1. INTRODUCTION

Consider functions \( f \) meromorphic in \( \mathbb{C} \), i.e. \( f = f_1/f_2 \) with \( f_j \) entire. Value distribution theory begins with:

**Picard’s theorem (ca. 1879)**
If \( f \) is meromorphic in the plane and omits each of the distinct values \( a, b, c \) then \( f \) is constant.

Refined and generalized by R. Nevanlinna: \( T(r, f), m(r, f) \), first and second fundamental theorems etc. Consider here:

1. Critical points and critical values of functions (connections to iteration)
2. Higher derivatives

**Critical points**

\( f \) non-constant, meromorphic. Then \( f \) is one-one on a nbd of \( z_0 \) and the inverse function \( f^{-1} \) exists on a nbd of \( w_0 = f(z_0) \)

iff \( f'(z_0) \neq 0 \) or \( z_0 \) is a simple pole.

If \( f'(z_0) = 0 \) or \( z_0 \) is a multiple pole,

\( z_0 \) is a **critical (multiple) point**, and \( w_0 = f(z_0) \) a critical value.

\( f^{-1} \) has an algebraic singularity over \( w_0 \) ( \( n \) "sheets").

Critical points and values countable.

**Examples**

1. The Weierstrass doubly periodic function \( p \) solves

\[
(p')^2 = 4(p-e_1)(p-e_2)(p-e_3),
\]

with \( e_j \) distinct constants. Critical values are \( e_j, \infty \).
2. \( f(z) = 1/(e^z + 1) \) has no critical points. But if we continue \( f^{-1} \) along a path on which \( w \to 1 \), then \( f^{-1}(w) \) tends to infinity in the left half plane.

**Asymptotic values**

If \( f \) is transcendental and we have a path \( \gamma \to \infty \) such that

\[
f(z) \to a, \text{ with } a \text{ finite or } \infty, \text{ as } z \to \infty \text{ on } \gamma,
\]

then \( a \) is an asymptotic value of \( f \) (transcendental singularity of \( f^{-1} \)).

**Iversen's theorem (1913)**

If \( f \) is transcendental meromorphic and \( f(z) = a \) only finitely often in the plane, then \( a \) is an asymptotic value of \( f \).

Converse false: \( e^z(e^z - 1) \).

**More examples**

1. \( \tan z \) has 2 omitted values \( \pm i \), both asymptotic, and no critical values.
2. \( \tan^2 z \) has one asymptotic value \(-1\) and two critical values \( 0, \infty \).
3. \( \int_0^z e^{-t^2} dt \) has three asymptotic values (two finite) and no critical values.
4. Weierstrass \( p \) function has no asymptotic values.
5. Eremenko (1979): there exists a meromorphic function \( f \) for which every value in \( \mathbb{C} \cup \{\infty\} \) is asymptotic.

**How many singular values?**
Asymptotic and critical values called SINGULAR values of $f^{-1}$.

Must be at least 3, except in special cases.

Suppose $f^{-1}$ has singularities only over $0, \infty$.

Then $f/f'$ is entire. Either $f/f'$ is constant and $f(z) = \exp(az + b)$

or Iversen gives $\gamma \to \infty$ on which $f/f' \to \infty$.

If $f/f'$ is linear, $f(z) = (az + b)^k$.

If $f/f'$ non-linear, can get

$$\int_{\gamma} |f'(z)/f(z)| \ |dz| < \infty, \ f(z) \to a \neq 0, \infty, z \in \gamma.$$  

2. ORDER OF A FUNCTION $f$

Many questions depend on growth.

Write $f = f_1/f_2$, $f_j$ entire. Say $f$ has FINITE ORDER if there exists $k > 0$ such that

$$\log |f_j(z)| \leq O(|z|^k) \text{ as } z \to \infty, \ j = 1, 2.$$  

The order $\rho(f)$ is the infimum of such $k$.

ALTERNATIVELY, take any 3 distinct values $a,b,c$ in $\mathbb{C} \cup \{\infty\}$.

Let $n(r)$ be the number of points in $|z| \leq r$ at which $f(z) = a,b$ or $c$. Then (Nevanlinna)

$$\rho(f) = \lim_{r \to \infty} \frac{\log n(r)}{\log r}.$$  

Examples:

$$\rho(e^z) = 1, \ \rho(\cos(\sqrt{z})) = 1/2,$$
\[ \rho(\tan(e^z)) = \infty. \]

E.g. note that \( \tan(e^z) = \infty \) whenever \( e^z = (k + 1/2)\pi \), with \( k \) integer.

3. GROWTH AND CRITICAL POINTS

1. F. Nevanlinna, 1920s. Take \( f \) transcendental, of order \( \rho \), and assume that \( f \) has no multiple points (locally one-one).

The Schwarzian derivative

\[ S(f) = f^{(3)}/f' - (3/2)(f''/f')^2 \]

is ENTIRE. Also \( f \) can be written \( f_1/f_2 \), where \( f_j \) solve

\[ y'' + A(z)y = 0, \quad A(z) = S(f)/2. \]

If \( \rho < \infty \) then \( A \) is a polynomial

\[ \rho(f) = (\deg(A) + 2)/2. \]

Example:

\[ f(z) = (\int_0^z e^{-t^2} dt)/(C + \int_0^z e^{-t^2} dt), \]

\[ C \text{ constant}, \quad A(z) = -1 - z^2. \]

\( f(z) \) tends to different asymptotic values in sectors, depending (on \( C \) and) on whether \( \exp(t^2) \) big or small.

Eremenko’s theorem (1994)

Suppose that \( f \) has finite order \( \rho \) and very few multiple points. Then \( 2\rho \) is an integer \( \geq 2 \), and \( f \) behaves asymptotically like examples above.

Growth and critical points, cont’d

\( f \) transcendental meromorphic, order \( \rho \).

2. If \( \rho < 1 \) then \( f' \) has infinitely many zeros. (Eremenko, Langley, Rossi ’94) (cf. tan \( z \))
3. If \( f \) is entire and \( \rho < 1/2 \), then \( f \) has infinitely many critical values (folklore). Sharp, due to \( \cos(\sqrt{z}) \).

4. For meromorphic \( f \), if \( n(r) = o(\log r) \) as \( r \to \infty \), then \( f \) has infinitely many critical values. (Langley, '94) Again sharp.

**4. GROWTH AND ASYMPTOTIC VALUES**

Recall: a transcendental singularity \( w \) of \( f^{-1} \) arises from a path \( \gamma \) tending to infinity on which \( f(z) \to w \).

**Denjoy-Carleman-Ahlfors Theorem (ca. 1930)**

A non-constant entire function of finite order can have at most \( 2\rho(f) \) finite asymptotic values.

Note: \( \infty \) asymptotic value (Iversen)

Examples: \( z^{-1/2} \sin(z^{1/2}) \) ; \( \int_0^z \sin t \; dt/t \).

**Reason for DCA**

More asymptotic values means narrower regions between paths, hence \( f \) must grow faster.

Note: no analogue of DCA for meromorphic \( f \): can have EVERY complex value asymptotic. (Eremenko 1979)

**Bergweiler-Eremenko Theorem (1995)**

If \( f \) is meromorphic of finite order and \( f \) has only finitely many critical values, then \( f \) has at most \( 2\rho(f) \) asymptotic values.

Note: Order assumption necessary in both DCA and Bergweiler-Eremenko theorems:

\[
 f(z) = \int_0^z e^{e^t} \; dt \text{ is entire, with no critical values, but with infinitely many asymptotic values.}
\]

Here \( \rho = \infty \).
An application of Bergweiler-Eremenko theorem

Several results have easier proofs using the last theorem.

**Theorem.** If $f$ is transcendental entire with $\rho(f) < 1$ then $f'/f$ has at least one zero. (Clunie, Eremenko, Rossi 1993)

Suppose no zeros. Then the only possible critical value of $f$ is 0. Also $\infty$ is an asymptotic value of $f$ (Iversen).

By Bergweiler-Eremenko, $\infty$ is the only asymptotic value of $f$.

But then $f^{-1}$ has at most two singular values.

**5. APPLICATIONS OF SINGULARITIES OF $f^{-1}$.**

1. If $f$ is meromorphic, $u = \log |f|$ is a potential, harmonic except $\mp \infty$ at zeros/poles.

The gradient $\nabla u$ models e.g. fluid or heat flow; zeros and poles of $f$ correspond to sources, sinks.

$\nabla u$ vanishes where $f'/f = 0$. (cf. papers by Clunie, Eremenko, Rossi, Langley, Shea)

2. Iteration theory

Given meromorphic $f$, look at iterates $f_n$, where $f_{n+1} = f(f_n)$.

A "stable point" $z_0$ means roughly: a small change in $z_0$ makes a small change in the forward orbit (equicontinuity)

$$O^+(z_0) = \{ f_n(z_0): n \geq 0 \}. $$

$z_0$ is in the (stable set) Fatou set if:

The iterates $f_n$ are all defined on a nbd $V$ of $z_0$ and form a normal family on $V$. 
(normal: every sequence has a convergent subsequence, possibly with limit $\infty$.)

The complement is the Julia set, non-empty if $f$ is not Moebius.

**Montel Criterion**

If a family $F$ of functions $f$ meromorphic on a domain $D$ is such that each $f$ omits 3 fixed values $a,b,c$ in $D$, then $F$ is normal in $D$.

(Local analogue of the Picard theorem)

**Periodic points**

Suppose $f_n(z_0) = z_0$ and $|f_n'(z_0)| < 1$ (attracting periodic point). Then $z_0$ lies in a component of Fatou set, which contains a singularity of $f^{-1}$.

If $|f_n'(z_0)| = 1$ then $z_0$ may lie on boundary of components each containing a singularity.

**Example** $f(z) = \lambda \tan z$, $0 < \lambda < 1$

$f(z)$ maps upper half plane, lower half plane, real axis into themselves.

$f(0) = 0$ (fixpoint) and $f'(0) = \lambda$ (attracting fixpoint)

Thus 0 and $\mathbb{C} \setminus \mathbb{R}$ lie in a component $U$ of the Fatou set, as do all pre-images of 0.

Julia set is a totally disconnected subset of $\mathbb{R}$.

The Fatou set is $U$ and contains two singularities of $f^{-1}$ (namely $\pm \lambda i$).

Fatou set and Julia set invariant under $f$.

If $U$ is a component of Fatou set, then $f_n$ maps $U$ into a component $U_n$ of Fatou set.

$U$ is WANDERING if all the $U_n$ are distinct (otherwise eventually periodic).
Rational functions can’t have wandering domains (Sullivan).

Transcendental entire $f$ can (Baker, Herman).

If $f^{-1}$ has only finitely many singular values (Class $S$) then $f$ has no wandering domains. (Goldberg, Keen, Eremenko, Lyubich, Baker, Kotus, Lu).

6. HIGHER DERIVATIVES

Pólya Shire Theorem (1922)

If $f$ is meromorphic with at least one pole, and $w$ is in $\mathbb{C}$, then:

there exist $z_n \to w$ and $k_n \to \infty$ such that $f^{(k_n)}(z_n) = 0$

iff the nearest pole of $f$ to $w$ is not unique.

Example 1: For $f(z) = \tan z$, the point $w$ attracts zeros of higher derivatives iff $\text{Re}(w) = k\pi$, $k$ in $\mathbb{Z}$. The strips in between are the "shires".

Only gives information about zeros of $f^{(k)}$, $k$ large.

Must $f^{(k)}$ have zeros if $f$ has poles?

Example 2:

Let $m \geq 0$ be integer. Then there exists a transcendental meromorphic $f$ such that $f^{(m)}$ and $f^{(m+1)}$ have no zeros, but $f$ has infinitely many poles.

Take any entire $h$ with only simple zeros, and define $f$ by

\[
\frac{f^{(m+1)}}{f^{(m)}} = \frac{e^g}{h}
\]

with $g$ entire, chosen (Mittag-Leffler interpolation) so $f$ meromorphic and $h = 0$ implies $f = \infty$.

Note: $\rho(f) = \infty$. 
Theorem (conjectured by Hayman 1959)

Suppose that $m \geq 0, k \geq 2$, and that $f$ is transcendental meromorphic in the plane such that $f^{(m)}$ and $f^{(m+k)}$ both have only finitely many zeros.

Then $f^{(m)}$ is of the form $Re^P$ with $R$ rational, $P$ a polynomial.

In particular, $f$ has only finitely many poles.

Essentially says that two "non-adjacent" zero-free derivatives determine $f$.

For $k \geq 3$ proved by Frank (1976) using Nevanlinna theory, DEs, Wronskians. For $k = 2$, proved by Langley (1993).

Sketch of proof for $k = 2$:

Assume $m=0$ and $f$ and $f''$ have no zeros. Use the Newton function

$$H(z) = z - f(z)/f'(z).$$

(Edrei 1950s et al.) If $H$ is rational, solve for $f$. Assume $H$ transcendental.

$$H'(z) = f(z)f''(z)/f'(z)^2 \neq 0.$$ 

Also, poles of $H$ are simple. Thus $H$ has no multiple points, so (F. Nevanlinna)

$$H = h_1/h_2, \quad h_j'' + A(z)h_j = 0,$$

$A(z)$ entire. Work near maximum modulus of $A$. Show:

$H$ has fixpoints $H(z) = z$ with $H'(z)$ large.

But $H(z) = z$ implies $f(z) = \infty$ and $H'(z) = (q+1)/q$ with $q$ integer.

7. THE GOL’DBERG CONJECTURE

Gol’dberg has conjectured that

$$\bar{N}(r,f) \leq N(r,1/f^{(k)}) + S(r,f), \quad k \geq 2,$$

with $S(r,f)$ a relatively small error term.
i.e. if \( k \geq 2 \) the frequency of distinct poles of \( f \) controlled by frequency of zeros of \( f^{(k)} \) (cf. Pólya).

Proof of this would have major impact on several other conjectures.

Not true for \( k = 1 \) ( \( \tan z \) again! ).

**WHAT'S KNOWN**

(i) Gol’dberg conjecture is true if \( f \) has only simple poles (Mues 1971).

(ii) If \( f \) has finite order and \( f'' \) has only finitely many zeros then the number of distinct poles of \( f \) in \( |z| \leq r \) is at most \( O(\log r)^2 \).

If, in addition, \( f' \) has finitely many zeros, or the poles of \( f \) have bounded multiplicities, then \( f \) has finitely many poles. (Langley, 1995-7).

Methods for (ii): Bergweiler-Eremenko theorem, harmonic measure, length-area principles, ideas from iteration.