a sector \( |\arg z| < \alpha < \pi/2 \), as shown by the example \( \exp(-\beta z)\sin \pi z \), with \( \beta \) a large positive constant.

**2 The forward difference method**

Let \( h \) be a function which is defined at the points \( a, a+1, \ldots \). The forward differences are defined in the standard way [19, p. 52] by

\[
\Delta^0 h(a) = h(a), \quad \Delta h(a) = \Delta^1 h(a) = h(a+1) - h(a), \quad \Delta^{n+1} h(a) = \Delta^n h(a+1) - \Delta^n h(a),
\]

and the well known formula

\[
\Delta^n h(a) = h(a+n) - nh(a+n-1) + \cdots + (-1)^n h(a) = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} h(a+k), \quad n \geq 0,
\]

is easily proved by induction, as is the formula, for \( a = 0 \),

\[
(\Delta - 1)^k h(n) = 2^{n+k}\Delta^k H(n), \quad h(n) = 2^n H(n), \quad k = 0, \ldots .
\]  

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If $P$ is a polynomial of degree $Q \geq 1$ then $\Delta P(n)$ is a polynomial of degree at most $Q - 1$, and conversely the general solution of the difference equation $\Delta F(n) = P(n)$, $n = 0, 1, \ldots$, is a polynomial of degree at most $Q + 1$. The following fact is fundamental to Pólya’s method in [12]: a proof is included for completeness.

**Lemma 2.1** Suppose that $f$ is a function defined at the points $0, 1, \ldots$, and that $L$ and $N$ are positive integers such that

$$\Delta^n (\Delta - 1)^L f(0) = 0 \quad \text{for all } n \geq N.$$  

Then there exist polynomials $P_1$ and $P_2$ such that $f(n) = P_1(n) 2^n + P_2(n)$ for $n = 0, 1, \ldots$.

**Proof.** Let $g(m) = (\Delta - 1)^L f(m)$ and choose a polynomial $S_1$, of degree at most $N - 1$, which equals $g$ at the $N$ points $0, \ldots, N - 1$. Then $\Delta^n (g - S_1)(0) = 0$ for $n \geq N$ and so it follows using (8) that $g(n) = S_1(n)$ for all integers $n \geq 0$. Now choose a polynomial $S_2$ such that

$$\Delta^n (\Delta - 1)^L S_2(n) = S_1(n) = g(n) \quad \text{for } n = 0, 1, \ldots,$$

which is easily achieved by setting

$$S_2(n) = (-1)^L (1 + \Delta + \Delta^2 + \ldots)^L S_1(n)$$

and using the fact that $\Delta^m S$ vanishes identically if $S$ is a polynomial and $m$ is large enough. Finally, write

$$h(n) = f(n) - S_2(n) = 2^n H(n), \quad 0 = \Delta^n (\Delta - 1)^L h(n) = 2^n L \Delta^L H(n) \quad \text{for } n = 0, 1, \ldots,$$

using (9). The general solution of this difference equation for $H$ is a polynomial, and the conclusion of Lemma 2.1 follows at once. \hfill $\Box$

### 3 Lemmas required for Theorem 1.2

**Lemma 3.1** There exists a positive constant $c$ with the following property. Let $n$ be a positive integer and let

$$\phi(y) = \phi_n(y) = \frac{2^y}{\exp(2y \arctan(n/y))(1 + (y/n)^2)^{n}}.$$  

Then

$$\phi(y) \leq e^{-cy} \quad \text{for } 0 < y \leq 2n.$$  

**Proof.** Set

$$y = un, \quad h(u) = \frac{\ln \phi(y)}{n} = 2u \ln 2 - 2u \arctan \left( \frac{1}{u} \right) - \ln(1 + u^2)$$

for $u > 0$. Then

$$h(u) \to 0 \quad \text{as } u \to 0+, \quad h(2) = 4 \ln 2 - 4 \arctan \frac{1}{2} - \ln 5 < 0.$$
Differentiating (12) gives
\[ h'(u) = 2 \ln 2 - 2 \arctan \left( \frac{1}{u} \right), \]
so that if \( h'(u) = 0 \) then \( h(u) = -\ln(1 + u^2) < 0 \). Hence \( h(u) < 0 \) for \( 0 < u \leq 2 \), using (13). Moreover
\[ h'(u) \to 2 \ln 2 - \pi < 0 \quad \text{as} \quad u \to 0+, \]
and so \( h(u) < (\ln 2 - \pi/2)u \) for small positive \( u \). Hence there exists a positive constant \( c \) such that \( h(u) < -cu \) for \( 0 < u \leq 2 \), so that
\[ \ln \phi(y) = nh(u) < -cny = -cy \]
for \( 0 < y \leq 2n \), which is (11). \( \Box \)

**Lemma 3.2** Let \( \mu \) and \( s \) be real numbers with \( \mu \geq 0 \) and \( s > 1 \). Let \( n \in \mathbb{N} \) be large and let
\[ R = R_n = 2n, \quad S = S_n = \sqrt{R^2 - s^2}. \] (14)

Let \( C_n \) be the contour consisting of the arc \( \Omega_n \) of the circle \(|t| = R \) from \(-s - iS \) to \(-s + iS \) via \( R \), described once counter-clockwise, followed by the straight line segment \( T_n \) from \(-s + iS \) to \(-s - iS \). Then
\[ I_n = I_n(\mu) = \int_{\Omega_n} \frac{n!2^{|t|}}{|t(t-1) \ldots (t-n)|} \left| \frac{t-2n}{n} \right|^\mu |dt| \leq c(\mu)n^{-\mu/2} \] (15)
and
\[ J_n = J_n(\mu) = \int_{T_n} \frac{n!2^{|t|}}{|t(t-1) \ldots (t-n)|} \left| \frac{t-2n}{n} \right|^\mu |dt| \leq d(\mu, s)n^{1/2 - s} \] (16)
as \( n \to \infty \), where \( c(\mu) \) and \( d(\mu, s) \) denote positive constants depending at most on \( \mu \), and on \( \mu \) and \( s \), respectively.

**Proof.** The estimate (15) may be found in [12, Hilfsatz, p.5] but the proof is included for completeness. The arc \( \Omega_n \) is parametrized by \( t = Re^{i\theta}, -a_n \leq \theta \leq a_n \), where \( a_n \to \pi/2 \) as \( n \to \infty \). Then for \( 0 \leq k \leq n \) standard inequalities give
\[ |t-k|^2 = (R-k)^2 + 2Rk(1 - \cos \theta) \geq (R-k)^2 + CRk^2 \geq (R-k)^2 \exp \left( \frac{CRk^2}{(R-k)^2} \right), \] (17)
using \( C \) to denote positive constants depending at most on \( \mu \), and the fact that \( Rk(R-k)^{-2} \leq 2 \). Combining (14) and (17) leads to
\[ \prod_{k=0}^n |t-k| \geq \exp \left( \sum_{k=0}^n \frac{CRk^2}{(R-k)^2} \right) \prod_{k=0}^n (R-k) \geq \exp \left( Cn\theta^2 \right) \prod_{k=0}^n (R-k). \] (18)
Moreover, for \( t \in \Omega_n \),
\[ |t-2n|^2 = |t-R|^2 = 2R^2(1 - \cos \theta) \leq CR^2\theta^2, \quad \left| \frac{t-2n}{n} \right|^\mu \leq C\theta^\mu. \] (19)
Hence it follows from (7), (14), (18), (19) and the change of variables $x = \theta \sqrt{\mu}$ that
\[
I_n \leq CR^2 R \left( \frac{n!}{R(R-1) \ldots (R-n)} \right) \int_{-\pi}^{\pi} \theta^m \exp (-Cn\theta^2) \, d\theta \\
\leq CR^2 R \left( \frac{(n-1)!}{(R-1) \ldots (R-n)} \right) n^{-\mu/2-1/2} \int_{-\infty}^{\infty} x^\mu \exp(-Cx^2) \, dx \\
\leq C2^R n^{-\mu/2+1/2} \left( \frac{\Gamma(n)\Gamma(R-n)}{\Gamma(R)} \right).
\] (20)

But Stirling’s formula gives
\[
\Gamma(x) \sim \sqrt{2\pi} x^{x-1/2} e^{-x} \quad \text{as} \ x \to +\infty,
\] (21)
and so (14) and (20) lead to
\[
I_n \leq C2^R n^{-\mu/2+1/2} \left( \frac{n^{-1/2}e^{-n}(R-n)^{R-n-1/2}e^{n-R}}{R^{R-1/2}e^{-R}} \right) \\
\leq C2^R n^{-\mu/2+1/2} \left( \frac{n^{-1/2}(R-n)^{R-n-1/2}}{R^{R-1/2}} \right) \\
\leq C2^R n^{-\mu/2+1/2} n^{-1/2} 2^{-R} = Cn^{-\mu/2},
\]
which proves (15).

To prove (16) requires a different approach. Let $0 \leq k \leq n$ and $t \in T_n$ and write $t = -s + iy$ where
\[
-S \leq y \leq S, \quad |t - k|^2 = (s + k)^2 + y^2.
\] (22)
Combining (22) with (14) gives
\[
J_n \leq D \int_0^S \frac{n!2^y \, dy}{\prod_{k=0}^n \sqrt{(s + k)^2 + y^2}}.
\] (23)
using $D$ to denote positive constants depending at most on $\mu$ and $s$. Note next that for $y > 0$ integration by parts leads to
\[
L_n := \sum_{k=0}^n \ln (s^2 + y^2) \geq \ln(s^2 + y^2) + \int_s^{n+s} \ln(t^2 + y^2) \, dt \\
= (n + s) \ln((n + s)^2 + y^2) + (1 - s) \ln(s^2 + y^2) - 2 \int_s^{n+s} \frac{t^2}{t^2 + y^2} \, dt \\
= (n + s) \ln((n + s)^2 + y^2) + (1 - s) \ln(s^2 + y^2) - 2n + 2 \int_s^{n+s} \frac{y^2}{t^2 + y^2} \, dt \\
= (n + s) \ln((n + s)^2 + y^2) + (1 - s) \ln(s^2 + y^2) - 2n + \\
+ 2y \arctan \left( \frac{n + s}{y} \right) - 2y \arctan \left( \frac{s}{y} \right) \\
\geq (n + s) \ln((n + s)^2 + y^2) + (1 - s) \ln(s^2 + y^2) - 2n + \\
+ 2y \arctan \left( \frac{n + s}{y} \right) - D.
\] (24)
With \( \phi(y) \) as in (10) and using Lemma 3.1, (21) and (24) it now follows that

\[
\frac{(n!)^{2}2^{2y}}{\prod_{k=0}^{n}((s+k)^{2}+y^{2})} \leq \frac{Dn^{2n+1}e^{-2n(s^{2}+y^{2})^{n}e^{2n+2y}}}{((n+s)^{2}+y^{2})^{n}e^{2y\arctan((n+s)/y)}} \leq \frac{Dn^{2n+1}(s^{2}+y^{2})^{s-1}2^{2y}}{(n^{2}+y^{2})^{n}e^{2y\arctan(n/y)}} \leq Dn^{2n+1-2n-2s}(s^{2}+y^{2})^{s-1}\phi(y) \leq Dn^{1-2s}(s^{2}+y^{2})^{s-1}e^{-cy}
\]

for \( 0 < y \leq S \), since \( S \leq R = 2n \). Substituting this estimate into (23) gives

\[
J_{n} \leq Dn^{1/2-s} \int_{0}^{\infty} (s^{2}+y^{2})^{(s-1)/2}e^{-cy/2} \, dy = Dn^{1/2-s},
\]

which establishes (16) and proves Lemma 3.2. \( \Box \)

The next lemma simplifies somewhat the approach of [12, Hilfsatz, p.4 and Korollar, p.5].

**Lemma 3.3** Let \( L \) be a positive integer. For \( x, y \in \mathbb{C} \) and \( p = 0, \ldots, L \) write

\[
G(p, x, y) = \left( \prod_{0 \leq q \leq p-1} (1+x+qy) \right) \left( \prod_{p \leq q \leq L-1} (1-qty) \right) = (1+x)(1+x+y) \ldots (1+x+(p-1)y)(1-ty) \ldots (1-(L-1)y), \tag{25}
\]

with the convention that a product is unity if the corresponding range of \( q \) is empty. Then:

(i) each \( G(p, x, y) \) is a polynomial

\[
G(p, x, y) = \sum_{\mu, \nu} B_{\mu, \nu}(p)x^{\mu}y^{\nu} \tag{26}
\]

in \( x \) and \( y \) of degree at most \( L \) in \( x \) and at most \( L-1 \) in \( y \);

(ii) for each pair \( \mu, \nu \) with \( 0 \leq \mu \leq L \) and \( 0 \leq \nu \leq L-1 \) there exists a polynomial \( C_{\mu, \nu}(p) \) in \( p \) of degree at most \( \mu + 2\nu \) such that \( B_{\mu, \nu}(p) = C_{\mu, \nu}(p) \) for \( p = 0, \ldots, L \).

**Proof.** Assertion (i) is obvious, and it is convenient to regard the sum in (26) as being over all integers \( \mu, \nu \), with \( B_{\mu, \nu}(p) = 0 \) unless \( 0 \leq \mu \leq L \) and \( 0 \leq \nu \leq L-1 \).

Assertion (ii) will now be proved by induction on \( m = \mu + \nu \), and it is obvious that \( B_{0,0}(p) = 1 \) for every \( p \). Assume now that \( m \) is a positive integer and that assertion (ii) holds whenever \( 0 \leq \mu + \nu < m \). Suppose that \( \mu + \nu = m \). There is nothing to prove if \( \mu \) or \( \nu \) is negative, so assume that \( \mu \geq 0 \) and \( \nu \geq 0 \). From (25) it follows that, for \( p = 0, \ldots, L-1 \),

\[
(1-ty)G(p+1, x, y) = G(p, x, y)(1+x+py).
\]

Comparing the coefficients of \( x^{\mu}y^{\nu} \) shows that

\[
B_{\mu, \nu}(p+1) - pB_{\mu, \nu-1}(p+1) = B_{\mu, \nu}(p) + B_{\mu-1, \nu}(p) + pB_{\mu, \nu-1}(p),
\]
and this may be written in the form
\[ B_{\mu,\nu}(p + 1) - B_{\mu,\nu}(p) = pB_{\mu,\nu-1}(p + 1) + B_{\mu-1,\nu}(p) + pB_{\mu,\nu-1}(p). \] (27)

The induction hypothesis gives a polynomial \( g(p) \) of degree at most \( \mu + 2\nu - 1 \) which equals the
right hand side of (27) for \( p = 0, \ldots, L - 1 \). Thus (27) may be viewed as a difference equation
\[ \Delta F(p) = g(p), \] (28)
satisfied by \( B_{\mu,\nu}(p) \) for \( p = 0, \ldots, L - 1 \). There exists a polynomial \( F \) of degree at most
\( 1 + \deg g \leq \mu + 2\nu \) satisfying (28) for all integers \( p \), and \( F \) may be chosen so that \( F(0) = B_{\mu,\nu}(0) \).
It then follows that \( B_{\mu,\nu}(p) = F(p) \) for \( p = 0, \ldots, L \), and Lemma 3.3 is proved. \( \square \)

The next lemma is a special case of Carlson’s theorem for a half-plane [2], or may be proved
directly using the Nevanlinna characteristic in a half-plane [4, p.38].

**Lemma 3.4** Let \( F(z) \) be analytic and satisfy (6) in the half-plane \( H \) given by \( \Re z \geq 0 \). If
\( F(n) = 0 \) for every \( n \in \mathbb{N} \) then \( F \) vanishes identically.


4 Proof of Theorem 1.2

To prove the first part of Theorem 1.2 let \( f \) satisfy the hypotheses there. Fix a positive integer
\( L \) satisfying \( L > 2M \), and assume without loss of generality that \( A < -L - 2 \). Following Pólya
[12] the aim is to show that
\[ D_n = \Delta^{n-L}(\Delta - 1)^L f(0) = \sum_{p=0}^{L} \left( \frac{L!}{p!(L-p)!} \right) (-1)^p \Delta^{n-p} f(0) \]
\[ = \Delta^nf(0) - L\Delta^{n-1}f(0) + \ldots + (-1)^L \Delta^{n-L}f(0) \] (29)
vanishes for all sufficiently large positive integers \( n \). Once this has been established it follows from
Lemma 2.1 that there exists a function \( h(z) = P_1(z)^2 + P_2(z) \), with \( P_1 \) and \( P_2 \) polynomials,
such that \( f(n) = h(n) \) for \( n = 0, 1, \ldots \). Applying Lemma 3.4 to \( F = f - h \) then shows that
\( f - h \) vanishes identically.

Let \( n \) be a large positive integer, and let \( R \) and \( S \) be defined by (14), with \( s = 1 + L \). Let \( C_n \)
be the contour defined in Lemma 3.2. Then (29) and the formula [19, pp. 52-53]
\[ \Delta^{n-p} f(0) = \frac{1}{2\pi i} \int_{C_n} \frac{(n-p)!f(t)}{t(t-1)\ldots(t-n+p)} \, dt \]
\[ = \frac{1}{2\pi i} \int_{C_n} \frac{n!f(t)}{t(t-1)\ldots(t-n)} \prod_{0 \leq q \leq p-1} \left( \frac{t-n+q}{n-q} \right) \, dt \]
for \( p = 0, \ldots, L \) imply that
\[ D_n = \frac{1}{2\pi i} \int_{C_n} \frac{n!f(t)}{t(t-1)\ldots(t-n)} \sum_{p=0}^{L} \left( \frac{(-1)^p L!}{p!(L-p)!} \right) \prod_{0 \leq q \leq p-1} \left( \frac{t-n+q}{n-q} \right) \, dt. \]
With Pólya’s notation

\[ x = \frac{t - 2n}{n}, \quad y = \frac{1}{n}, \quad \frac{t - n + q}{n - q} = \frac{1 + x + qy}{1 - qy}, \]

this gives

\[
D_n = \frac{1}{2\pi i} \int_{C_n} \frac{n!f(t)}{t(t-1)\ldots(t-n)} \sum_{p=0}^{L} \left( \frac{(-1)^p L!}{p!(L-p)!} \right) \prod_{0 \leq q \leq p-1} \left( \frac{1 + x + qy}{1 - qy} \right) \, dt.
\]

Let \( G(p, x, y) \) be the function defined in Lemma 3.3. Then

\[
D_n \prod_{q=0}^{L-1} (1 - qy) = \frac{1}{2\pi i} \int_{C_n} \frac{n!f(t)}{t(t-1)\ldots(t-n)} \sum_{p=0}^{L} \left( \frac{(-1)^p L!}{p!(L-p)!} \right) G(p, x, y) \, dt.
\]

It now follows using the formula (compare (8))

\[
\Delta^L G(0, x, y) = \sum_{p=0}^{L} \left( \frac{L!}{p!(L-p)!} \right) (-1)^{L-p} G(p, x, y)
\]

that

\[
D_n \prod_{q=0}^{L-1} (1 - qy) = \left( \frac{-1}{2\pi i} \right)^L \int_{C_n} \frac{n!f(t)}{t(t-1)\ldots(t-n)} \Delta^L G(0, x, y) \, dt.
\]

But Lemma 3.3 shows that the function \( G(p, x, y) \) has a representation (26), in which \( B_{\mu,\nu}(p) \) is a polynomial in \( p \) of degree at most \( \mu + 2\nu \), and \( B_{\mu,\nu}(p) = 0 \) unless \( 0 \leq \mu \leq L \) and \( 0 \leq \nu \leq L-1 \).

This gives constants \( A_{\mu,\nu} \), independent of \( n \), such that

\[
\Delta^L G(0, x, y) = \sum_{0 \leq \mu \leq L, 0 \leq \nu \leq L} A_{\mu,\nu} x^\mu y^\nu,
\]

where \( A_{\mu,\nu} = 0 \) for \( \mu + 2\nu < L \). Hence (31) may be written in the form

\[
D_n \prod_{q=0}^{L-1} (1 - qy) = \left( \frac{-1}{2\pi i} \right)^L \sum_{0 \leq \mu \leq L, 0 \leq \nu \leq L} A_{\mu,\nu} \int_{C_n} \frac{n!f(t)x^\mu y^\nu}{t(t-1)\ldots(t-n)} \, dt.
\]

Recalling the notation (30) and the fact that \(|t| = O(n)| on \( C_n \) leads to

\[
\left| D_n \prod_{q=0}^{L-1} (1 - q/n) \right| \leq d \sum_{0 \leq \mu \leq L, 0 \leq \nu \leq L} \left| A_{\mu,\nu} \right| n^{M-\nu} \int_{C_n} \frac{n!2^l}{|t(t-1)\ldots(t-n)|} \left| \frac{t-2n}{n} \right|^\mu |dt|
\]

\[
\leq d \sum_{0 \leq \mu \leq L, 0 \leq \nu \leq L} \left| A_{\mu,\nu} \right| n^{M-\nu} \left( n^{-\mu/2} + n^{1/2-\nu} \right)
\]

\[
= d \sum_{0 \leq \mu \leq L, 0 \leq \nu \leq L} \left| A_{\mu,\nu} \right| \left( n^{M-\nu-\mu/2} + n^{M-\nu-1/2-L} \right),
\]
where the constants $d$ are independent of $n$, using (15), (16) and the fact that $s = 1 + L$. Since $L > 2M$ and $A_{\mu, \nu} = 0$ unless $\mu + 2\nu \geq L$ this now gives, on letting $n \to \infty$,

$$|D_n| \leq \frac{dnM^{L/2}}{2} \to 0.$$  

But each $D_n$ is an integer by (5) and (29), and so it follows that $D_n = 0$ for all sufficiently large $n$ as required, and the first part of Theorem 1.2 is proved.

To prove the last part of Theorem 1.2 assume now that $A = 0$ and that $f$ satisfies (7). Following Pólya [12, p.8], it is clear from the first part that $f(z) = a2^z + P_2(z)$ with $a \in \mathbb{C}$ and $P_2$ a polynomial. But then, for large $n \in \mathbb{N}$,

$$\Delta^n f(0) = \Delta^n(a2^z)(0) = a,$$

so that $a \in \mathbb{Z}$, and using (7) again this forces $a = 0$. \hfill \square

References


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