Non-real zeros of derivatives of real meromorphic functions of infinite order

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Abstract

Let f be a real meromorphic function of infinite order in the plane, with finitely many zeros and non-real poles. Then f'' has infinitely many non-real zeros.
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1 Introduction

Around 1911 Wiman conjectured [1, 2] that if a transcendental entire function f is real (that is, maps $\mathbb{R}$ into itself) and f and f'' have only real zeros, then f belongs to the Laguerre-Pólya class LP of entire functions which are locally uniform limits of real polynomials with real zeros, from which it follows that all derivatives of f have only real zeros. This was proved by Sheil-Small [29] for f of finite order and in [6] for infinite order (see also [23]). Furthermore, for an entire function $f = Ph$, where h is a real entire function with real zeros and P is a real polynomial, the number of non-real zeros of $f^{(k)}$ is 0 for large k if $h \in LP$ [7, 8, 16, 17], and tends to infinity with k otherwise [5, 18]: these results proved a conjecture of Pólya [26].

For meromorphic functions which are real (that is, map $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$) there are less complete results. Meromorphic functions f in the plane for which all derivatives $f^{(k)}$ $(k \geq 0)$ have only real zeros were determined by Hinkkanen [13, 14, 15], while functions for which some derivatives have only real zeros have been considered in several papers including [11, 12, 27]. The following theorem was proved in [21] (see also [20]).

Theorem 1.1 ([21]) *Let f be a real meromorphic function in the plane, not of form $f = Se^P$ with S a rational function and P a polynomial. Let μ and k be integers with $1 \leq \mu < k$. Assume that all but finitely many zeros of f and $f^{(k)}$ are real, and that $f^{(\mu)}$ has finitely many zeros. Then $\mu = 1$ and $k = 2$ and f satisfies*

$$f(z) = \frac{R(z)e^{icz} - 1}{AR(z)e^{icz} - \overline{A}},$$

where $c \in (0, \infty)$, $A \in \mathbb{C} \setminus \mathbb{R}$, and R is a rational function with $|R(x)| = 1$ for all $x \in \mathbb{R}$. Moreover, all but finitely many poles of f are real.

For the related problem where f, instead of $f^{(\mu)}$, is assumed to have finitely many zeros, the following theorem is a combination of results of Hellerstein and Williamson [11] and Rossi [27].

Theorem 1.2 ([11, 27]) *Let f be a real meromorphic function in the plane with real poles and no zeros, and assume that all zeros of f' are real. If f has infinite order then f'' has infinitely many non-real zeros. The same conclusion holds if f has finite order and infinitely many poles.*

For functions of finite lower order, this result was improved in [22] as follows.

Theorem 1.3 ([22]) *Let f be a real meromorphic function of finite lower order in the plane, with finitely many zeros and non-real poles, and let $a \in \mathbb{R}$ satisfy $a < 1$. Assume that $N_0(r) = o(T(r, f'/f))$ as $r \to \infty$, where $N_0(r)$ counts the non-real zeros of $ff''/(f')^2 - a$. Then $f = Se^P$, with S a rational function and P a polynomial.*

Theorem 1.3 applies in particular if $N_0(r) = O(\log r)$ as $r \to \infty$ [22]. If $a = 0$ then $N_0(r)$ counts non-real zeros of ff'' which are not zeros of f'. The case $0 < a < 1$ of Theorem 1.3 complements certain results of Nicks [25] for real entire functions f, and setting

$$g = \frac{1}{f}, \quad \frac{gg''}{(g')^2} - a = 2 - a - \frac{ff''}{(f')^2}, \tag{1}$$

with f as in Theorem 1.3, leads easily to an analogous result for real entire functions with real zeros and $a > 1$ (see [22]). The case $a = 1$ is exceptional, since if $f = e^h$ and h is a real entire function such that h'' has only real zeros then so has $ff''/(f')^2 - 1 = h''/(h')^2$ [3, 24].

The aim of the present paper is a result in the direction of Theorem 1.3, but for functions of infinite order. Now the methods of [22] depend on [22, Proposition 3.1], which only holds for functions of finite lower order, and so an alternative approach is developed here, based on a counterpart of a result proved in [6] for the infinite order case of the Wiman conjecture. Let L be a real transcendental meromorphic function in the plane such that all but finitely many poles of L are real and simple and have positive residues. Then L has, as in [6, pp.978-979], a Levin-Ostrovskii factorisation

$$L = \phi\psi, \tag{2}$$

where ϕ and ψ are real meromorphic functions such that:
(A) the function ϕ has finitely many poles;
(B) every pole of ψ is real and simple and is a simple pole of L;
(C) either $\psi \equiv 1$ or $\psi(H^+) \subseteq H^+$, where $H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.
Such a factorisation may be derived as follows. If L has finitely many poles, set $\psi = 1$. For L with infinitely many poles, denote by α_p the real simple poles of L with positive residues, ordered so that $\alpha_p < \alpha_{p+1}$. For $|p| \geq p_0$, where p_0 is large, α_p and α_{p+1} are of the same sign, and there is a zero β_p of L in the interval (α_p, α_{p+1}). Write

$$\psi(z) = \prod_{|p| \geq p_0} \frac{1 - z/\beta_p}{1 - z/\alpha_p}, \quad 0 < \sum_{|p| \geq p_0} \arg \frac{1 - z/\beta_p}{1 - z/\alpha_p} = \sum_{|p| \geq p_0} \arg \frac{\beta_p - z}{\alpha_p - z} < \pi \quad \text{for} \quad z \in H^+.$$

The product ψ then converges by the alternating series test, and satisfies $\psi(H^+) \subseteq H^+$.

Theorem 1.4 ([6]) *Let L be a real transcendental meromorphic function in the plane such that all but finitely many poles of L are real and simple and have positive residues. Let ϕ and ψ be as in (2) and (A), (B) and (C). If ϕ is transcendental then $L + L'/L$ has infinitely many non-real zeros.*

Now suppose that L is a real transcendental meromorphic function in the plane such that all but finitely many poles of L are real and simple and have negative residues. Then applying the above construction to $-L$ shows that L has a factorisation (2) satisfying (A), (B) and (C). The following direct counterpart of Theorem 1.4 is the main result of this paper.

Theorem 1.5 *Let L be a real transcendental meromorphic function in the plane such that all but finitely many poles of L are real and simple and have negative residues. Let ϕ and ψ be as in (2) and (A), (B) and (C). If ϕ is transcendental then $L + L'/L$ has infinitely many non-real zeros.*

Theorem 1.5 will lead at once to:

Theorem 1.6 *Let f be a real meromorphic function of infinite order in the plane with finitely many zeros and non-real poles, and let $a \in \mathbb{R}$ satisfy $a < 1$. Then f''/f' and $ff''/(f')^2 - a$ each have infinitely many non-real zeros.*

Nicks [25] proved that the same conclusions hold if $a < 1$ and f is a real entire function of infinite order with finitely many non-real zeros: this was deduced in [25] from Theorem 1.4, and the same method is used here to derive Theorem 1.6 from Theorem 1.5. On combination with Theorem 1.3, the assertion concerning non-real zeros of f''/f' in Theorem 1.6 shows that the hypothesis in Theorem 1.2 that f' has only real zeros may be dispensed with. Again the transformation (1) shows that $gg''/(g')^2 - a$ has infinitely many non-real zeros if $a > 1$ and g is a real entire function of infinite order with real zeros.

2 Proof of Theorem 1.5

To prove Theorem 1.5 assume that L, ϕ and ψ are as in the hypotheses, with ϕ transcendental, but that $L + L'/L$ has finitely many non-real zeros. Several of the initial steps of the proof will be similar to arguments from [6]. Write

$$H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad H^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\},$$

as well as

$$F(z) = z - \frac{1}{L(z)}, \quad F'(z) = 1 + \frac{L'(z)}{L(z)^2} = \frac{1}{L(z)} \left(L(z) + \frac{L'(z)}{L(z)} \right). \tag{3}$$

Lemma 2.1 *If x_0 is a real simple pole of L with negative residue then x_0 does not lie in the closure of the set $Y^- = \{z \in H^+ : L(z) \in H^-\}$.*

Proof. This follows from a now standard argument due to Sheil-Small [29], using the fact that L is univalent on a neighbourhood of x_0. The details may be found in [6, p.987]. □

The next lemma requires the Tsuji characteristic [30] (see also [6, p.980]).

Lemma 2.2 *The Tsuji characteristic of L satisfies $\mathfrak{T}(r, L) = O(\log r)$ as $r \to \infty$.*

Proof. This is identical to the proof of [6, Lemma 3.1]. The method is a combination of the Tsuji characteristic with Hayman's alternative [9, p.60], starting from the fact that $g = 1/L$ and $g' - 1 = -F'$ have finitely many zeros in H^+, using (3). □

If h is a transcendental meromorphic function in the plane then a path γ tending to infinity on which $h(z)$ tends to infinity defines a direct transcendental singularity of h^{-1} over ∞ [4] if there exists $M > 0$ with the property that an unbounded subpath of γ lies in a component C_M of the set $\{z \in \mathbb{C} : |h(z)| > M\}$ such that C_M contains no poles of h. This singularity will be referred to as lying in H^+ if there exists such an M with $C_M \subseteq H^+$.

Lemma 2.3 *The inverse function L^{-1} has at most one direct transcendental singularity over ∞ lying in H^+, and there exist at most finitely many values $\alpha \in \mathbb{C}$ such that $L(z) \to \alpha$ on a path tending to infinity in $\mathbb{C} \setminus \mathbb{R}$. Finally, F has finitely many non-real critical and asymptotic values.*

Proof. The first assertion holds by Lemma 2.2 and [19, Lemma 2.4]. Next, because L is real and has finitely many non-real poles, the existence of infinitely many $\alpha \in \mathbb{C}$ such that $L(z) \to \alpha$ on a path tending to infinity in $\mathbb{C} \setminus \mathbb{R}$ would give infinitely many direct transcendental singularities of L^{-1} over ∞ lying in H^+ and an immediate contradiction. Now F has finitely many non-real asymptotic values, because $F(z) \to \alpha \in \mathbb{C}$ on a path tending to infinity implies (as in [6, pp.984-5]) that

$$z^2 L(z) - z = \frac{zF(z)}{z - F(z)} \to \alpha,$$

and $z^2 L(z) - z$ again has finitely many non-real poles. The fact that F has finitely many non-real critical values follows from (3). □

The next lemma is an immediate consequence of Lemma 2.3.

Lemma 2.4 *For real $K > 0$ let*

$$H_K^+ = \{z \in H^+ : |z| > K\}, \quad W_K = \{z \in H^+ : F(z) \in H_K^+\}. \tag{4}$$

Then there exists a large positive real number K such that F^{-1} has no singular values in H_K^+, and F maps each component of W_K conformally onto H_K^+.

Lemma 2.5 *Let A be a component of W_K which contains a path γ tending to infinity on which $L(z)$ tends to infinity. Then there is no zero of L in the closure of A.*

Proof. This is again by an argument due to Sheil-Small [29]. A zero a of L in the closure of A must be a pole of F on the boundary of A, and if M is positive and sufficiently large then all values $w \in H_M^+$ are taken by F at points in A which lie close to a. But $F(z)$ tends to infinity as z tends to infinity on γ, by (3), and this contradicts the fact that F is univalent on A by Lemma 2.4. □

Lemma 2.6 *The function ϕ has order at most 1.*

Proof. This follows from Lemma 2.2; the argument is identical to that of [6, Lemma 3.2]. □

If r is large and positive then the function $g(z) = g_r(z) = 1/rL(rz)$ has no zeros in the domain $D = \{z \in H^+ : 1/2 < |z| < 2\}$, and nor has $g'(z) - 1 = -F'(rz)$. Applying Gu's normal family counterpart of Hayman's alternative [10] (see also [28]) to the family $\{g_r\}$, in conjunction with Lemma 2.6, then gives the following, which is proved exactly as in [6, pp.982-3].

Lemma 2.7 *Let δ and K_0 be positive real numbers. Then L satisfies, for all $r \geq 1$ in a set of logarithmic density 1,*

$$|zL(z)| > K_0 \quad \text{for } |z| = r,\ \delta < |\arg z| < \pi - \delta.$$

The next lemma is deduced from Lemmas 2.6 and 2.7 as in [6, pp.982-3].

Lemma 2.8 *The function ϕ has infinitely many zeros.*

Lemma 2.9 *The function L has infinitely many non-real zeros, or infinitely many real zeros x with $L'(x) \leq 0$.*

Proof. This is similar to [6, Lemma 3.5]. Assume that L has finitely many non-real zeros and finitely many real zeros x with $L'(x) \leq 0$. Since ϕ has infinitely many zeros by Lemma 2.8, infinitely many zeros of ϕ must be real. If a real interval contains infinitely many zeros of ϕ but no poles of L, then it contains infinitely many zeros x of L, of which infinitely many must be such that $L'(x) \leq 0$, contrary to assumption. Hence there must exist infinitely many open intervals $I_k = (a_k, a_{k+1})$ such that a_k and a_{k+1} are real simple poles of L and of ψ, in both cases with negative residues, such that I_k contains no poles of L but at least one zero of ϕ. But then I_k contains a zero of ψ and so at least two zeros x of L, counting multiplicities, of which at least one must have $L'(x) \leq 0$. This prove Lemma 2.9. □

The remainder of the proof of Theorem 1.5 departs from the methods of [6]. Denote by $B(a, r)$ the open disc of centre a and radius r, and by $S(a, r)$ its boundary circle.

Lemma 2.10 *There exist $\theta \in (0, \pi)$ and $N_0 \in \mathbb{N}$ with the following properties. First, L^{-1} has no singular values in $R^+ \cup R^-$, where*

$$R^+ = \{re^{i\theta} : 0 < r < \infty\}, \quad R^- = \{re^{-i\theta} : 0 < r < \infty\}. \tag{5}$$

Next, let

$$x = \frac{K}{\sin \theta}, \tag{6}$$

where the real number K is as chosen in Lemma 2.4. Then there exist at most N_0 points z on the circle $S(0, 2x)$ with $L(z) \in R^+$.

Proof. The existence of θ follows from Lemma 2.3 and the fact that L is real, and the second assertion holds since L is transcendental. □

Lemma 2.11 *Let $N > N_0$ be a large positive integer. Then there exists a set of $2N+1$ pairwise distinct zeros z_n of L satisfying exactly one of the following alternatives:*

$$\text{(i) every } z_n \text{ lies in } H^+\text{; (ii) every } z_n \text{ is real and } L'(z_n) \leq 0. \tag{7}$$

Moreover, for $1 \leq n \leq 2N+1$ there exist pairwise disjoint simple paths $\gamma_n : [0, \infty) \to \mathbb{C}$ such that $\gamma_n(0) = z_n$, while $\gamma_n(t) \in H^+$ and $L(\gamma_n(t)) = te^{-i\theta}$ for $t > 0$, where θ is as in Lemma 2.10, and $\gamma_n(t) \to \infty$ as $t \to \infty$. In addition, for $1 \leq n \leq 2N$ there exists a simply connected domain $D_n \subseteq H^+$ which is bounded by γ_n, γ_{n+1} and a simple arc σ_n joining z_n to z_{n+1}, with the following properties:
(a) each arc σ_n contains no poles of L and no zeros of L apart from z_n and z_{n+1}, and the σ_n are pairwise disjoint apart from the fact that $\sigma_n \cap \sigma_{n+1} = \{z_{n+1}\}$, and they satisfy

$$\sigma_n \setminus \{z_n, z_{n+1}\} \subseteq H^+, \quad \sigma_n \cap \gamma_n = \{z_n\}, \quad \sigma_n \cap \gamma_{n+1} = \{z_{n+1}\};$$

(b) with R^+ as in (5), each σ_n intersects $L^{-1}(R^+)$ finitely often, and never tangentially;
(c) the function L has no poles in D_n.

Proof. Lemma 2.9 shows that an arbitrarily large number N_1 of distinct zeros of L may be chosen so as to satisfy exactly one of (i) and (ii). The existence of the paths γ_n, and the fact that they are pairwise disjoint, follow from analytic continuation and the choice of θ. Letting $t \to +\infty$ forces $\gamma_n(t)$ to tend either to infinity or to a pole z^* of L. In the latter case z^* must be non-real, or a multiple pole, or a real simple pole with positive residue, using Lemma 2.1. Since all but finitely many poles of L are real and simple and have negative residues, it may be assumed by Lemma 2.9 that $\gamma_n(t)$ tends to infinity for $1 \leq n \leq N_1$.

In case (i) of (7) each path γ_n may be extended to a simple path Γ_n in $H^+ \cup \{0\}$ in such a way that the Γ_n are pairwise disjoint apart from a common starting point at the origin, and the z_n may be re-labelled so that Γ_n separates Γ_{n-1} from Γ_{n+1} in H^+, for $n = 2, 3, \ldots, N_1 - 1$. On the other hand if case (ii) of (7) holds then clearly it may be assumed that $z_n < z_{n+1}$. In either case it is then possible to choose simple arcs σ_n satisfying (a). Each σ_n may be chosen so that $\sigma_n \cap L^{-1}(R^+)$ is finite (for example, by initially forming σ_n as a union of circular arcs, using the fact that L is transcendental). Distorting σ_n slightly, if necessary, leads to (b). Since L has finitely many non-real poles, reducing N_1 if necessary and re-labelling gives (c). □

Using Lemma 2.11(a), choose $M > 0$ such that

$$|L(z)| < M \quad \text{for all} \quad z \in \bigcup_{n=1}^{2N} \sigma_n. \tag{8}$$

Lemma 2.12 *There exist $P > 0$ and a set $X \subseteq \{1, \ldots, 2N\}$, with at least N elements and the following properties. For each $n \in X$ there exists a path $\mu_n : \mathbb{R} \to H^+$ such that:*
(a) L maps μ_n injectively onto the ray R^+ in (5), and these paths μ_n are pairwise disjoint;
(b) the subpath $\mu_n^+ = \{z \in \mu_n : |L(z)| \geq P\}$ lies in D_n and tends to infinity;
(c) there exists a component A_n of the set W_K defined in Lemma 2.4, such that $\mu_n \subseteq A_n$;
(d) the closure of A_n contains no zero of L.

It is not asserted that these components A_n must be distinct.

Proof. Use Lemma 2.3 to choose P such that the following conditions are all satisfied: P/M is large and positive; L has no critical values on $S(0,P)$; there does not exist $\alpha \in S(0,P)$ such that $L(z)$ tends to α on a path tending to infinity in $\mathbb{C} \setminus \mathbb{R}$. For each $n \in \{1,\ldots,2N\}$ take the branch of L^{-1} which maps $Pe^{-i\theta}$ to $\gamma_n(P)$ and continue L^{-1} along $S(0,P)$, in the direction which takes $z = L^{-1}(w)$ into D_n: this is possible since $S(0,P)$ is perpendicular to the ray R^- in (5). By the choice of P and the fact that (8) gives

$$L(\partial D_n) \subseteq B(0,M) \cup R^-, \tag{9}$$

this continuation leads to $\zeta_n \in D_n$ with $L(\zeta_n) = Pe^{i\theta}$.

Each ζ_n lies on a path $\mu_n : \mathbb{R} \to H^+$ as in (a), and these paths μ_n are pairwise disjoint, by the choice of θ and the fact that $\zeta_n \in D_n$ and the D_n are pairwise disjoint. Follow μ_n, starting from ζ_n, in the direction in which $|L(z)|$ increases. The resulting path μ_n^+ cannot exit D_n, by (9). Moreover, since L has no poles in D_n, it follows that μ_n^+ is a path tending to infinity in D_n on which $L(z)$ tends to infinity, which proves (b).

Since $N > N_0$, Lemma 2.10 now gives a set $X \subseteq \{1,\ldots,2N\}$, with at least N elements, such that for $n \in X$ the path μ_n lies in H^+_{2x} as defined by (4). Now if $z \in \mu_n$ and $|L(z)| \geq 1/x$ then (3) and (6) give

$$\operatorname{Im} F(z) > 0, \quad |F(z)| \geq |z| - \frac{1}{|L(z)|} > 2x - x = \frac{K}{\sin\theta} > K.$$

On the other hand, if $z \in \mu_n$ and $|L(z)| = r < 1/x$ then $F(z)$ satisfies

$$|F(z)| \geq \operatorname{Im} F(z) > -\operatorname{Im} \frac{1}{L(z)} = \frac{\sin\theta}{r} > x\sin\theta = K.$$

This proves (c), and (d) now follows from (b) and Lemma 2.5, since $\mu_n^+ \subseteq \mu_n \subseteq A_n$. $\square$

Now let $n \in X$, and follow μ_n, starting from $\zeta_n \in D_n$, this time in the direction in which $|L(z)|$ decreases. The resulting path μ_n^- must tend to infinity in A_n, by the choice of θ and Lemma 2.12, parts (c) and (d), but cannot meet $\gamma_n \cup \gamma_{n+1}$. In case (ii) of (7) it now follows that for each $n \in X$ there exists a path tending to infinity in the part of H^+ bounded by γ_n, γ_{n+1} and the interval (z_n, z_{n+1}), such that $L(z)$ tends to 0 as z tends to infinity on this path. Since $L(z)$ tends to infinity on the path γ_{n+1}, this gives at least $N-1 \geq 2$ direct singularities of L^{-1} over ∞ lying in H^+, contradicting Lemma 2.3. This completes the proof of Theorem 1.5 in case (ii).

Assume henceforth that case (i) of (7) applies, and take $n \in X$. Lemma 2.12(d) and (3) imply that there exists a large positive real number Q such that

$$|L(z)| \geq \frac{2}{Q} \quad \text{and} \quad |F(z)| \leq Q \quad \text{for all} \quad z \in A_n \cap \sigma_n. \tag{10}$$

Choose S large and positive, in particular with $S\sin\theta > 2Q$, and choose u_n, v_n on μ_n such that

$$L(u_n) = Se^{i\theta}, \quad L(v_n) = S^{-1}e^{i\theta}. \tag{11}$$

Then $u_n \in \mu_n^+ \subseteq D_n$ by Lemma 2.12(b), while $v_n \in \mu_n^-$. Moreover, u_n and v_n satisfy

$$|F(u_n)| > 2Q, \quad |F(v_n)| > 2Q, \tag{12}$$

the first of these holding because S is large and z and $F(z)$ tend to infinity as $L(z)$ tends to infinity on μ_n^+, using (3), while the second holds because (3) and (11) give

$$F(v_n) = v_n - Se^{-i\theta}, \quad \operatorname{Im} F(v_n) > S \sin\theta > 2Q.$$

In particular, u_n and v_n lie in $A_n \setminus \sigma_n$ by Lemma 2.12(c) and (10).

Lemma 2.13 *The points u_n and v_n lie in the same component of $A_n \setminus \sigma_n$, and v_n lies in D_n.*

Proof. Using (4) and (12), join $F(u_n)$ to $F(v_n)$ by a simple path p_n in H_Q^+. Then, since Q is large and F maps A_n univalently onto H_K^+, by Lemma 2.4, there exists a simple path τ_n in A_n joining u_n to v_n, with $F(\tau_n) = p_n$, and τ_n cannot meet σ_n by (10), which proves the first assertion. Now suppose that $v_n \notin D_n$. The points v_n and u_n are joined by a simple subpath λ_n of μ_n, which cannot meet $\gamma_n \cup \gamma_{n+1}$ and so must cross σ_n, since $u_n \in \mu_n^+ \subseteq D_n$ and $v_n \notin \partial D_n$, by (10) and (12). On the other hand τ_n joins u_n to v_n in A_n but does not meet σ_n. Now $\sigma_n \cap \lambda_n$ is finite, and λ_n does not intersect σ_n tangentially, by Lemma 2.11(b). It may be assumed that $\tau_n \cap \lambda_n$ is also finite, since τ_n can be distorted slightly. Since u_n lies in D_n but v_n outside, λ_n must intersect σ_n an odd number of times. This gives a component λ_n^* of $\lambda_n \setminus \tau_n$ which intersects σ_n an odd number of times. Hence the union of λ_n^* with a simple subpath of τ_n gives a Jordan curve Λ_n in A_n which intersects σ_n an odd number of times. But the endpoints z_n and z_{n+1} of σ_n are zeros of L and so do not lie in A_n, by Lemma 2.12(d), and they cannot both lie in the unbounded component of the complement of Λ_n. Since A_n is simply connected by Lemma 2.4, this is a contradiction. □

Lemma 2.13 may now be applied for all sufficiently large S in (11), and implies that for each $n \in X$ the domain D_n contains a path tending to infinity on which $L(z)$ tends to 0. Since $L(z)$ tends to infinity on the path γ_{n+1}, this again gives at least $N - 1 \geq 2$ direct singularities of L^{-1} over ∞ lying in H^+, which contradicts Lemma 2.3. This completes the proof of Theorem 1.5. □

3 Proof of Theorem 1.6

Let the function f be as in the hypotheses. Then $L = f'/f$ has only simple poles, all but finitely many of which are real with negative residue. Thus L has a Levin-Ostrovskii factorisation (2), and ϕ must be transcendental, by an argument identical to that of [6, Lemma 5.1]. The assertion that f''/f' has infinitely many non-real zeros now follows at once from Theorem 1.5, since $L + L'/L = f''/f'$. The second assertion is proved using a method from [25]. Define M and N by $M = (1-a)L$ and

$$N = M + \frac{M'}{M} = (1-a)L + \frac{L'}{L} = \frac{f''}{f'} - a\frac{f'}{f} = \frac{f'}{f}\left(\frac{ff''}{(f')^2} - a\right) = \frac{M}{1-a}\left(\frac{ff''}{(f')^2} - a\right).$$

Applying Theorem 1.5 to M shows that N has infinitely many non-real zeros, and since a zero of M is a pole of N these must be zeros of $ff''/(f')^2 - a$. □

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