Lemma 0.1 Let $G$ be a domain in $\mathbb{C}$, whose boundary $\partial G$ consists of countably many pairwise disjoint curves $\gamma_k$, each either simple closed or simple and going to infinity in both directions, and assume that these do not accumulate at finite points in the following sense: each $a \in \mathbb{C}$ has $r_a > 0$ such that the open disc $B(a, r_a)$ meets at most one $\gamma_k$. Let $z_1, z_2$ be in $G$, and let $\eta$ be a simple path from $z_1$ to $z_2$. Then there exists a simple path $\sigma$ from $z_1$ to $z_2$ such that $\sigma$ is contained in $G \cup \partial G$ and consists of sub-paths of $\eta$ and arcs of boundary curves of $G$.

Note that the hypotheses of the lemma are clearly satisfied if $G$ is a component of the set $\{z \in \mathbb{C} : |f(z)| < R\}$, where $f$ is a meromorphic function on $\mathbb{C}$ and $R$ is such that $|f(z)| = R$ implies $f'(z) \neq 0$.

Proof. Assume that $\eta$ is defined on $I = [0, 1]$ and that $\eta$ meets $\partial G$, since otherwise there is nothing to prove. By compactness, $\eta$ is covered by finitely many open discs $B(a, r_a/2)$ such that each $B(a, r_a)$ meets at most one $\gamma_k$. Hence there exists $\varepsilon > 0$ (the minimum of finitely many $r_a/2$) such that if $a \in \eta$ then $B(a, \varepsilon)$ meets at most one $\gamma_k$. Moreover, there exists $\delta > 0$ such that $|\eta(s) - \eta(t)| < \varepsilon$ for all $s, t \in I$ with $|s - t| \leq \delta$. If $\gamma_k$ is bounded, then by the Jordan curve theorem its complement in $\mathbb{C}$ consists of two disjoint domains, $E_k$ and $F_k$. On the other hand, if $\gamma_k$ is unbounded, adjoin to it the point at infinity, so that this time the complement of $\gamma_k$ on the Riemann sphere consists of two disjoint domains, $E_k$ and $F_k$. For each $k$, whether or not $\gamma_k$ is bounded, we may assume that $G \subseteq E_k$.

Thus $G \subseteq E = \cap E_k$, and $E$ is a connected subset of $\mathbb{C}$, by the chaining lemma. Moreover, $E$ is open: to see this, take $a \in E$. Then $a \in \mathbb{C}$ and there exists $r > 0$ such that $B(a, r)$ meets at most one $f \gamma_k$. Hence $B(a, r)$ lies in all but at most one $E_k$, and reducing $r$ if necessary gives an open disc of centre $a$ lying in $E$. It follows that $E$ is a domain, with $G \subseteq E$. Suppose that $G \neq E$. Then there exists a path in $E$ joining $z_1 \in G$ to $z_3 \not\in G$, and this path must meet $\partial G$ and so meet some $\gamma_k$, a contradiction. Hence $G = E$.

Because $z_1 \in G$ and $\partial G$ is closed, $t_0 = \min\{t \in [0, 1] : \eta(t) \in \partial G\}$ exists and is positive: choose $\gamma_k$ such that $\eta(t_0) \in \gamma_k$, and let $t_1 = \max\{t \in [0, 1] : \eta(t) \in \gamma_k\}$. Since $z_1, z_2 \in G \subseteq E_k$, we have $\eta(t) \in E_k$ for $0 \leq t < t_0$ and $t_1 < t \leq 1$. Choose a bounded arc $\lambda$ of $\gamma_k$ joining $\eta(t_0)$ to $\eta(t_1)$ and replace the part of $\eta$ for $t_0 \leq t \leq t_1$ by $\lambda$. By the choice of $t_0, t_1$, this does not affect the injectivity of the path.

For $t_1 < t \leq t_2 = \min\{t_1 + \delta, 1\}$ we have $\eta(t) \in E_k \cap B(\eta(t_1), \varepsilon)$ and so $\eta(t) \not\in \cup \gamma_j$, by the choice of $\varepsilon$ and $t_1$. Assume that $t_3 \in (t_1, t_2]$ has $\eta(t_3) \not\in G$. Then $\eta(t_3) \not\in E$, and so there exists $m \neq k$ with $\eta(t_3) \not\in E_m$ and hence $\eta(t_3) \in F_m$. Since $\eta(t) \not\in \gamma_m$ for $t_1 \leq t \leq t_3$, we must have $\eta(t_1) \in F_m$. But $\eta(t_1) \in \gamma_k \subseteq \partial G$, and so $G$ meets $F_m$, a contradiction.

Hence we have $\eta(t) \in G$ for $t_1 < t \leq t_2$. If $\eta(t) \not\in G$ for some $t \in [t_2, 1]$ then $t_2 < 1$; hence we may take $t_4 = \min\{t \in [t_2, 1] : \eta(t) \in \partial G\}$ and repeat the process, but the fact that $t_4 - t_1 \geq \delta$ means that repetition cannot occur infinitely many times.