The second derivative of a meromorphic function of finite order

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Abstract

Let \( f \) be meromorphic of finite order in the plane, such that the second derivative \( f'' \) has finitely many zeros. Then \( f \) has finitely many poles. This result was conjectured by the author in 1996, and an example shows that the theorem is sharp.

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1 Introduction

By a theorem of Pólya [9, 22], if \( f \) is meromorphic with at least two poles in the plane then for each sufficiently large \( k \) the \( k' \)th derivative \( f^{(k)} \) has at least one zero. The following theorem established a conjecture of Hayman [8, 9, 10] from 1959:

**Theorem 1.1** ([5, 7, 13]) Suppose that \( m \geq 0 \) and \( k \geq 2 \) and that \( f \) is meromorphic in the plane such that \( f^{(m)} \) and \( f^{(m+k)} \) each have finitely many zeros. Then \( f^{(m+1)}/f^{(m)} \) is a rational function. In particular, \( f \) has finite order and finitely many poles.

Related results may be found in [3, 6, 14, 18] and elsewhere. Now Gol'dberg has conjectured that the frequency of distinct poles of \( f \) is governed by the frequency of zeros of a single derivative \( f^{(k)} \), provided \( k \geq 2 \), and in the present paper we prove a related conjecture of the author [15].

**Theorem 1.2** Suppose that \( f \) is meromorphic of finite order in the plane and that \( f^{(k)} \) has finitely many zeros, for some \( k \geq 2 \). Then \( f \) has finitely many poles.

Obviously it suffices to prove Theorem 1.2 for \( k = 2 \). No such result holds for functions of infinite order [15]. Indeed, we observe in Section 6 that there exists meromorphic \( f \) with infinitely many poles and with \( f^{(k)} \) zero-free, of arbitrarily slow growth subject to infinite order.

A number of partial results in the direction of Theorem 1.2 appear in [15, 16, 17, 18, 19], but the proof here is self-contained.
2 Lemmas needed for the theorem

Throughout this paper we denote by $B(z_0, r)$ the disc $\{z : |z - z_0| < r\}$ and by $S(z_0, r)$ the circle $\{z : |z - z_0| = r\}$. We will need Tsuji’s well known estimate for harmonic measure [24, p.116].

Lemma 2.1 ([24]) Let $D$ be a simply connected domain not containing the origin, and let $z_0$ lie in $D$. Let $r$ satisfy $0 < 4r < |z_0|$ or $4|z_0| < r < \infty$. Let $\theta(t)$ denote the angular measure of $D \cap S(0, t)$, and let $D_r$ be the component of $D \setminus S(0, r)$ which contains $z_0$. Then the harmonic measure of $S(0, r)$ with respect to the domain $D_r$, evaluated at $z_0$, satisfies

$$\omega(z_0, S(0, r), D_r) \leq C \exp \left( -\pi \int_1 I \frac{dt}{t \theta(t)} \right),$$

in which $C$ is an absolute constant, and $I = [2|z_0|, r/2]$ if $r > 4|z_0|$, with $I = [2r, |z_0|/2]$ if $4r < |z_0|$.

Note that (1) for $4r < |z_0|$ is obtained from the same estimate for the case $r > 4|z_0|$ by the substitution $\zeta = 1/z$. Next, we require the well known Cartan lemma:

Lemma 2.2 ([12], p.366) Let $n$ be a positive integer and let $a_1, \ldots, a_n$ be complex numbers. For each $\rho > 0$ we have

$$\left| \prod_{j=1}^n (z - a_j) \right| \geq \left( \frac{\rho}{2\pi} \right)^n$$

for all $z$ outside a union $H_\rho$ of discs having sum of radii at most $\rho$.

We need a lemma on the maximum $M(r, g)$ and minimum $m_0(r, g)$ of $|g(z)|$ on $|z| = r$, when $g$ is a transcendental meromorphic function of small growth. If $g$ is entire and

$$T(r, g) = O(\log r)^2$$

then [1] we have $M(r, g)/m_0(r, g) < C < \infty$ on a set of lower logarithmic density arbitrarily close to 1. Further results may be found in [1, Theorem 5 et al.], but these do not seem to suffice for our purposes here. For meromorphic $g$, and with an extra assumption on the concentration of zeros and poles, we prove the following.

Lemma 2.3 Let $g$ be transcendental and meromorphic in the plane, satisfying (2), and with the following property. For every fixed constant $K > 1$ the number $N_r$, counting multiplicities, of zeros and poles of $g$ in

$$B(K) = \{z : r/K \leq |z| \leq Kr\}$$

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satisfies \(N_x = o(\log r)\) as \(r \to \infty\). Let \(\delta > 0\). Then for all sufficiently large positive \(r\) we have

\[
\frac{M(s, g)}{m_0(s, g)} \leq s^\delta
\]

(4)

for all \(s \in [r/2, 2r]\) and lying outside a set of measure at most \(\delta r\).

Proof. We may clearly assume that \(g(0) = 1\). In addition, we assume for the time being that \(g\) is entire. By (2) there exists a positive constant \(C\) such that

\[
n(r^2) = n(r^2, 1/g) \leq C \log r
\]

for all large \(r\). Let the positive constant \(K\) be large, in particular with

\[
C \log \left( \frac{1 + 1/K}{1 - 1/K} \right) < \delta/2.
\]

(6)

Let \(r\) be large and let \(b_1, \ldots, b_N\), with \(N = o(\log r)\), be the zeros of \(g\) in \(B(K^2)\), with \(a_n\) the remaining zeros of \(g\), in both cases with repetition according to multiplicity. By Lemma 2.2 we have

\[
|Q(z)| \geq \left( \frac{\delta r}{4e} \right)^N, \quad Q(z) = \prod_{j=1}^{N} (z - b_j),
\]

(7)

for all \(z\) outside a union \(H\) of discs having sum of diameters at most \(\delta r\). Let \(s \in [r/2, 2r]\), such that the circle \(S(0, s)\) fails to meet \(H\). Then we have

\[
\log \frac{M(s, Q)}{m_0(s, Q)} \leq N \log(2K^2 r) + N \log(4e/\delta r) = N \log(8K^2 e/\delta) = o(\log r).
\]

(8)

Next, if \(|a_n| < r/K^2\) then

\[
(1 - 1/K) \frac{s}{|a_n|} \leq |1 - z/a_n| \leq (1 + 1/K) \frac{s}{|a_n|}, \quad |z| = s.
\]

(9)

Also, if \(|a_n| > K^2 r\) then

\[
1 - 1/K \leq 1 - s/|a_n| \leq |1 - z/a_n| \leq 1 + s/|a_n| \leq 1 + 1/K, \quad |z| = s.
\]

(10)

Combining (7), (8), (9) and (10), we obtain

\[
\log \frac{M(s, g)}{m_0(s, g)} \leq n(r^2) \log \left( \frac{1 + 1/K}{1 - 1/K} \right) + o(\log s) + I,
\]

(11)

in which

\[
I = \int_{r^2}^{\infty} \log \left( \frac{1 + s/t}{1 - s/t} \right) dn(t) \leq 4 \int_{r^2}^{\infty} \frac{s}{t} dn(t) \leq 4s \int_{r^2}^{\infty} \frac{n(t)}{t^2} dt = o(1).
\]

Using (5), (6) and (11), we now obtain (4).
If \( g \) is meromorphic we write \( g = g_1/g_2 \) with \( g_j \) entire and \( T(r; g_j) = O(\log r)^2 \), using (2) and [9, Theorem 1.11, p.27]. Then

\[
\frac{M(s, g)}{m_0(s, g)} \leq \frac{M(s, g_1)M(s, g_2)}{m_0(s, g_1)m_0(s, g_2)}.
\]

Proving the lemma for the \( g_j \), with \( \delta \) replaced by \( \delta/2 \), establishes (4) for \( g \). This proves Lemma 2.3.

We require some standard facts from the Wiman-Valiron theory [11]. Let \( F \) be a transcendental entire function. Provided \( r \) is normal for \( F \), that is provided \( r \) lies outside an exceptional set \( E \) of finite logarithmic measure, we have, for \( z_0 \) with \( |z_0| = r \) and \( |F(z_0)| > (1 - o(1))M(r, F) \),

\[
F'(z_0)/F(z_0) = \nu(r)z_0^{-1}(1 + o(1)),
\]

in which \( \nu(r) = \nu(r, F) \) is the central index of \( F \). In addition, with \( \mu(r) \) the maximum term,

\[
\log M(r/2, F) \leq \log \mu(r) + \log 2 \leq \nu(r) \log r + O(1), \quad \nu(r) \leq \log \mu(r) \leq \log M(r, F)
\]

so that \( \nu(r) \) and \( \log M(r, F) \) have the same order and lower order. Suppose now that \( G \) is transcendental and meromorphic in the plane, with finitely many poles \( b_1, \ldots, b_q \), repeated according to multiplicity. Then \( F(z) = G(z) \prod_{j=1}^q (z - b_j) \) is entire and the estimate (12) holds with \( F \) replaced by \( G \). Thus, abusing notation slightly, we may regard \( \nu(r, F) \) as the central index of \( G \).

3 Preliminaries

Suppose that \( h \) is transcendental and meromorphic in the plane, and that \( h(z) \) tends to the finite complex number \( a \) as \( z \) tends to infinity along a path \( \gamma \). Then \( a \) is an asymptotic value of \( h \), and the inverse function \( h^{-1} \) has a transcendental singularity over \( a \) [4, 21]. For each \( t > 0 \), the domain \( C(t) \) is that component of \( C'(t) = \{ z : |h(z) - a| < t \} \) which contains an unbounded subpath of \( \gamma \). Here \( C(t) \subset C(s) \) if \( 0 < t < s \), and the intersection of all the \( C(t), t > 0 \), is empty. The singularity of \( h^{-1} \) over \( a \) corresponding to \( \gamma \) is said to be direct if \( C(t) \), for some \( t > 0 \), contains finitely many zeros of \( h(z) - a \), in which case \( C(t) \), for sufficiently small \( t \), contains no zeros of \( h(z) - a \). Singularities over \( \infty \) are classified analogously.

Suppose next that \( H \) is meromorphic of finite order \( \rho(H) \) in the plane with finitely many critical values, that is, values taken by \( H \) at multiple points of \( H \). By a theorem of Bergweiler and Eremenko [4], all transcendental singularities of \( H^{-1} \) are direct and, by the Denjoy-Carleman-Ahlfors theorem [4, 21], there are at most \( 2\rho(H) \) of them.
We need next some standard facts from [21, p.287] which are discussed in detail in [17, 19]. Suppose that $F$ is a transcendental meromorphic function with no asymptotic or critical values in $0 < c_1 < |w| < \infty$. Then every component $C_0$ of the set $\{z : |F(z)| > c_1\}$ is simply connected, and there are two possibilities. Either (i) $C_0$ contains a single pole $z_0$ of multiplicity $k$, in which case $F^{-1/k}$ maps $C_0$ univalently onto $B(0, c_1^{-1/k})$, or (ii) $C_0$ contains no pole of $F$, but instead a path tending to infinity on which $F$ tends to infinity.

**Lemma 3.1 ([19])** Suppose that $G$ is a transcendental meromorphic function of finite order $\rho$ and that $G'$ has no asymptotic or critical values in $0 < |w| < d_1 < \infty$. Let $D$ be a component of the set $\{z : |G'(z)| < d_1\}$ on which $G'$ has no zeros, but such that $D$ contains a path $\gamma \to \infty$ on which $G'(z) \to 0$ as $z \to \infty$.

Then there exists a positive constant $S_1$ depending on $G$ and $D$ such that for $z_1$ in $D$ with $|G'(z_1)| < e^{-1} d_1$ we have

$$|G(z_1)| \leq S_1 + \frac{C|z_1 G'(z_1)|}{\log |d_1/G'(z_1)|},$$

in which $C$ is a positive absolute constant, in particular not depending on $d_1, G$ or $D$.

Lemma 3.1 is proved in [19], but in a more complicated form than we need here, and so, partly in order to keep the present paper self-contained, we sketch a proof. We may assume that 0 is not in $D$, if necessary replacing $G$ by $G(z + b)$, for some constant $b$. Let $N$ be an integer with $N > 2 + \rho$. Choose $d_2$ with $0 < d_2 < d_1$ such that $|G'(z)| > d_2$ on some circle $S(0, \sigma)$ with $1 \leq \sigma \leq 2$, and let $D_1 = \{z \in D : |z| > \sigma, |G'(z)| < d_2\}$. Choose $d$ as in Lemma 2 of [20], with $0 < d < d_2$, such that $z^N G'(z)$ has no multiple points with $|z^N G'(z)| = d$, while the length of the level curves $|z^N G'(z)| = d$ lying in $|z| \leq r$ is $O(r^{2+\rho})$ for all sufficiently large $r$. Define a function $u$, subharmonic in the plane, by $u(z) = \log^+ |d/z^N G'(z)|$ for $z$ in $D_1$, with $u(z) = 0$ otherwise. Let $W_j$ be a component of the set $\{z : u(z) > 0\}$, and fix $z^* \in W_j$. Since $G'$ has finite order and

$$u(z) \leq \frac{3}{2\pi} \int_0^{2\pi} u(2re^{i\theta}) d\theta \leq 3m(2r, 1/G'), \quad |z| = r \to \infty,$$

the number of $W_j$ is finite [12, p.562]. For $z$ in $W_j$ we join $z^*$ to $z$ by a path $\gamma_z$ in the closure of $W_j$ consisting of part of the ray $\arg t = \arg z^*$, part of the circle $|t| = |z|$, and part of the boundary $\partial W_j$ of $W_j$. Partitioning $\partial W_j$ into its intersections with annuli $\{t : 2^q-1 < |t| \leq 2^q\}$ we have

$$\int_{\gamma_z} |G'(t)| \ |dt| \leq \int_{|z^*|}^{\infty} dt^{N} dt + 2\pi d|z|^{1-N} + \sum_{q=1}^{\infty} d2^{-N(q-1)} O(2^{q(2+\rho)}) \leq O(1).$$

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Now choose \( z_0 \in D \). Let \( g = G' \) and let \( \psi = g^{-1} \) be that branch of the inverse function mapping \( w_0 = g(z_0) \) to \( z_0 \). Choose \( v_0 \) such that \( e^{-v_0} = w_0 \) and set \( \phi(v) = \psi(e^{-v}) = g^{-1}(e^{-v}) \). Since \( G' \neq 0 \) on \( D \), the function \( \phi \) extends to be analytic and univalent on \( H = \{ v : \Re(v) > c_0 = \log 1/d_1 \} \). Also, \( \phi(H) = D \), and \( D \) is simply connected. We apply a logarithmic change of variables as used in [3] and elsewhere. Since \( 0 \notin D \), we may define an analytic and univalent branch of \( \zeta = \log \phi(v) \) on \( H \). By the Koebe one-quarter theorem [23, p.9], we thus have

\[
|d\zeta/dv| = |\phi'(v)/\phi(v)| \leq 8\pi/(\Re(v) - c_0) < 32/(\Re(v) - c_0)
\]  

(15)

for \( v \) in \( H \). Let \( z_1 \) be as in the statement of the lemma. Then there exists \( v_1 \) in \( H \) with

\[
z_1 = \phi(v_1), \quad v_1 = Q + iy, \quad Q = \log |1/G'(z_1)| > c_0 + 1.
\]  

(16)

Let \( L \) be the horizontal half line given by \( v = s + iy, s \geq Q \). For \( s \geq Q \), by (15),

\[
|\phi(s + iy)| \leq |\phi(Q + iy)| \exp\left( \int_Q^s 32(t - c_0)^{-1}dt \right) = |\phi(Q + iy)|(s - c_0)^{32}(Q - c_0)^{-32}.
\]  

(17)

We have \( \phi(s + iy) \to \infty \) as \( s \to +\infty \) and

\[
\int_{\phi(L)} |G'(z)| \ |dz| = \int_L \exp(-\Re(v))|\phi'(v)| \ |dv| = \int_Q^\infty e^{-s}|\phi'(s + iy)|ds.
\]

Using (15), (16) and (17) we get

\[
\int_{\phi(L)} |G'(z)| \ |dz| \leq |z_1| \int_Q^\infty e^{-s}32(s - c_0)^{31}(Q - c_0)^{-32}ds \leq C_1|z_1|e^{-Q(Q - c_0)^{-1}},
\]  

(18)

after integrating by parts, in which \( C_1 \) is a positive absolute constant. But (17) gives

\[
\log |1/G'(z)| = s \geq c_0 + (Q - c_0)|z/z_1|^{1/32}
\]

and so \( z = \phi(s + iy) \), for large \( s \), is in some \( W_j \). It follows that a sub-path of \( \phi(L) \) joins \( z_1 \) to a point in one of the finitely many \( W_j \). Since (14) implies that \( G \) is bounded on each of the \( W_j \), (13) now follows from (18).

4 Critical points and asymptotic values

Suppose that \( F \) is meromorphic of finite order in the plane, and that \( F \) has infinitely many poles, but \( F' \) has finitely many zeros. Then, by Section 3, \( F \) has finitely many asymptotic values, and each corresponds to finitely many direct transcendental singularities [4, 21] of the inverse function.
Let $J$ be a simple closed polygonal path, such that every finite critical or asymptotic value of $F$ lies on $J$, but is not a vertex of $J$. Then $J$ divides its complement into two simply connected domains $B_1$ and $B_2$, such that $B_1$ is bounded, while $\infty \in B_2$. Fix conformal mappings $h_m : B_m \to B(0, 1)$, for $m = 1, 2$, with $h_2(\infty) = 0$. By the reflection principle, if $I$ is a closed line segment contained in $J$ and not meeting any vertex of $J$ then for $m = 1, 2$ there are positive constants $d_m$ such that

$$d_m \leq |h'_m(w)| \leq 1/d_m, \quad w \in I. \quad (19)$$

Let $J'$ be the set of vertices of $J$ and critical and asymptotic values of $F$, and let $J'' = J \setminus J'$. For each component $J^*$ of $J''$ we choose a closed line segment $I_q$ contained in $J^*$, and hence not meeting $J'$, and for each such $I_q$ there are constants $d_m$ as in (19).

Take a quasiconformal homeomorphism $\psi_1$ of the extended plane onto itself such that $\psi_1(\infty) = \infty$ and $\psi_1 = h_1$ on $B_1$ [23, p.94]. There exist a function $g$ meromorphic in the plane and a quasiconformal mapping $\psi$ such that $\psi(\infty) = \infty$ and $\psi_1 \circ F = g \circ \psi$. This $g$ has finitely many critical and asymptotic values $b_\nu$, all of modulus 1, and $g'$ has finitely many zeros.

Let $A_1 = B(0, 1)$ and $A_2 = \{w : 1 < |w| \leq \infty\}$. Then, as in Section 3, for each component $T$ of $g^{-1}(A_2)$, either $T$ contains just one pole of $g$, or $T$ contains no pole of $g$, but instead a path tending to infinity on which $g(z)$ tends to infinity. Because the inverse function $g^{-1}$ has finitely many singularities, there are only finitely many components $T$ of the latter type. Each component of $g^{-1}(A_1)$ is conformally equivalent under $g$ to the unit disc $B(0, 1)$. Using Section 3 and the fact that $g'$ has finitely many zeros, all components of $g^{-1}(A_j), j = 1, 2$, are simply connected, and all but finitely many are unbounded.

Let $S$ be a component of $g^{-1}(A_1)$ having no zero of $g'$ in its closure in the finite plane. Then $g$ is univalent on the finite boundary $\partial S$, which consists of finitely many simple level curves of $g$, each going to infinity in both directions. As $z$ tends to infinity along a boundary arc of $\partial S$, the image $g(z)$ tends to one of the $b_\nu$, and we call $S$ type I if there is only one such asymptotic value of $g$ approached along a boundary arc of $S$, and type II if there are at least two distinct such values. Clearly a type I component $S$ with no zero of $g'$ on its boundary $\partial S$ is such that $\partial S$ consists of just one simple analytic curve going to infinity in both directions, and such an $S$ cannot separate the plane. We shall call an unbounded component $S'$ of the set $F^{-1}(B_1)$ type I or II if $S = \psi(S')$ is a type I or II component of $g^{-1}(A_1)$.

Let $z_1$ be a pole of $g$, of multiplicity $p$, with $|z_1|$ large, lying in a component $T_1$ of $g^{-1}(A_2)$. Then $T_1$ is simply connected and unbounded and $g^{-1/p}$ maps $T_1$ conformally onto $B(0, 1)$. Again the finite boundary $\partial T_1$ consists of finitely many simple level curves of $g$, each going to infinity in
both directions, of which at least one must be a boundary curve of a type II component of $g^{-1}(A_1)$. In particular, $g$ has at least two distinct finite asymptotic values and so has $F$.

5 Proof of Theorem 1.2

We assume that $f$ is meromorphic of finite order $\rho(f)$, and that $f$ has infinitely many poles, while $f''$ has finitely many zeros, and apply the reasoning of Section 4, with $F = f'$, retaining the notation there. We may assume that $0 \in B_1$, and that $h_1(0) = 0$, since if this is not the case we may replace $f$ by $f(z) - \lambda z$, for some constant $\lambda$. Let the finite asymptotic values of $f'$ be $a_n$, repeated according to how often they occur as direct transcendental singularities of $(f')^{-1}$. In particular, $f'$ has at least two distinct finite asymptotic values.

**Lemma 5.1** The lower order of $f''$ is at least $\frac{1}{2}$.

**Proof.** If this is not the case then the function $1/f''$ has finitely many poles and is transcendental of lower order less than $\frac{1}{2}$. The cos $\pi \lambda$ theorem [2] now gives $r_n \to +\infty$ such that $f''(z) = O(r_n^{-2})$ on $|z| = r_n$, and integrating around $|z| = r_n$ we obtain a contradiction, since $f'$ has at least two distinct finite asymptotic values. This proves Lemma 5.1.

Next choose $\varepsilon_0 > 0$ such that, for each $n$, there are no critical or asymptotic values of $f'$ in $0 < |w - a_n| \leq 4\varepsilon_0$. The following lemma is an immediate consequence of Lemma 3.1 and the discussion preceding it.

**Lemma 5.2** There exist $\varepsilon_1 > 0$ and, for each $n$, an unbounded simply connected domain $U_n$, a component of the set $\{z : |f'(z) - a_n| < \varepsilon_0\}$, such that $U_n$ contains a path tending to infinity on which $f'(z)$ tends to $a_n$. Further, $f'(z) \neq a_n$ on $U_n$ and $|f(z) - a_n z| < \varepsilon_0|z|$ for all large $z$ in $U_n$ with $|f'(z) - a_n| < \varepsilon_1$.

Now let $\varepsilon_2$ be such that, for each $n$, if $|h_1(v) - h_1(a_n)| \leq \varepsilon_2$ then $|v - a_n| < \varepsilon_1$, in which $\varepsilon_1$ is as determined in Lemma 5.2.

**Lemma 5.3** There exist positive constants $C_1, C_2$ with the following property. If $D$ is a type II component of the set $(f')^{-1}(B_1)$ and $z_0 \in D, f'(z_0) = 0$, then provided $|z_0|$ is large enough we have

$$B(z_0, C_1|z_0|) \subseteq \{z \in D : |h_1(f'(z))| < \frac{1}{2}\}$$

(20)
and
\[ |f'''(z)|/f''(z)| \leq C_2/|z_0|, \quad z \in B(z_0, \frac{1}{2}C_1|z_0|). \] (21)

Proof. Assume that there is no positive constant $C_1$ such that (20) holds for all but finitely many type II components. Since $V = h_1 \circ f'$ maps each such $D$ conformally onto $B(0, 1)$, it follows from the Koebe one-quarter theorem [23, p.9] that we may choose arbitrarily large $z_0$ such that the inverse function $G$ of $V$ maps $B(0, 1)$ conformally onto $D$, with $G(0) = z_0$ and $G'(0) = o(|z_0|)$. Since $D$ is a type II component, we may assume that $a_1 \neq a_2$ and that $a_1, a_2$ are each asymptotic values of $f'$ on $\partial D$. For $n = 1, 2$, let $\mu_n$ be a path in $D$, close to $\partial D$ and tending to infinity, on which $f'(z) \to a_n$. Then $V(\mu_n)$ is a path in $B(0, 1)$ tending to $h_1(a_n)$, since $h_1$ extends to be continuous in the plane. Hence, if $z \in D$ and $|V(z) - h_1(a_n)| < \varepsilon_2$, we have $z \in U_n$.

By the Koebe distortion theorem [23, p.9], we have $|G'(w)| = o(|z_0|)$ for $|w| \leq 1 - \frac{3}{4}\varepsilon_2$, and the image under $G$ of the line segment $w = th_1(a_n)$, $0 \leq t \leq 1 - \frac{3}{4}\varepsilon_2$, has length $o(|z_0|)$. This allows us to choose a path $\gamma^*$ in $D$, of length $o(|z_0|)$, such that $\gamma^*$ joins $\eta_1$ to $\eta_2$, with $\eta_n \in U_n$ for $n = 1, 2$. Thus Lemma 5.2 gives
\[ |f(\eta_n) - a_n\eta_n| \leq \varepsilon_0|\eta_n|, \quad n = 1, 2. \] (22)

But $\gamma^*$ has length $o(|z_0|)$, and $f'$ maps $\gamma$ into the bounded domain $B_1$, and so $f(\eta_2) - f(\eta_1) = \int_\gamma f'(z)dz = o(|\eta_1|)$. Since $a_1 \neq a_2$, this contradicts (22), and the first assertion of Lemma 5.3 is proved. Finally, (21) follows by applying [23, Proposition 1.2, p.9] to $g_1(z) = f'(z_0 + C_1|z_0|z)$.

**Lemma 5.4** Let $L(r) \to \infty$ with $L(r) \leq \frac{1}{8}\log r$ as $r \to \infty$, and for $k > 0$ and large $r$ let $A(k) = \{z : re^{-kL(r)} \leq |z| \leq re^{kL(r)}\}$. Then the number $N_1$ of distinct poles of $f$ in $A(1)$ satisfies
\[ N_1 = O(\phi(r)), \quad \phi(r) = L(r) + \frac{\log r}{L(r)}, \quad r \to \infty. \] (23)

Proof. Assume that $r$ is large and that $A(1)$ contains $N_1$ distinct poles $w_1, \ldots, w_{N_1}$ of $f$, with $\phi(r) = o(N_1)$. Let $D_j$ be the component of $(f')^{-1}(B_2)$ in which $w_j$ lies, and let $\theta_j(t)$ be the angular measure of $D_j \cap S(0, t)$. Since $r$ is large the $D_j$ are simply connected.

We shall use in this proof $c$ to denote positive constants, not necessarily the same at each occurrence, but in particular not depending on $r, L(r)$ or $N_1$. By the discussion in Section 4, we may assume that at least $16N$ of these $D_j$, say $D_1, \ldots, D_{16N}$, with $N$ an integer satisfying
\[ N \geq cN_1, \quad \phi(r) = o(N), \] (24)

are such that, with $a_1, a_2$ distinct finite asymptotic values of $f'$, the following is true. To each $D_j$ corresponds a type II component $E_j$ of $(f')^{-1}(B_1)$, the boundaries of $D_j$ and $E_j$ sharing a
component $K_j$. Each $K_j$ is a simple piecewise smooth curve going to infinity in both directions and mapped by $f'$ onto a fixed sub-path $J_1$ of the curve $J$, the closure of $J_1$ joining $a_1$ to $a_2$. Since $f'$ is univalent on each $E_j$, we have $E_j \neq E_k$ for $1 \leq j < k \leq 16N$.

A standard application of the Cauchy-Schwarz inequality gives $M^2 \leq 2\pi \sum_{j=1}^{M} 1/\theta_j(t)$ so that, as in [17, 19], at least $4N$ of the $D_j$, say $D_1, \ldots, D_{4N}$, have

$$
\int_{2r(e^{-L(r)})}^{(1/2)re^{2L(r)}} dt/\theta_j(t) > cNL(r), \quad \int_{2r(e^{-L(r)})}^{(1/2)re^{-L(r)}} dt/\theta_j(t) > cNL(r).
$$

We fix a sub-arc $J_0$ of $J_1$, one of the closed line segments $I_\theta$ chosen following (19). We write $p_j$ for the multiplicity of the pole of $f'$ at $w_j$, and for $1 \leq j \leq 4N$ define $v_j = (h_2 \circ f')^{1/p_j}$, so that $v_j$ maps $D_j$ conformally onto $B(0, 1)$, with $v_j(w_j) = 0$. The path $K_j$ forming the boundary between $D_j$ and $E_j$ has a sub-path $\lambda_j$ mapped onto $J_0$ by $f'$. As $z$ describes the arc $\lambda_j$, the image $(h_2 \circ f')(z)$ describes an arc of the unit circle of length at least $c$, using (19), so that $v_j(z)$ describes an arc of the unit circle of length at least $c/p_j \geq cr^{-2p(j')}^{-1}$. This gives a harmonic measure estimate

$$
\omega(w_j, \lambda_j, D_j) \geq c/p_j \geq cr^{-2p(j')}^{-1}.
$$

Set $\sigma_j = \lambda_j \setminus A(2)$. Since $w_j$ lies in $A(1)$, Lemma 2.1 and (25) imply that

$$
\omega(w_j, \sigma_j, D_j) \leq c \exp \left( -\pi \int_{2r(e^{-L(r)})}^{(1/2)re^{2L(r)}} dt/\theta_j(t) \right) + c \exp \left( -\pi \int_{2r(e^{-L(r)})}^{(1/2)re^{-L(r)}} dt/\theta_j(t) \right) \leq c \exp(-cNL(r)).
$$

Since (23) and (24) give $\log r = o(NL(r))$, the estimate (26) implies, provided $r$ is large enough, that

$$
\omega(w_j, \lambda_j^*, D_j) \geq c/p_j \geq cr^{-2p(j')}^{-1}, \quad \lambda_j^* = \lambda_j \cap A(2).
$$

(27)

By (27), $\lambda_j^*$ is mapped by $v_j$ into a finite union of sub-arcs of the unit circle of total length at least $c/p_j$ and so is mapped by $f'$ into a union of sub-arcs of $J_0$ of total length at least $c$, using (19) again. Let $\phi_j(t)$ be the angular measure of the intersection of $E_j$ with the circle $S(0, t)$. The same argument as for (25) gives at least $N$ of the $E_j$, say $E_1, \ldots, E_N$, each having

$$
\int_{2r(e^{L(r)})}^{(1/2)re^{3L(r)}} dt/\phi_j(t) > cNL(r), \quad \int_{2r(e^{-L(r)})}^{(1/2)re^{-3L(r)}} dt/\phi_j(t) > cNL(r).
$$

We know that $V = h_1 \circ f'$ maps $E_j$ univalently onto $B(0, 1)$, with $\lambda_j^*$ mapped onto a union $\mu_j$ of sub-arcs of the unit circle of total length at least $c$, by (19). Hence $\omega(w, \mu_j, B(0, 1)) \geq c(1 - |w|)$.
for $|w| < 1$. If $z$ lies in $E_j \setminus A(3)$ then, because $\lambda_j^*$ lies in $A(2)$, Lemma 2.1 and (28) imply that
\[
\omega(V(z), \mu_j, B(0, 1)) = \omega(z, \lambda_j^*, E_j)
\leq c \exp \left( -\pi \int_{(1/2)e^{\alpha L(r)}}^{(1/2)e^{-2L(r)}} dt/t\phi_2(t) \right) + c \exp \left( -\pi \int_{2e^{-3L(r)}}^{e^{-2L(r)}} dt/t\phi_2(t) \right)
\leq c \exp(-c N L(r)) = o(1).
\]
Thus for $j = 1, \ldots, N$ the annulus $A(3)$ contains the set $H_j = \{ z \in E_j : |V(z)| < \frac{1}{2} \}$. But each $H_j$ contains, by Lemma 5.3, a disc $B(u_j, C_1 |u_j|)$ and so
\[
cN \leq \sum_{j=1}^{N} \int_{H_j} |z|^{-2} dxdy \leq \int_{A(3)} |z|^{-2} dxdy \leq cL(r),
\]
which contradicts (23) and (24). Lemma 5.4 is proved.

Choosing $L(r) = \frac{1}{8} \log r$ gives
\[
\pi(r^{9/8}, f) - \pi(r^{7/8}, f) = O(\log r),
\]
and so
\[
\overline{N}(r, f) = O(\log r)^2, \quad r \to \infty.
\]

**Lemma 5.5** Let $H(z) = f'''(z)/f''(z)$. Then $T(r, H) = O(\log r)^2$ as $r \to \infty$.

**Proof.** We have, by (29),
\[
T(r, H) = N(r, H) + m(r, H) \leq \overline{N}(r, f''') + \overline{N}(r, 1/f'') + O(\log r) = O(\log r)^2.
\]

**Lemma 5.6** There exists a subset $E_0$ of $[1, \infty)$, of finite logarithmic measure, such that for all $s \in [1, \infty) \setminus E_0$ there exists $\zeta_s$ with $|\zeta_s| = s$ and $|H(\zeta_s)| > s^{-2/3}$.

Lemma 5.6 follows immediately from (12), applied to $1/f''$, and Lemma 5.1.

**Lemma 5.7** Let $K$ be a large positive constant and let $P(r)$ be the number of zeros and poles, counting multiplicity, of $H$ in $B(K)$, with $B(K)$ defined as in (3). Then $P(r) = O(\log r)^{1/2}$ as $r \to \infty$.

**Proof.** Since $f''$ has finite order, standard estimates based on [9, p.22] give a subset $E_1$ of $[1, \infty)$, of finite logarithmic measure, and a positive constant $T$, such that
\[
|H(z)| \leq s^T, \quad |z| = s \in [1, \infty) \setminus E_1.
\]
For large \( r \), choose \( R = R(r) \) with \( L(r) = \log R = (\log r)^{1/2} + O(1) \), such that the estimate (30) holds for \( s = r/R, Rr \). By Lemma 5.4, the poles \( c_1, \ldots, c_M \) of \( H \) in \( B(R) \), all of which are poles of \( f \), are such that \( M = O(\log r)^{1/2} \).

Assume next that \( r \) is large, and that \( H \) has \( Q \) zeros \( b_1, \ldots, b_Q \) in \( B(K) \), repeated according to multiplicity, with

\[
(\log r)^{1/2} = o(Q), \quad M + T = o(Q). \tag{31}
\]

Let

\[
h(z) = H(z) \prod_{m=1}^{M} (z - c_m) \prod_{q=1}^{Q} (z - b_q)^{-1}, \tag{32}
\]

so that \( h \) is analytic in \( B(R) \), and non-zero in \( B(K) \). On \( |z| = r/R \) we have \( |z - b_q| \geq r/2K \) and \( |z - c_m| \leq 2Rr \), since \( K \) and \( R/K \) are large, and using (30) we get

\[
|h(z)| \leq (r/R)^T (2Rr)^M (2K/r)^Q \leq SK^Q, \quad |z| = r/R, \quad S = r^{T+M-Q} 2^{M+Q} R^{M+T}. \tag{33}
\]

Next, on \( |z| = Rr \) we have \( |z - b_q| \geq Rr/2 \) and \( |z - c_m| \leq 2Rr \), so that (30) gives

\[
|h(z)| \leq (Rr)^T (2Rr)^M (Rr/2)^{-Q} = r^{T+M-Q} 2^{M+Q} R^{T+M-Q} = SR^{-Q}, \quad |z| = Rr. \tag{34}
\]

The function \( u(z) = \log |h(z)/S| \) is subharmonic on \( B(R) \), and by (33) and (34) satisfies

\[
u(z) \leq Q \log K \left( \frac{\log |rR/z|}{2 \log R} \right) - Q \log R \left( \frac{\log |rz/r|}{2 \log R} \right) \tag{35}
\]

on the boundary of \( B(R) \). The maximum principle then gives (35) for all \( z \) in \( B(R) \). For \( |z| \geq r/K \) we have

\[
\frac{\log |rz/r|}{2 \log R} \geq \frac{\log R/K}{2 \log R} \geq \frac{1}{4},
\]

and so

\[
u(z) \leq \frac{3}{4} Q \log K - \frac{1}{4} Q \log R \leq -\frac{1}{8} Q \log R, \quad z \in B(K),
\]

using in both steps the fact that \( R/K \) is large. This gives, recalling (31) and (33),

\[
|h(z)| \leq SR^{-Q/8} = r^{T+M-Q} 2^{M+Q} R^{M+T} R^{-Q/8} \leq r^{T+M-Q} R^{-Q/16}, \quad z \in B(K). \tag{36}
\]

Using Lemmas 2.2 and 5.6, choose \( w \) with

\[
3r/4 \leq |w| \leq 5r/4, \quad |H(w)| \geq |w|^{-2/3} \geq 1/r, \quad \left| \prod_{m=1}^{M} (w - c_m) \right| \geq \left( \frac{r}{16e} \right)^M. \tag{37}
\]

We have \( |w - b_q| \leq 2Kr \) and so (32), (36) and (37) give

\[
1/r \leq |H(w)| \leq r^{T+M-Q} R^{-Q/16} (2Kr)^Q (16e/r)^M,
\]

12
which leads, using (31) and the fact that $R$ and $R/K$ are large, to

$$1 \leq r^{T+1} 2^{Q+6M} K^Q R^{-Q/16} \leq r^{T+1} R^{-Q/32}, \quad Q \log R \leq 32(T + 1) \log r.$$  

But this contradicts (31), by the choice of $R$, and Lemma 5.7 is proved.

We now complete the proof of Theorem 1.2. By Lemma 5.3, there exist positive constants $C_1, C_2$ and arbitrarily large $z_0$ satisfying (21). Let $\delta$ be small and positive. By Lemmas 5.5 and 5.7, the function $H = f''' / f''$ satisfies the hypotheses of Lemma 2.3, which on combination with Lemma 5.6 gives $s > 0$ such that: (i) the circle $S(0, s)$ meets $B(z_0, C_1|z_0|/2)$; (ii) $|H(\zeta)| > s^{-2/3}$ for some $\zeta$ on $S(0, s)$; (iii) we have (4). But (ii) and (iii) give $|H(z)| \geq s^{-3/4}$ on $S(0, s)$, provided $\delta$ was chosen small enough, and this contradicts (21). Theorem 1.2 is proved.

6 An example

Take a positive integer $k$ and a positive sequence $(a_n)$ with $a_1 > 100$ and $a_{n+1}/a_n > 100$ for $n \geq 1$. We use a fairly standard argument of Mittag-Leffler type, and define

$$L(z) = g(z)^{k} H(z), \quad g(z) = \prod_{n=1}^{\infty} (1 - z/a_n), \quad H(z) = \sum_{n=1}^{\infty} \sum_{j=0}^{k} d_{j,n}(z - a_n)^{j-k-1},$$  

in which the uniformly bounded constants $d_{j,n}$ are to be determined. Standard arguments as in [9, p.27] and Lemma 2.3 give

$$M(r, g)/m_0(r, g) \leq r^{C}, \quad \log M(r, g) \leq C n(r) \log r, \quad 2a_n \leq r \leq \frac{1}{2} a_{n+1},$$  

in which $n(r) = n(r, 1/g)$ and $C$ denotes a positive constant, not necessarily the same at each occurrence, but independent of $r$ and $n$. With $h(z) = g(z)(z - a_n)^{-1}$, we obtain $|h(a_n)|^{-1} \leq C$ from (39), and using the formula $h'(z)/h(z) = \sum_{m \neq n} 1/(z - a_m)$ we get $|h^{(j)}(a_n)/h(a_n) | \leq C$ for $1 \leq j \leq k$. This gives an expansion

$$g(z)^{k} = (z - a_n)^{k} (A_0 + A_1(z - a_n) + \ldots), \quad |A_0|^{-1} \leq C, \quad |A_j/A_0| \leq C, \quad 1 \leq j \leq k.$$  

The $d_{j,n}$ are determined by

$$d_{0,n} A_0 = -k - 1, \quad 0 = \sum_{p=0}^{j} A_p d_{j-p,n} = A_0 (d_{j,n} + \ldots + d_{0,n} A_j/A_0), \quad 1 \leq j \leq k,$$

and by (40) we get $|d_{j,n}| \leq C$. As $z \to a_n$ we thus have $L(z) = -(k+1)/(z - a_n) + O(|z - a_n|^{k})$ and so $f^{(k+1)}/f^{(k)} = L$ defines a meromorphic function $f$ with simple poles at the $a_n$ and $f^{(k)}$ zero-free.
For $z$ outside the union of the discs $B(a_n, \frac{1}{2}a_n)$ we have $|H(z)| \leq C \sum_{n=1}^{\infty} \sum_{j=0}^{k} a_n^{j-k-1} \leq C$, and so integrating along the negative real axis and around arcs of circles we get, using (38) and (39),

$$|f^{(k)}(z)| \leq C \exp(C r \exp(C n(r) \log r)), \quad |f(z)| \leq C r^k \exp(C r \exp(C n(r) \log r)),$$

for $2a_n \leq |z| = r \leq \frac{1}{2}a_{n+1}$. Since $N(r, f) = N(r, 1/g)$, we thus have $T(r, f) \leq \exp(C n(4r) \log r)$ for all large $r$, and $n(r)$ may be chosen so as to tend to infinity arbitrarily slowly.

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References


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