Linear differential equations with entire coefficients of small growth

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Abstract

We prove that if \( n \geq 3 \) and \( A_0, \ldots, A_{n-2} \) are entire functions of small growth, not all polynomials, then the linear differential equation

\[
w^{(n)} + \sum_{j=0}^{n-2} A_j w^{(j)} = 0
\]

cannot have a fundamental set of solutions each with few zeros.

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1 Introduction

Our starting point is the following theorem, the notation as in [7].

Theorem 1.1 ([1, 14, 15]) Suppose that \( A \) is a transcendental entire function with order

\[
\rho(A) = \limsup_{r \to \infty} \frac{\log^+ T(r, A)}{\log r} < \frac{1}{2}.
\]

Then the linear differential equation

\[
w^n(z) + A(z)w(z) = 0
\]

cannot have linearly independent solutions \( f_1, f_2 \) each with

\[
\lambda(f_j) = \limsup_{r \to \infty} \frac{\log^+ N(r, 1/f_j)}{\log r} < \infty.
\]

Theorem 1.1 was proved by Bank and Laine [1] for \( \rho(A) < \frac{1}{2} \), and by Rossi [14] and Shen [15] independently for \( \rho = \frac{1}{2} \), the proofs depending heavily on a formula [1] for \( A \) in terms of the product of linearly independent solutions of (1). It has been conjectured that if \( A \) is transcendental entire and \( \rho(A) \) is finite but is not a positive integer then (1) cannot have linearly independent solutions \( f_1, f_2 \) satisfying (2).

The present paper is concerned with an analogue of Theorem 1.1 for the higher order equation

\[
w^{(n)}(z) + \sum_{j=0}^{n-2} A_j(z)w^{(j)}(z) = 0,
\]

in which \( n \geq 3 \) and \( A_0, \ldots, A_{n-2} \) are entire functions, not all polynomials. If one coefficient \( A_s \) is assumed to have dominant growth in some sense compared to the other \( A_j \), then it is possible [2, 10, 11, 12] to prove some results concerning the maximum number of linearly independent solutions with few zeros, the proofs being based upon local representations for solutions in terms of \( A_s \). With no such assumption on the relative growth of the \( A_j \), these representations are not available, but the following was conjectured in [10].
Conjecture 1.1 To each integer \( n \geq 2 \) corresponds a positive real number \( L(n) \) such that if \( A_0, \ldots, A_{n-2} \) are entire functions of order less than \( L(n) \), not all polynomials, then the equation (3) cannot have linearly independent solutions \( f_1, \ldots, f_n \) each satisfying (2).

Of course, Theorem 1.1 shows at once that \( L(2) \geq \frac{1}{2} \), and it was proved in [10] that \( L(3) \geq \frac{1}{3} \), this improved to \( L(3) \geq \frac{1}{4} \) in [4, 5]. It seems likely that \( L(n) = 1 \) for every \( n \geq 2 \), and the following well known example (see e.g. [6]) shows that this would be sharp. If \( n \) is a positive integer and \( D = d/dz \) then

\[
f_j(z) = e^{-nz} \exp \left( e^{2\pi i j/n} e^z \right), \quad j = 1, \ldots, n,
\]

are linearly independent solutions of

\[
(D + 1) \ldots (D + n) w = e^{nz} w,
\]

and a standard transformation leads to an equation of form (3).

In the present paper we show that \( L(n) \geq \frac{1}{2(n-1)} \) for every \( n \geq 3 \).

**Theorem 1.2** Let \( n \) be an integer not less than 3, and let \( A_0, \ldots, A_{n-2} \) be entire functions, not all polynomials, of order less than \( \frac{1}{2(n-1)} \). Then the linear differential equation (3) cannot have linearly independent solutions \( f_1, \ldots, f_n \) each satisfying (2).

As already noted, it does not seem possible to prove Theorem 1.2 by asymptotic methods. Instead we use the following Wronskian identity [13, p.663], which the author first came across in a paper of Steinmetz [16].

**Lemma 1.1 ([13])** Let \( m \) and \( n \) be positive integers and let \( f_1, \ldots, f_m \) and \( g_1, \ldots, g_n \) be functions meromorphic on a domain \( D \). Then

\[
W(f_1, \ldots, f_m, g_1, \ldots, g_n) W(f_1, \ldots, f_m)^{n-1} = W(W(f_1, \ldots, f_m, g_1), \ldots, W(f_1, \ldots, f_m, g_n)) \quad (4)
\]
on \( D \).

The identity (4) may be proved by induction on \( m \), and it follows from (4) that the functions \( W(f_1, \ldots, f_m, g_j) \) are linearly independent on \( D \) if and only if the \( f_1, \ldots, f_m, g_1, \ldots, g_n \) are.

## 2 Lemmas needed for Theorem 1.2

We begin with the following lemma, the proof of which is obvious.

**Lemma 2.1** If \( h_1, \ldots, h_n \) are entire functions of finite order, and \( W_1, \ldots, W_n \) are functions meromorphic of finite order in the plane, then we may write

\[
W(W_1 e^{h_1}, \ldots, W_n e^{h_n}) = V e^h, \quad h = h_1 + \ldots + h_n, \quad (5)
\]
in which \( V \) is meromorphic of finite order in the plane.

Let \( H \) be analytic on an unbounded subset \( G \) of the plane. We shall say that \( H \) grows transcendently on \( G \) if

\[
\lim_{|z| \to \infty, z \in G} \frac{\log |H(z)|}{\log |z|} = \infty. \quad (6)
\]
Following [9], we shall use the term R-set to denote a countable union of discs $B(z_j, r_j)$, in which $z_j \to \infty$ as $j \to \infty$ and $\sum r_j < \infty$, and we will use the well known fact [9, p.87] that if $V \neq 0$ is meromorphic of finite order in the plane then there exists $M > 0$ such that

$$V'(z)/V(z) = O(|z|^M)$$  \hspace{1cm} (7)

for all $z$ outside an $R$-set $U_V$.

**Lemma 2.2** Let $h_1, \ldots, h_n$, with $n \geq 2$, be entire functions of finite order, and let $W_1, \ldots, W_n$ be functions meromorphic of finite order in the plane, and set $f_j = W_j e^{h_j}$ for $j = 1, \ldots, n$. Suppose that $G$ is an unbounded subset of the plane such that $h'_1 - h'_j$ grows transcendentally on $G$ for $j = 2, \ldots, n$, and that $f_1, \ldots, f_n$ are linearly independent. Then

$$H = \frac{W(f_1, \ldots, f_n)}{f_1 W(f_2, \ldots, f_n)}$$  \hspace{1cm} (8)

grows transcendentally on $G \setminus G_1$, in which $G_1$ is an $R$-set.

**Proof.** For $n = 2$ we need only write

$$\frac{W(f_1, f_2)}{f_1 f_2} = f'_2/f_2 - f'_1/f_1 = h'_2 - h'_1 + W'_2/W_2 - W'_1/W_1$$

and apply (6) to $h'_2 - h'_1$ and (7) to $W_2/W_1$.

Assume next that $n \geq 3$, and that the lemma is true for $n - 1$. Lemma 1.1 gives

$$W(f_1, \ldots, f_n)W(f_2, \ldots, f_{n-1}) = \pm W(W(f_1, \ldots, f_{n-1}), W(f_2, \ldots, f_n)).$$  \hspace{1cm} (9)

But, by Lemma 2.1, we may write

$$W(f_1, \ldots, f_{n-1}) = V_1 e^{g_1}, \quad W(f_2, \ldots, f_n) = V_2 e^{g_2},$$

in which $V_1, V_2$ are meromorphic of finite order, not identically zero, and $g_1, g_2$ are entire of finite order, with $g'_1 - g'_2 = h'_1 - h'_n$. Using the fact that the lemma has been established for $n = 2$, and the induction hypothesis, there exist $R$-sets $G_2, G_3$ such that

$$H_1 = \frac{W(W(f_1, \ldots, f_{n-1}), W(f_2, \ldots, f_n))}{W(f_1, \ldots, f_{n-1})W(f_2, \ldots, f_n)}$$

grows transcendentally on $G \setminus G_2$, while

$$H_2 = \frac{W(f_1, \ldots, f_{n-1})}{f_1 W(f_2, \ldots, f_{n-1})}$$

grows transcendentally on $G \setminus G_3$. Since (8) and (9) give $H = \pm H_1 H_2$, Lemma 2.2 is proved.

The next lemma is the main step in the proof of Theorem 1.2.

**Lemma 2.3** Suppose that $f_1, \ldots, f_m, g_1, \ldots, g_n$ are linearly independent and given by

$$f_j = V_j e^{h}, \quad g_k = W_k e^{H_k}, \quad j = 1, \ldots, m, \quad k = 1, \ldots, n,$$  \hspace{1cm} (10)

in which $h$ and the $H_k$ are entire functions of finite order, and the $V_j$ and $W_k$ are meromorphic of finite order in the plane. Suppose further that $G$ is an unbounded subset of the plane on which each $h' - H'_k, k = 1, \ldots, n$, grows transcendentally.
Then there exists an R-set $G_1$ such that

$$L = \frac{W(f_1, \ldots, f_m, g_1, \ldots, g_n)}{W(f_1, \ldots, f_m, g_1, \ldots, g_n)} \quad (11)$$

grows transcendentally on $G \backslash G_1$.

If $m = 1$ then Lemma 2.3 follows at once from Lemma 2.2. For $m \geq 2$ we require the following lemma.

**Lemma 2.4** Let $f_1, \ldots, f_m, g_1, \ldots, g_n$ and $G$ be as in Lemma 2.3, and assume that $m \geq 2$. Set

$$Q_p = \frac{W(f_1, \ldots, f_m, g_p, \ldots, g_n)}{W(f_1, \ldots, f_m, g_p, \ldots, g_n)}, \quad p = 1, \ldots, n, \quad Q_{n+1} = \frac{W(f_1, \ldots, f_m)}{W(f_1, \ldots, f_m)}. \quad (12)$$

Then for $p = 1, \ldots, n$ there exists an R-set $U_p$ such that $Q_p/Q_{p+1}$ grows transcendentally on $G \backslash U_p$.

**Proof.** Suppose first that $1 \leq p \leq n - 1$. By Lemma 1.1 we have

$$Q_p = \pm \frac{W(W(f_1, \ldots, f_m, g_{p+1}, \ldots, g_n), W(f_2, \ldots, f_m, g_p, \ldots, g_n))}{W(f_2, \ldots, f_m, g_{p+1}, \ldots, g_n) W(f_2, \ldots, f_m, g_p, \ldots, g_n)} \quad (13)$$

But we may write

$$W(f_1, \ldots, f_m, g_{p+1}, \ldots, g_n) = V_1 e^{h_1}, \quad W(f_2, \ldots, f_m, g_p, \ldots, g_n) = V_2 e^{h_2} \quad (14)$$

in which $V_1, V_2$ are meromorphic of finite order in the plane, not identically zero, and $h_1, h_2$ are entire of finite order, with $h_1 - h_2 = h - H_p$. But then, by Lemma 2.2, there exists an R-set $U_p$ such that

$$\frac{W(V_1 e^{h_1}, V_2 e^{h_2})}{V_1 e^{h_1} V_2 e^{h_2}}$$

grows transcendentally on $G \backslash U_p$ so that, by (13) and (14), $Q_p$ is large compared to

$$\frac{V_1 e^{h_1} V_2 e^{h_2}}{W(f_2, \ldots, f_m, g_{p+1}, \ldots, g_n) W(f_2, \ldots, f_m, g_p, \ldots, g_n)} = Q_{p+1}.$$ 

Thus Lemma 2.4 is proved for $p \leq n - 1$. For $p = n$ the proof is the same, except that (13) is replaced by

$$Q_n = \pm \frac{W(W(f_1, \ldots, f_m), W(f_2, \ldots, f_m, g_n))}{W(f_2, \ldots, f_m) W(f_2, \ldots, f_m, g_n)}.$$ 

This completes the proof of Lemma 2.4.

We now prove Lemma 2.3 by induction on $m$. Repeated use of Lemma 2.4 and its notation shows that as $z$ tends to infinity in $G \backslash U_1$, in which $U_1$ is an R-set,

$$L_1 = \frac{W(f_1, \ldots, f_m, g_1, \ldots, g_n)}{W(f_2, \ldots, f_m, g_1, \ldots, g_n)} = Q_1$$

is large compared to

$$L_2 = \frac{W(f_1, \ldots, f_m)}{W(f_2, \ldots, f_m)} = Q_{n+1}.$$
Assuming the lemma true for \( m - 1 \), there is an \( R \)-set \( U_2 \) such that

\[
L_3 = \frac{W(f_2, \ldots, f_m, g_1, \ldots, g_n)}{W(f_2, \ldots, f_m)W(g_1, \ldots, g_n)}
\]
grows transcendentally on \( G \setminus U_2 \). Since (11) gives

\[
L = \frac{L_1L_3}{L_2}
\]
we need only set \( G_1 = U_1 \cup U_2 \) and Lemma 2.3 is proved for \( n \).

3 Proof of Theorem 1.2

Suppose that \( n \geq 3 \) and that \( A_0, \ldots, A_{n-2} \) are entire functions, not all polynomials, of order at most \( \sigma < \frac{1}{2(n-1)} \). Suppose further that the equation (3) has a fundamental set of solutions \( f_1, \ldots, f_n \) each satisfying (2). We write \( f_j = W_j e^{h_j} \) for \( j = 1, \ldots, n \), in which the \( W_j \) and \( h_j \) are entire functions, and each \( W_j \) has finite order. By [10, p.520], each \( h_j \) has order at most \( \sigma \), and by [2, Theorem 1] the product \( f_1 \ldots f_n \) has finite order, so that \( h_1 + \ldots + h_n \) is a polynomial.

It follows that there exist entire functions \( H_1, \ldots, H_q \), with \( q \leq n \), and with the following properties. Each \( f_j \) may be written in the form

\[
f_j = V_j \exp \left( H_{mj} \right),
\]
in which \( V_j \) is an entire function of finite order and \( 1 \leq m_j \leq q \). Further, the \( H_j \) all have order at most \( \sigma \), and all the differences \( H_j - H_k, j \neq k \), are transcendental. Since at least one of the \( h_j \) is transcendental, and since the sum of the \( h_j \) is a polynomial, we have \( q \geq 2 \).

Choose \( \alpha \) with \( (n-1)\sigma < \alpha < \frac{1}{2} \). By the \( \cos \pi \rho \) theorem [3, 8] there exists, for each \( p \) with \( 2 \leq p \leq q \), a subset \( E_p \) of \((1, \infty)\) of lower logarithmic density at least \( 1 - \frac{\sigma}{\alpha} > 1 - \frac{1}{(n-1)} \), such that for \( |z| = r \in E_p \) we have

\[
\log |H_1'(z) - H_p'(z)| > (\cos \pi \alpha) \log M(r, H_1' - H_p').
\]
The intersection \( E \) of the \( E_p \), \( 2 \leq p \leq q \), has positive lower logarithmic density, and we denote by \( U \) the set of \( z \) such that \( |z| \) lies in \( E \). Thus all of the \( H_1' - H_p', 2 \leq p \leq q \), grow transcendentally on \( U \).

We partition the \( f_j \) as follows, to form classes \( F_1, \ldots, F_k \) and \( G_1, \ldots, G_{n-k} \). The \( F_\mu \) are those \( f_j \) for which \( H_j = H_1 \) in (15), while the \( G_\nu \) are the remaining \( f_j \). Since \( W(f_1, \ldots, f_n) \) is a non-zero constant, it follows from Lemma 2.3 that there is an \( R \)-set \( U_1 \) such that the entire function

\[
F = W(F_1, \ldots, F_k)W(G_1, \ldots, G_{n-k})
\]
tends to 0 as \( z \) tends to infinity in \( U \setminus U_1 \). But \( U \setminus U_1 \) contains the circle \( |z| = r \), for all \( r \) in a set of positive lower logarithmic density, from which it follows that \( F \equiv 0 \). This contradicts the linear independence of \( F_1, \ldots, F_k, G_1, \ldots, G_{n-k} \) and Theorem 1.2 is proved.

References


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