Integer points of analytic functions in a half-plane

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Abstract

It is shown that if $f$ is an analytic function of sufficiently small exponential type in the right half-plane, which takes integer values on a subset of the positive integers having positive lower density, then $f$ is a polynomial. MSC2000: 30D20, 30D35.

1 Introduction

A classical theorem of Pólya (see [14] and [21, p.55]) shows that $2^z$ is the slowest growing transcendental entire function which takes integer values at the non-negative integers. That is, let $f$ be entire and take integer values on $\mathbb{N} \cup \{0\}$. Pólya shows that if

$$\limsup_{r \to \infty} \frac{M(r,f)}{2^r} < 1,$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$, then $f$ is a polynomial and, further, that if

$$M(r,f) = O(r^N2^r)$$

(1)

as $r \to \infty$ for some $N > 0$, then there exist polynomials $P_1$ and $P_2$ such that $f(z) \equiv P_1(z)2^z + P_2(z)$. Further results on integer-valued entire functions may be found in [1, 2, 4, 6, 11, 12, 16, 17, 18, 19, 20].

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This article is concerned with similar results for analytic functions in a half-plane. It was proved in [8, Lemma 5] that if \( f \) is analytic of polynomial growth in the right half-plane and takes integer values at the positive integers, then \( f \) is a polynomial. This result has several applications to value distribution theory and differential equations [8, 9, 10]. In [13], an analogue of Pólya’s result for a half-plane is given. That is, let \( f \) be analytic in the closed right half-plane \( \Omega = \{ z \in \mathbb{C} : \text{Re}(z) \geq 0 \} \) with maximum modulus

\[
M_\Omega(r, f) = \max\{|f(z)| : z \in \Omega, |z| \leq r \},
\]

and assume that \( f(n) \) is an integer for all sufficiently large positive integers \( n \). If \( f \) satisfies (1) as \( r \to \infty \) for some \( N > 0 \), with \( M(r, f) \) replaced by \( M_\Omega(r, f) \), then again there exist polynomials \( P_1 \) and \( P_2 \) with \( f(z) \equiv P_1(z)z^2 + P_2(z) \).

Further, if \( f \) takes integer values at all the non-negative integers and

\[
\limsup_{|z| \to \infty, z \in \Omega} \frac{|f(z)|}{2|z|} < 1,
\]

then \( f \) is a polynomial.

We remark that a result was proved in [22] for functions holomorphic on the product \( \Omega^n \) of \( n \) half-planes and taking integer values on \( \mathbb{N}^n \). This result contains [8, Lemma 5], but not the theorem from [13]. We are very grateful to the referee for drawing our attention to this reference and to others such as [1, 2, 23, 24].

In order to state our result some terminology will be required. Let \( f \) be analytic in \( \Omega \), and let \( 0 \leq \lambda < \infty \). Then \( f \) is of exponential type \( \lambda \) in \( \Omega \) if

\[
\limsup_{r \to \infty} \frac{\log^+ M_\Omega(r, f)}{r} = \lambda,
\]

where \( \log^+ x = \max\{0, \log x\} \) and \( M_\Omega(r, f) \) is as in (2). This is of course in direct analogy with the definition of exponential type for entire functions.

The main result to be proved is the following half-plane analogue of a theorem of Waldschmidt for entire functions [17].

**Theorem 1.1.** Let \( d, J, \lambda \) satisfy

\[
0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad 16 \left( \frac{1 + \log(1 + J/2)}{J} \right) + 8(J-1)\lambda < d^2.
\]
Let $E \subset \mathbb{N}$ have lower linear density

$$D(E) = \liminf_{n \to \infty} \frac{|E \cap \{1, \ldots, n\}|}{n} > d,$$

where $|X|$ denotes the number of elements of the set $X$. Let the function $f$ be analytic of exponential type less than $\lambda$ in the closed right half-plane $\Omega$, and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then $f$ is a polynomial.

2 Lemmas used in the proof of Theorem 1.1

2.1 Linear forms

The following lemma is a slight modification of [6, Lemma I, p.11]: a proof is given for completeness.

Lemma 2.1. Let $A \geq 1$ and $N \geq 2$ be integers. Suppose that $L_1, \ldots, L_m$ are linear forms in the $n$ variables $x_1, \ldots, x_n$, with real coefficients $a_{j,k}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, that is,

$$L_j = a_{j,1}x_1 + \ldots + a_{j,n}x_n.$$

Suppose further that $n > m$ and

$$\max_{j,k} |a_{j,k}| \leq A.$$

Then there exist integers $x_1, \ldots, x_n$, not all zero, such that

$$|L_j| \leq \frac{1}{N}$$

for $j = 1, \ldots, m$, and

$$|x_k| \leq 2 \cdot (2nAN)^{\frac{m}{n-m}}$$

for $k = 1, \ldots, n$.

Proof. Define $X$ by

$$X = \left\lceil (2nAN)^{\frac{m}{n-m}} \right\rceil,$$

where $\lceil x \rceil$ denotes the greatest integer not exceeding $x$. An $n$-tuple of integers $(x_1, \ldots, x_n)$, in which each $x_k$ has absolute value no greater than $X$,
gives rise to a point \((L_1,\ldots,L_m)\) lying in the closed \(m\)-dimensional cube of centre \((0,\ldots,0)\) and side length \(2nAX\). Divide this cube into \((2nAXN)^m\) closed subcubes each of side length \(1/N\). The number of distinct \(n\)-tuples \((x_1,\ldots,x_n)\) is evidently

\[(2X + 1)^n \geq (2 \cdot (2nAN)^{m/n} - 1)^n > (2nAN)^{m/n} \geq (2nAXN)^m,\]

since if this is not the case then we get

\[(2nAN)^n < (2nAXN)^{n-m} \leq (2nAN)^{n-m} \cdot (2nAN)^m,\]

which is impossible. Hence there are distinct \(n\)-tuples giving rise to points \((L_1',\ldots,L_m')\) and \((L_1'',\ldots,L_m'')\) lying in the same subcube. But then we may write

\[
\left| \sum_{k=1}^{n} a_{j,k}(x_k' - x_k'') \right| = \left| L_j' - L_j'' \right| \leq \frac{1}{N}
\]

for \(j = 1,\ldots,m\), where

\[|x_k' - x_k''| \leq 2X \leq 2 \cdot (2nAN)^{m/n} \]

and \(x_k = x_k' - x_k'' \neq 0\) for at least one \(k\). This completes the proof. \(\square\)

### 2.2 An application of the maximum principle

**Lemma 2.2.** Let \(d, M, L, K\) satisfy

\[0 < d < 1, \ M > 0, \ 1 < K < L < \infty, \ ML^2K < d^2(L - K). \quad (5)\]

Let \(G \subseteq \mathbb{N}\) and let \(F\) be analytic in the closed right half-plane \(\Omega\) such that \(F(z) \in \mathbb{Z}\) for all \(z \in G\). Let \(s > 0\) be such that \(M_\Omega(Ls, F) \leq e^{MLs}\) and \(F\) has \(m \geq ds\) distinct zeros in \(G \cap [1,s]\). Then \(F(z) = 0\) for all \(z \in G \cap [s,KS]\).

**Proof.** Let \(x_1,\ldots,x_m\) be distinct zeros of \(F\) in \(G \cap [1,s]\). For \(0 < x \leq s\) let

\[p(z) = p(z,x) = \frac{z - x}{z + x}.\]

Then \(p\) satisfies

\[|p(z)| = 1 \ (z \in i\mathbb{R}), \ |p(z)| \geq \frac{Ls - x}{Ls + x} \ (|z| = Ls),\]
and

$$|p(z)| \leq \frac{Ks - x}{Ks + x} \quad (z \in [s, Ks] \subseteq \mathbb{R}),$$

the last estimate following from monotonicity. Next, let

$$g(x) = \log \left[ \left( \frac{Ls + x}{Ls - x} \right) \left( \frac{Ks - x}{Ks + x} \right) \right].$$

Then, for $0 \leq x \leq s$,

$$g'(x) = \frac{1}{Ls + x} + \frac{1}{Ls - x} - \frac{1}{Ks + x} - \frac{1}{Ks - x}$$

$$= \frac{Ls^2 - x^2 - Ks^2 - x^2}{2Ls}$$

$$= \frac{2s^3KL(K - L) + 2x^2s(K - L)}{(Ls^2 - x^2)(Ks^2 - x^2)}$$

$$\leq \frac{2(K - L)}{LKs}$$

and hence

$$g(x) \leq \frac{2x(K - L)}{LKs}.$$  

The function

$$F_1(z) = F(z) \prod_{j=1}^{m} \frac{1}{p(z, x_j)}$$

is analytic in $\Omega$ and satisfies

$$|F_1(z)| \leq M_\Omega(Ls, F) \prod_{j=1}^{m} \frac{Ls + x_j}{Ls - x_j}$$

on the boundary of the region given by $z \in \Omega, |z| \leq Ls$, and this estimate also holds for $z \in [s, Ks]$, by the maximum principle. For $z \in [s, Ks]$ it
therefore follows that

\[
|F(z)| \leq M_\Omega(Ls, F) \prod_{j=1}^{m} \left[ \frac{(Ls + x_j)}{(Ls - x_j)} \left( \frac{Ks - x_j}{Ks + x_j} \right) \right]
\]

\[
= M_\Omega(Ls, F) \exp \left( \sum_{j=1}^{m} g(x_j) \right)
\]

\[
\leq M_\Omega(Ls, F) \exp \left( \frac{2(K - L)}{LKs} \sum_{j=1}^{m} x_j \right)
\]

\[
\leq M_\Omega(Ls, F) \exp \left( \frac{2(K - L)}{LKs} \cdot \frac{m(m + 1)}{2} \right)
\]

\[
\leq \exp \left( MLs + \frac{d^2(K - L)s}{LK} \right) < 1,
\]

using (5) and the fact that the \(x_j\) are distinct positive integers, which proves the lemma.

In order to apply Lemma 2.2, it is necessary for a given \(d\) to choose \(M\), \(L\) and \(K\) with (5) in mind. Evidently if

\[
ML^2 < d^2(L - 1)
\]

then \(K\) may be chosen with \(K - 1\) small and positive so that (5) is satisfied. Since elementary calculus gives

\[
q(L) = \frac{L - 1}{L^2} \leq q(2) = \frac{1}{4}
\]

for \(1 < L < \infty\), the appropriate condition is \(4M < d^2\).

**Lemma 2.3.** Let \(0 < d < 1\) and \(0 < 4M < d^2\). Let \(G \subseteq \mathbb{N}\) and let \(F\) be analytic in \(\Omega\) such that \(F(z) \in \mathbb{Z}\) for all \(z \in G\). Let \(S > 0\) be such that

\[
Q(r) = |G \cap [0, r]| \geq dr \quad \text{and} \quad M_\Omega(r, F) \leq e^{Mr}
\]

for all \(r \geq S\), and assume that \(F(z) = 0\) for all \(z\) in \(G \cap [1, S]\). Then \(F(z) = 0\) for all \(z \in G\).
Proof. Choose \( L = 2 \) and \( K \in (1, 2) \) such that (5) is satisfied. Then \( F \) has at least \( dS \) distinct zeros in \( G \cap [1, S] \). Applying Lemma 2.2 with \( s = S \) then shows that \( F(z) = 0 \) for all \( z \in G \cap [S, KS] \), from which it follows at once that \( F \) has at least \( Q(KS) \geq dKS \) distinct zeros in \([1, KS]\). Hence Lemma 2.2 may again be applied, this time with \( s = KS \). Repetition of this argument proves Lemma 2.3.

2.3 The Nevanlinna characteristic in a half-plane

This section provides a brief overview of a half-plane characteristic analogous to the Nevanlinna characteristic in the plane, the details of which may be found in [7, p.38]. Let \( f \) be meromorphic in the closed upper half-plane

\[
\mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) \geq 0 \}
\]

with poles at \( \rho_n e^{i\psi_n} \), where \( \rho_n \geq 0 \) and \( 0 \leq \psi_n \leq \pi \). The counting function of the poles is

\[
c(r, f) = \sum_{1<\rho_n \leq r} \sin \psi_n,
\]

and the integrated counting function takes the form

\[
C(r, f) = 2 \int_1^r c(t, f) \left( \frac{1}{t^2} + \frac{1}{r^2} \right) dt = 2 \sum_{1<\rho_n \leq r} \left( \frac{1}{\rho_n} - \frac{\rho_n}{r^2} \right) \sin \psi_n.
\]

The analogue of the Nevanlinna proximity function consists of the following two functions:

\[
A(r, f) = \frac{1}{\pi} \int_1^r \left( \frac{1}{t^2} - \frac{1}{r^2} \right) \left[ \log^+ |f(t)| + \log^+ |f(-t)| \right] dt;
\]

\[
B(r, f) = \frac{2}{\pi r} \int_0^\pi \log^+ |f(re^{i\phi})| \sin \phi \, d\phi.
\]

The half-plane characteristic is then given by

\[
S(r, f) = A(r, f) + B(r, f) + C(r, f)
\]

and satisfies, for non-constant \( f \) and \( a \in \mathbb{C} \),

\[
S \left( r, \frac{1}{f-a} \right) = S(r, f) + O(1)
\]
The following lemma uses the half-plane characteristic and is in the spirit of Carlson’s theorem [5]. For generalisations in other directions see [23, 24].

**Lemma 2.4.** Let $E \subseteq i\mathbb{N} = \{i, 2i, ...\}$ have lower density $D$. Let $f$ be analytic in $\mathbb{H}$, of exponential type $\lambda < \pi D$, with $f(z) = 0$ for all $z \in E$. Then $f \equiv 0$.

Here the lower density of $E$ and exponential type relative to the upper half-plane are defined in straightforward analogy with §1.

**Proof.** Assume that $f$ is not identically zero. As $r \to \infty$,

$$B(r, f) \leq \frac{2}{\pi r} \int_0^\pi (\lambda + o(1)) r \sin \phi \, d\phi = O(1)$$

and

$$A(r, f) \leq \frac{1}{\pi} \int_1^r \left( \frac{1}{t^2} - \frac{1}{r^2} \right) 2(\lambda + o(1)) t \, dt + O(1) \leq \frac{2(\lambda + o(1))}{\pi} \log r.$$  

Since $f$ has no poles in $\mathbb{H}$ applying (6) with $a = 0$ now gives

$$S(r, 1/f) \leq A(r, f) + B(r, f) + O(1) \leq \frac{2(\lambda + o(1))}{\pi} \log r. \tag{7}$$

But since the lower density of $E$ is $D$ we have

$$c(r, 1/f) \geq \sum_{n \in \mathbb{N} \cap [1,r], in \in E} 1 \geq (D - o(1)) r$$

as $r \to \infty$. Integrating this yields

$$S(r, 1/f) \geq C(r, 1/f) \geq 2 \int_1^r (D - o(1)) t \left( \frac{1}{t^2} + \frac{1}{r^2} \right) \, dt \geq 2(D - o(1)) \log r$$

as $r \to \infty$, which on combination with (7) gives $\lambda \geq \pi D$, a contradiction. Therefore $f$ must be identically zero. \qed
2.4 A class of polynomials

The following lemma summarises some basic facts from [17] concerning a class of polynomials which are key to the proof of Theorem 1.1.

Lemma 2.5. Define polynomials \( p_0, p_1, \ldots \) by

\[
p_0(z) = 1, \quad p_1(z) = z, \quad p_h(z) = \frac{z(z-1)\ldots(z-h+1)}{h!} \quad (h = 2, 3, \ldots).
\]  

(8)

Then \( p_h(\mathbb{Z}) \subseteq \mathbb{Z} \) and for \( R > 0 \) and \( H \in \mathbb{N} \) we have

\[
|p_h(z)| \leq e^H \left( \frac{R}{H} + 1 \right)^H \quad \text{for } |z| \leq R, \ h = 0, \ldots, H.
\]  

(9)

Proof. It is easy to check that \( p_h(\mathbb{Z}) \subseteq \mathbb{Z} \). To prove (9) write, following [17],

\[
|p_h(z)| \leq \frac{(R + H)^h}{h!} \leq H^h h! \left( \frac{R}{H} + 1 \right)^H \leq e^H \left( \frac{R}{H} + 1 \right)^H.
\]

\( \square \)

2.5 Algebraic functions mapping integers to integers

Proposition 2.6. Let the algebraic function \( f \) be analytic in \( \Omega \) and satisfy \( f(E) \subseteq \mathbb{Z} \) for some set \( E \subseteq \mathbb{N} \) of positive lower density. Then \( f \) is a polynomial.

To prove Proposition 2.6, let \( E \) and \( f \) be as in the hypotheses, and assume that the lower density of \( E \) exceeds \( D > 0 \). We assert that \( f \) maps the positive real axis into \( \mathbb{R} \). To see this observe that the functions \( \overline{f(z)} \) and \( f(z) - \overline{f(z)} \) are algebraic because \( f \) is. Since \( f(z) \in \mathbb{R} \) for \( z \in E \) and since an algebraic function having a sequence of zeros tending to infinity must vanish identically, the assertion follows.

Again since \( f \) is algebraic there exists a positive integer \( m \) such that, for all sufficiently large \( r \),

\[
M_\Omega(r, f) \leq r^m.
\]  

(10)

Let \( n \) and \( N \) be integers with \( n/m \) and \( N/n \) large, and in particular with

\[
DN \geq n + 1.
\]  

(11)
Lemma 2.7. There exist arbitrarily large \( r \in \mathbb{N} \) such that

\[
|E \cap \{r, r+1, \ldots, r+N-1\}| \geq n+1. \tag{12}
\]

Proof. Assume that there exists \( p_0 \in \mathbb{N} \) such that, for every \( p \geq p_0 \),

\[
|E \cap \{Np, \ldots, N(p+1)-1\}| \leq n.
\]

Since the lower density of \( E \) exceeds \( D \) this gives, for large \( p \),

\[
DNp \leq |E \cap \{1, \ldots, Np\}| \leq (p - p_0)n + O(1) \leq (n + o(1))p,
\]

which contradicts (11). \( \square \)

Let \( \varepsilon \) be small and positive and choose a large positive integer \( r \) satisfying (12). Let \( \Gamma = \Gamma_r \) be the circle of centre \( r \), radius \( \varepsilon r \), described once counterclockwise. Choose distinct

\[
a_0, \ldots, a_n \in E \cap \{r, r+1, \ldots, r+N-1\}. \tag{13}
\]

Then \( a_0, \ldots, a_n \) lie inside \( \Gamma \), since \( r \) is large.

For \( k = 0, \ldots, n \) it follows from Cauchy’s integral formula and the identity

\[
\frac{1}{t - z} = \frac{1}{t - a_0} + \frac{z - a_0}{(t - a_0)(t - a_1)} + \cdots + \frac{(z - a_0) \cdots (z - a_k)}{(t - a_0) \cdots (t - a_k)(t - z)},
\]

which is easily proved by induction, that

\[
f(z) = P_k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{(z - a_0) \cdots (z - a_k) f(t) \, dt}{(t - a_0) \cdots (t - a_k)(t - z)} \tag{14}
\]

for \( z \) inside \( \Gamma \), where

\[
P_k(z) = A_0 + A_1(z - a_0) + \cdots + A_k(z - a_0) \cdots (z - a_{k-1}) \tag{15}
\]

is given by

\[
A_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t) \, dt}{(t - a_0) \cdots (t - a_j)}. \tag{16}
\]

Thus \( P_k(z) \) is the interpolating polynomial of degree at most \( k \) which equals \( f(z) \) at the \( k + 1 \) points \( a_0, \ldots, a_k \) [6, p.103].

10
Next, let
\[ Q = \prod_{0 \leq j < k \leq n} |a_k - a_j| \leq C = N^{(n+1)^2}, \]  
and observe that $C$ is independent of $r$. Since $f(a_j) \in \mathbb{Z}$ for $j = 0, \ldots, n$ it follows from (16) and the residue theorem that
\[ QA_j \in \mathbb{Z} \quad \text{for } j = 0, \ldots, n. \]  
(18)

On the other hand since $r$ is large (13) gives
\[ |t - a_j| \geq \frac{\varepsilon r}{2} \]
for $t \in \Gamma$. Thus combining (10), (16) and (17) yields for $m < j \leq n$, again since $r$ is large,
\[ |QA_j| \leq \frac{C(\varepsilon r)(2r)^m}{(\varepsilon r/2)^{j+1}} < \frac{1}{2}, \]
which in conjunction with (18) gives $A_j = 0$.

Recalling (14) and the definition (15) of $P_k$ it now follows that $P_m = P_n$ and that $f - P_m$ has $n + 1$ zeros $a_0, \ldots, a_n$ in the interval $[r, r + N - 1]$. Hence $f^{(n)} = (f - P_m)^{(n)}$ has a zero in the same interval, using Rolle's theorem. Since $r$ may be chosen arbitrarily large the algebraic function $f^{(n)}$ must vanish identically, and $f$ is a polynomial. This proves Proposition 2.6.
\[ \square \]

3 Proof of Theorem 1.1

Let $E \subseteq \mathbb{N}$ and $d, J, \lambda, f$ be as in the hypotheses. Label the elements of $E$ as $1 \leq \alpha_1 < \alpha_2 < \ldots$. Let $R$ be a large positive integer such that
\[ H = \frac{n}{J} \in \mathbb{N}, \quad \text{where } m = |E \cap [1, R]| \quad \text{and } n = 2m. \]  
(19)

Form the functions
\[ q_{\mu, \nu}(z) = p_\mu(z)f(z)^\nu, \quad \mu = 0, 1, \ldots, H - 1, \quad \nu = 0, 1, \ldots, J - 1, \]
(20)
where $p_\mu$ is defined as in (8). This gives $HJ = n$ functions which we label $g_1, \ldots, g_n$, where
\[ g_k(z) = p_{\mu(k)}(z)f(z)^{\nu(k)}. \]
In order to prove Theorem 1.1, it suffices to show that the functions \(g_1, \ldots, g_n\)
are linearly dependent over \(\mathbb{C}\). Once such a relation
\[
\sum_{k=1}^{n} B_k g_k(z) \equiv 0
\]
is established with the \(B_k\) constants, not all zero, then it cannot be the case
that there is an integer \(q\) such that \(B_k \neq 0\) implies \(\nu(k) = q\), because \(p_h\)
has degree \(h\) in (8). Hence it follows that \(f\) is algebraic, and Proposition 2.6
shows that \(f\) is a polynomial.

In order to prove that the \(g_k\) are linearly dependent, observe first that
\[
a_{j,k} = g_k(\alpha_j) \in \mathbb{Z},
\]
using Lemma 2.5. Moreover, we have, for \(j = 1, \ldots, m\),
\[
|a_{j,k}| \leq e^H \left( \frac{R}{H} + 1 \right)^H \left( 1 + M\Omega(R, f) \right)^{J-1} \leq e^H \left( \frac{R}{H} + 1 \right)^H e^{(J-1)\lambda R} = J(R) \leq A = [J(R)] + 1,
\]
by (9), (20) and the fact that \(R\) is large. Applying Lemma 2.1 with \(N = 2\)
yields integers \(A_1, \ldots, A_n\), not all zero, such that
\[
\sum_{k=1}^{n} A_k g_k(\alpha_j) = 0 \quad (22)
\]
for \(j = 1, \ldots, m\), and
\[
|A_k| \leq 8nA, \quad (23)
\]
since \(n = 2m\). Set
\[
F(z) = \sum_{k=1}^{n} A_k g_k(z). \quad (24)
\]

**Lemma 3.1.** Choose a real number \(M\) with
\[
4 \left( \frac{1 + \log(1 + J/2)}{J} \right) + 2(J - 1)\lambda < M < \frac{d^2}{4}, \quad (25)
\]
using (4). Provided \(R\) was chosen large enough we have
\[
|E \cap [1, r]| \geq dr \quad \text{and} \quad \log^* M_{\Omega}(r, F) \leq Mr \quad \text{for} \quad r \geq R. \quad (26)
\]
Proof. The first inequality of (26) holds provided $R$ was chosen large enough, since $E$ has lower density greater than $d$. Let $c$ denote positive constants which do not depend on $r$ or $R$. Then we have

$$M_{\Omega}(r,F) \leq 8n^2 Ae^H \left(\frac{r}{H} + 1\right)^H (1 + M_{\Omega}(r,f))^{J-1}$$

$$\leq cr^2 e^{2H} \left(\frac{r}{H} + 1\right)^{2H} e^{2(J-1)\lambda_r}$$

using (9), (21), (23) and the fact that $R$ is large. Now (19) gives

$$\frac{r}{H} \geq \frac{R}{H} = \frac{RJ}{n} \geq \frac{J}{2}.$$ 

Since the function

$$\frac{1 + \log(x + 1)}{x}$$

is decreasing for $x > 0$ this yields, for $r \geq R$, 

$$\log^+ M_{\Omega}(r,F) \leq 2r \left(\frac{H}{r}\right) \left(1 + \log \left(\frac{r}{H} + 1\right)\right) + 2(J-1)\lambda_r + O(\log r)$$

$$\leq 4r \left(1 + \log(1 + J/2)\right) + 2(J-1)\lambda_r + O(\log r) < Mr$$

provided $R$ was chosen large enough. \hfill \Box

The function $F$ satisfies $F(z) \in \mathbb{Z}$ for all $z \in E$, and $F(z) = 0$ for all $z \in E \cap [1,R]$ by (22) and (24). It then follows from (25), (26) and Lemma 2.3, with $S = R$ and $G = E$, that $F(z) = 0$ for all $z \in E$. But (25) also gives

$$4M < d^2 < d, \quad M < \pi d,$$

and so (26) and Lemma 2.4, applied to the function $F(-iz)$, show that $F(z)$ vanishes identically, which completes the proof of Theorem 1.1. \hfill \Box

References


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