G12CAN Complex Analysis

Books: Schaum Outline book on *Complex Variables* (by M. Spiegel), or Churchill and Brown, *Complex Analysis and Applications*. There should be copies in Short Loan and Reference Only sections of the library. Notes are on *www.maths.nottingham.ac.uk/personal/jkl* (readable in PDF form).

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Office hours: displayed outside my office. (see notices and timetable outside my room).

AIMS AND OBJECTIVES:
Aims: to teach the introductory theory of functions of a complex variable; to teach the computational techniques of complex analysis, in particular residue calculus, with a view to potential applications in subsequent modules.
Objectives: a successful student will: 1. be able to identify analytic functions and singularities; 2. be able to prove simple propositions concerning functions of a complex variable, for example using the Cauchy-Riemann equations; 3. be able to evaluate certain classes of integrals; 4. be able to compute Taylor and Laurent series expansions.

SUMMARY: in this module we concentrate on functions which can be regarded as functions of a complex variable, and are differentiable with respect to that complex variable. These "good" functions include exp, sine, cosine etc. (but log will be a bit tricky). These are important in applied maths, and they turn out to satisfy some very useful and quite surprising and interesting formulas. For example, one technique we learn in this module is how to calculate integrals like $\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1}\, dx$ WITHOUT actually integrating.

PROBLEM CLASSES will be fortnightly, on Tuesdays at 11.00 and 12.00. You **must** be available for at least one of these times. Please see handouts for dates and further information.

COURSEWORK: Dates for handing in for G12CAN will be announced in the first handout (all will be Tuesdays). The problems will be made available at least one week before the work is due. Homework does not count towards the assessment, but its completion is strongly advised, and the work will emphasize the computational techniques which are **essential** to passing the module. Failure to hand in homework, poor marks, and non-attendance at problem classes will be reported to tutors.

ASSESSMENT: One 2-hour written exam. Section A is compulsory and is worth half the total marks. From Section B you must choose two out of three longer questions. For your revision, you may find it advantageous to look at old G12CAN papers, although there have been minor variations in content over the years.

The assessment will mainly be based on using the facts and theorems of the module to solve problems of a computational nature, or to derive facts about functions. You will not be expected to memorize the proofs of the theorems in the notes.

Proofs of some theorems will just be sketched in the lectures, with the details provided on handouts in case you wish to see them. **You will not be required to reproduce these proofs in the examination.**

1.1 Basic Facts on Complex Numbers from G1ALIM

All this section was covered in G1ALIM. Suppose we have two complex numbers $z = x+yi$ and $w = u+vi$ (where x, y, u, v are all real). Then $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$,

$$(x+yi)+(u+vi) = (x+u)+(y+v)i, \quad (x+yi)-(u+vi) = (x-u)+(y-v)i,$$

$$(x+yi)(u+vi) = xu-yv+(xv+yu)i$$

and, if $x+yi \neq 0+0i$,

$$\frac{u+vi}{x+yi} = \frac{(u+vi)(x-yi)}{x^2+y^2} = \frac{(u+vi)(x-yi)}{(x+yi)(x-yi)}.$$

With these rules, we've made a field called $\mathbb{C}$, which contains $\mathbb{R}$, as $x = x+0i$.

The Argand diagram, or complex plane

Think of the complex number $z = x+yi$, with $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$ both real, as interchangeable with the point (x, y) in the two dimensional plane. A real number x corresponds to $(x, 0)$ and the x axis becomes the REAL axis, while numbers iy, with y real (often called purely imaginary) correspond to points $(0, y)$, and the y axis becomes the IMAGINARY axis.

The complex conjugate

The complex conjugate of the complex number z is the complex number $\bar{z} = \mathrm{Re}(z) - i\,\mathrm{Im}(z)$. Some write z^* instead. E.g. $\overline{2+3i} = 2-3i$. In fact, $\bar{z}$ is the reflection of z across the real axis. The conjugate has the following easily verified properties:

$$\overline{(\bar{z})} = z, \quad \overline{z+w} = \bar{z}+\bar{w}, \quad \overline{zw} = \bar{z}\,\bar{w}, \quad z+\bar{z} = 2\mathrm{Re}(z), \quad z-\bar{z} = 2i\mathrm{Im}(z).$$

Modulus of a complex number

The modulus or absolute value of z is the non-negative real number $|z| = \sqrt{\mathrm{Re}(z)^2+\mathrm{Im}(z)^2}$. This is the distance from 0 to the point z in the complex plane. Note that

$$z\bar{z} = (\mathrm{Re}(z)+i\mathrm{Im}(z))(\mathrm{Re}(z)-i\mathrm{Im}(z)) = \mathrm{Re}(z)^2+\mathrm{Im}(z)^2 = |z|^2$$

so that a useful formula is $|z| = \sqrt{z\bar{z}}$. Also (i) $1/z = \bar{z}|z|^{-2}$ if $z \neq 0$ (ii) $|zw| = |z|\,|w|$.

Warnings (i) The rules $|z| = \pm z, z^2 = |z|^2$ are only true if z is real; (ii) The statement $z < w$ only makes sense if z and w are both real: you can't compare complex numbers this way.

Triangle Inequality

For all $z,w \in \mathbb{C}$, we have $|z+w| \leqslant |z|+|w|$ and $|z-w| \geqslant |z|-|w|$. Note that $0, z, w, z+w$ form the vertices of a parallelogram. The second inequality follows from $|z| \leqslant |w|+|z-w|$.

Note also that $0, w, z-w, z$ form the vertices of a parallelogram and hence $|z-w|$ is the distance from z to w.

Polar and exponential form

Associate the complex number z with the point $(\text{Re}(z)\,,\,\text{Im}(z))$ in $\mathbb{R}^2$.

If $z \neq 0$, then $\text{Re}(z)$ and $\text{Im}(z)$ aren't both zero, and $r = |z| \neq 0$. Let θ be the angle between the positive real axis and the line from 0 to z, measured counter-clockwise in radians. Then $x = \text{Re}(z) = r\cos\theta$, $y = \text{Im}(z) = r\sin\theta$. Writing

$$z = r\cos\theta + ir\sin\theta\,,$$

we have the POLAR form of z. The number θ is called an ARGUMENT of z and we write $\theta = \arg z$. Note that (1) $\arg 0$ does not exist. (2) If θ is one argument of z, then so is $\theta + k2\pi$ for any integer k. (3) From the Argand diagram, we see that $\arg z \pm \pi$ is an arg of $-z$.

We can always choose a value of $\arg z$ lying in $(-\pi, \pi]$ and we call this the PRINCIPAL ARGUMENT $\text{Arg}\, z$. Note that if z is on the negative real axis then $\text{Arg}\, z = \pi$, but $\text{Arg}\, z \to -\pi$ as z approaches the negative real axis from below (from the lower half-plane).

To compute $\text{Arg}\, z$ using a calculator: suppose $z = x + iy \neq 0$, with x, y real. If $x > 0$ then $\theta = \text{Arg}\, z = \tan^{-1}(y/x) = \arctan(y/x)$ but this gives the WRONG answer if $x < 0$. The reason is that calculators always give $\tan^{-1}$ between $-\pi/2$ and $\pi/2$. Thus if $x < 0$ then $\tan^{-1}(y/x) = \tan^{-1}(-y/(-x))$ gives $\text{Arg}\,(-z) = \text{Arg}\, z \pm \pi$. If $x = 0$ and $y > 0$ then $\text{Arg}\, z = \pi/2$, while if $x = 0$ and $y < 0$ then $\text{Arg}\, z = -\pi/2$.

Definition

For t real, we define $e^{it} = \cos t + i\sin t$. Using the trig. formulas

$$\cos(s+t) = \cos s\cos t - \sin s\sin t,\ \ \sin(s+t) = \sin s\cos t + \cos s\sin t,$$

we get, for s,t real,

$$e^{is}e^{it} = \cos s\,\cos t - \sin s\,\sin t + i(\cos s\,\sin t + \sin s\,\cos t) = e^{i(s+t)}.$$

Thus $e^{-it}e^{it} = e^{i0} = 1$. Also, $\overline{(e^{it})} = e^{-it}$ and, if z,w are non-zero complex numbers, we have

$$zw = |z|e^{i\arg z}|w|e^{i\arg w} = |zw|e^{i(\arg z + \arg w)}$$

and $\bar{z} = |z|e^{-i\arg z}$, $1/z = |z|^{-1}e^{-i\arg z}$. We get:
(a) $\arg z + \arg w$ is an argument of zw. (b) $-\arg z$ is an argument of $1/z$ and of $\bar{z}$.
Warning: it is not always true that $\text{Arg}\, z + \text{Arg}\, w = \text{Arg}\, zw$. Try $z = w = -1 + i$.

De Moivre's theorem

For t real, we have $e^{2it} = e^{it}e^{it} = (e^{it})^2$ and $e^{-it} = 1/(e^{it})$. Repeating this argument we get $(e^{it})^n = e^{int}$ for all real t and integer n (de Moivre's theorem). For example, for real t, we have $\cos 2t = \text{Re}(e^{2it}) = \text{Re}((e^{it})^2) = \text{Re}(\cos^2 t - 2i\cos t\sin t - \sin^2 t) = 2\cos^2 t - 1$.

Roots of unity

Let n be a positive integer. Find all solutions z of $z^n = 1$.

Solution: clearly $z \neq 0$ so write $z = re^{it}$ with $r = |z|$ and t an argument of z. Then $1 = z^n = r^n e^{int}$. So $1 = |z^n| = r^n$ and $r = 1$, while $e^{int} = \cos nt + i \sin nt = 1$. Thus $nt = k2\pi$ for some integer k, and $z = e^{it} = e^{k2\pi i/n}$. However, $e^{is} = e^{is + j2\pi i}$ for any integer j, so $e^{k2\pi i/n} = e^{k'2\pi i/n}$ if $k - k'$ is an integer multiple of n. So we just get the n roots $\zeta_k = e^{k2\pi i/n}$, $k = 0, 1, \ldots, n-1$. One of them ($k = 0$) is 1, and they are equally spaced around the circle of centre 0 and radius 1, at an angle $2\pi/n$ apart. The ζ_k are called the n'th roots of unity.

Solving some simple equations

To solve $z^n = w$, where n is a positive integer and w is a non-zero complex number, we first write $w = |w| e^{i \operatorname{Arg} w}$. Now $z_0 = |w|^{1/n} e^{(i/n) \operatorname{Arg} w}$, in which $|w|^{1/n}$ denotes the positive n'th root of $|w|$, gives $(z_0)^n = w$. This z_0 is called the principal root. Now if z is any root of $z^n = w$, then $(z/z_0)^n = w/w = 1$, so z/z_0 is an n'th root of unity. So the n roots of $z^n = w$ are $z_k = |w|^{1/n} e^{(i/n) \operatorname{Arg} w + k2\pi i/n}$, $k = 0, 1, \ldots, n-1$.

For example, to solve $z^4 = -1 - i = w$, we write $w = \sqrt{2} e^{-3\pi i/4}$ and $z_0 = 2^{1/8} e^{-3\pi i/16}$. The other roots are $z_1 = 2^{1/8} e^{-3\pi i/16 + \pi i/2} = 2^{1/8} e^{5\pi i/16}$ and $z_2 = 2^{1/8} e^{-3\pi i/16 + \pi i} = 2^{1/8} e^{13\pi i/16}$ and $z_3 = 2^{1/8} e^{-3\pi i/16 + 3\pi i/2} = 2^{1/8} e^{-3\pi i/16 - \pi i/2} = 2^{1/8} e^{-11\pi i/16}$.

Quadratics: we solve these by completing the square in the usual way. For example, to solve $z^2 + (2 + 2i)z + 6i = 0$ we write this as $(z + 1 + i)^2 - (1 + i)^2 + 6i = 0$ giving $(z + 1 + i)^2 = -4i = 4e^{-i\pi/2}$ and the solutions are $z + 1 + i = 2e^{-i\pi/4}$ and $z + 1 + i = 2e^{-i\pi/4 + i\pi} = 2e^{3i\pi/4}$.

In general, $az^2 + bz + c = 0$ (with $a \neq 0$) solves to give $4a^2 z^2 + 4abz + 4ac = 0$ and so $(2az + b)^2 = b^2 - 4ac$ and so $z = (-b + (b^2 - 4ac)^{1/2})/2a$ with, in general, two values for the square root.

For example, to solve $z^4 - 2z^2 + 2 = 0$ we write $u = z^2$ to get $(u-1)^2 + 1 = 0$ and so $u = 1 \pm i$. Now $z^2 = 1 + i = \sqrt{2} e^{i\pi/4}$ has principal root $z_1 = 2^{1/4} e^{i\pi/8}$ and second root $z_2 = z_1 e^{i\pi} = -z_1 = 2^{1/4} e^{i9\pi/8} = 2^{1/4} e^{-i7\pi/8}$, in which $2^{1/4}$ means the positive fourth root of 2. Two more solutions come from solving $z^2 = 1 - i = \sqrt{2} e^{-i\pi/4}$ and these are $z_3 = 2^{1/4} e^{-i\pi/8}$ and $z_4 = 2^{1/4} e^{i7\pi/8}$.

An example

Consider the straight line through the origin which makes an angle $\alpha, 0 \leqslant \alpha \leqslant \pi/2$, with the positive x-axis. Find a formula which sends each $z = x+iy$ to its reflection across this line.
If we do this first using the line with angle α, and then using the line with angle β ($0 < \beta < \pi/2$), what is the net effect?

1.2 Introduction to complex integrals

Suppose first of all that $[a,b]$ is a closed interval in $\mathbb{R}$ and that $g:[a,b] \to \mathbb{C}$ is continuous (this means simply that $u = \mathrm{Re}(g)$ and $v = \mathrm{Im}(g)$ are both continuous). We can just define

$$\int_a^b g(t)\ dt = \int_a^b \mathrm{Re}(g(t))\ dt + i\int_a^b \mathrm{Im}(g(t))\ dt.$$

Example Determine $\int_0^2 e^{2it}dt$.

Note that every complex number z can be written in the form $z = re^{it}$ with $r = |z| \geqslant 0$ and $t \in \mathbb{R}$, and so $|z| = ze^{-it}$. Thus we have, for some real s,

$$\left|\int_a^b g(t)\ dt\right| = e^{is}\int_a^b g(t)\ dt = \int_a^b e^{is}g(t)\ dt = \int_a^b \mathrm{Re}(\ e^{is}g(t)\ )\ dt$$

and this is, by real analysis,

$$\leqslant \int_a^b |\ \mathrm{Re}(\ e^{is}g(t)\ )\ |\ dt \leqslant \int_a^b |\ e^{is}g(t)\ |\ dt = \int_a^b |\ g(t)\ |\ dt.$$

Example: for $n \in \mathbb{N}$ set $I_n = \int_1^2 e^{it^3}(t+in)^{-1}dt$. Show that $I_n \to 0$ as $n \to +\infty$.

1.3 Paths and contours

Suppose that f_1, f_2 are continuous real-valued functions on a closed interval $[a,b]$. As the "time" t increases from a to b, the point $\gamma(t) = f_1(t) + if_2(t)$ traces out a curve (or path, we make no distinction between these words in this module) in $\mathbb{C}$. A path in $\mathbb{C}$ is then just a continuous function γ from a closed interval $[a,b]$ to $\mathbb{C}$, in which we agree that γ will be called continuous iff its real and imaginary parts are continuous.

Paths are not always as you might expect. There is a path $\gamma:[0,2] \to \mathbb{C}$ such that γ passes through every point in the rectangle $w = u+iv$, $u,v \in [0,1]$. (You can find this on p.224 of Math. Analysis by T. Apostol). There also exist paths which never have a tangent although (it's possible to prove that) you can't draw one.

Because of this awkward fact, we define a special type of path with good properties:

A smooth contour is a path $\gamma:[a,\ b] \to \mathbb{C}$ such that the derivative γ' exists and is continuous and never 0 on $[a,\, b]$. Notice that if we write $\mathrm{Re}(\gamma) = \sigma$, $\mathrm{Im}(\gamma) = \tau$ then $(\sigma'(t)\ ,\ \tau'(t))$ is the tangent vector to the curve, and we are assuming that this varies continuously and is never the zero vector.

For $a < t < b$ let $s(t)$ be the length of the part of the contour γ between "time" a and "time" t. Then if δt is small and positive, $s(t+\delta t)-s(t)$ is approximately equal to $|\gamma(t+\delta t)-\gamma(t)|$ and so

$$\frac{ds}{dt} = \lim_{\delta t \to 0+} \frac{|\gamma(t+\delta t)-\gamma(t)|}{\delta t} = |\gamma'(t)|.$$

Hence the length of the whole contour γ is $\int_a^b |\gamma'(t)|\, dt$, and is sometimes denoted by $|\gamma|$.

Examples

(i) A circle of centre a and radius r described once counter-clockwise. The formula is $z = a+re^{it}, 0 \leqslant t \leqslant 2\pi$.

(ii) The straight line segment from a to b. This is given by $z = a+t(b-a), 0 \leqslant t \leqslant 1$.

More on arc length (optional!)

Let $\gamma:[a,b] \to \mathbb{C}, \gamma(t) = f(t)+ig(t)$, with f,g real and continuous, be a path (not necessarily a smooth contour). The arc length of γ, if it exists, can be defined as follows. Let $a = t_0 < t_1 < t_2 < \ldots\ldots < t_n = b$. Then $P = \{t_0, \ldots\ldots, t_n\}$ is a partition of $[a,b]$ with vertices t_k (the notation and some ideas here have close analogues in Riemann integration), and

$$L(P) = \sum_{k=1}^{n} |\gamma(t_k)-\gamma(t_{k-1})|$$

is the length of the polygonal path through the $n+1$ points $\gamma(t_k), k = 0, 1, \ldots, n$. If we form P' by adding to P an extra point d, with $t_{j-1} < d < t_j$, then the triangle inequality gives

$$L(P')-L(P) = |\gamma(t_j)-\gamma(d)|+|\gamma(d)-\gamma(t_{j-1})|-|\gamma(t_j)-\gamma(t_{j-1})| \geqslant 0.$$

So as we add extra points, $L(P)$ can only increase, and if the arc length S of γ exists in some sense then it is reasonable to expect that $L(P)$ will be close to S if P is "fine" enough (i.e. if all the $t_k - t_{k-1}$ are small enough). With this in mind, we *define* the length S of γ to be

$$S = \Lambda(\gamma, a, b) = \text{l.u.b. } L(P),$$

with the supremum (l.u.b. i.e. least upper bound) taken over all partitions P of $[a,b]$. If the $L(P)$ are bounded above, then S is the least real number which is $\geqslant L(P)$ for every P, and γ is called *rectifiable*. If the set of $L(P)$ is not bounded above then $S = \infty$ and γ is non-rectifiable.

Suppose that $a < c < b$. Then every partition of $[a,b]$ which includes c as a vertex can be written as the union of a partition of $[a,c]$ and a partition of $[c,b]$. It follows easily that

$$\Lambda(\gamma, a, b) = \Lambda(\gamma, a, c)+\Lambda(\gamma, c, b).$$

The following theorem shows that, for a smooth contour, the arc length defined this way has the same value as the integral $\int_a^b |\gamma'(t)|\,dt$ which we derived earlier.

Theorem

Let $\gamma:[a,b] \to \mathbb{C}, \gamma = f+ig$, with f,g real, be a smooth contour. Then S as defined above satisfies

$$S = \Lambda(\gamma,a,b) = \int_a^b |\gamma'(t)|\,dt. \tag{1}$$

Proof: Let

$$S(v) = \Lambda(\gamma,a,v),\ a \leqslant v \leqslant b.$$

If we can show that $S'(v) = |\gamma'(v)|$ for $a < v < b$ then (1) follows by integration. So let $a < v < b$ and let $c = |\gamma'(v)| = \sqrt{f'(v)^2+g'(v)^2}$. We know that $c \neq 0$ (definition of smooth contour). Let $0 < \delta < c$, and choose $\varepsilon > 0$, so small that $-\varepsilon < p < \varepsilon$ and $-\varepsilon < q < \varepsilon$ imply that

$$c-\delta < \sqrt{(f'(v)+p)^2+(g'(v)+q)^2} < c+\delta. \tag{2}$$

Since $\gamma' = f'+ig'$ is continuous at v, we can choose $\rho > 0$ such that

$$|f'(s)-f'(v)| < \varepsilon,\ |g'(s^*)-g'(v)| < \varepsilon, \tag{3}$$

for $|s-v| < \rho$, $|s^*-v| < \rho$. So, for $|s-v| < \rho$, $|s^*-v| < \rho$, (2) and (3) give

$$c-\delta < \sqrt{f'(s)^2+g'(s^*)^2} < c+\delta. \tag{4}$$

Let $0 < h < \rho$, and let $P = \{t_0, t_1, \ldots, t_n\}$ be any partition of $[v, v+h]$. Then

$$L(P) = \sum_{k=1}^{n} |\gamma(t_k)-\gamma(t_{k-1})| = \sum_{k=1}^{n} \sqrt{(f(t_k)-f(t_{k-1}))^2+(g(t_k)-g(t_{k-1}))^2}$$

and the mean value theorem of real analysis gives us s_k and $s_k{}^*$ in (t_{k-1}, t_k) such that

$$L(P) = \sum_{k=1}^{n} (t_k-t_{k-1})\sqrt{f'(s_k)^2+g'(s_k{}^*)^2}.$$

Hence, by (4), we have

$$(t_n-t_0)(c-\delta) = \sum_{k=1}^{n} (t_k-t_{k-1})(c-\delta) < L(P) < \sum_{k=1}^{n} (t_k-t_{k-1})(c+\delta) = (t_n-t_0)(c+\delta).$$

Since P is an arbitrary partition of $[v, v+h]$ we get

$$h(c-\delta) \leqslant \Lambda(\gamma, v, v+h) = S(v+h)-S(v) \leqslant h(c+\delta)$$

and so, provided $0 < h < \rho$,

$$c-\delta \leqslant \frac{S(v+h)-S(v)}{h} \leqslant c+\delta.$$

Since δ may be chosen arbitrarily small we get $S'(v) = c = |\gamma'(v)|$.

1.4 Introduction to contour integrals

Suppose that $\gamma:[a, b] \to \mathbb{C}$ is a smooth contour. If f is a function such that $f(\gamma(t))$ is continuous on $[a, b]$ we set

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

1.5 a very important example!

Let $a \in \mathbb{C}$, let $m \in \mathbb{N}$ and $r > 0$, and set $\gamma(t) = a + re^{it}$, $0 \leqslant t \leqslant 2m\pi$. As t increases from 0 to $2m\pi$, the point $\gamma(t)$ describes the circle $|z-a| = r$ counter-clockwise m times. Now let $n \in \mathbb{Z}$. We have

$$\int_\gamma (z-a)^n\,dz = \int_0^{2m\pi} r^n e^{int}\,ire^{it}\,dt = \int_0^{2m\pi} i\,r^{n+1}\,e^{(n+1)it}\,dt.$$

If $n \neq -1$ this is 0, by periodicity of $\cos((n+1)t)$ and $\sin((n+1)t)$. If $n = -1$ then we get $2m\pi i$.

1.6 Properties of contour integrals

(a) If $\gamma:[a,b] \to \mathbb{C}$ is a smooth contour and λ is given by $\lambda(t) = \gamma(b+a-t)$ (so that λ is like γ "backwards") then

$$\int_\lambda f(z)\,dz = \int_a^b f(\gamma(b+a-t))\,(-\gamma'(b+a-t))\,dt = -\int_\gamma f(z)\,dz.$$

(b) A smooth contour is called SIMPLE if it never passes through the same point twice (i.e. it is a one-one function). Suppose that λ and γ are simple, smooth contours which describe the same set of points in the same direction. Suppose λ is defined on $[a,b]$ and γ on $[c,d]$. It is easy to see that there is a strictly increasing function $\phi:[a,b] \to [c,d]$ such that $\lambda(t) = \gamma(\phi(t))$ for $a \leqslant t \leqslant b$. Further, it is quite easy to prove that $\phi(t)$ has continuous non-zero derivative on $[a,b]$ and we can write

$$\int_\lambda f(z)\,dz = \int_a^b f(\lambda(t))\,\lambda'(t)\,dt = \int_a^b f(\gamma(\phi(t)))\,\gamma'(\phi(t))\,\phi'(t)\,dt =$$

$$= \int_c^d f(\gamma(s))\gamma'(s)\,ds = \int_\gamma f(z)\,dz.$$

Thus the contour integral is "independent of parametrization".

Here's the proof that $\phi'(t)$ exists (optional!). For t and t_0 in (a, b) with $t \neq t_0$ write

$$\frac{\lambda(t)-\lambda(t_0)}{t-t_0} = \frac{\gamma(\phi(t))-\gamma(\phi(t_0))}{\phi(t)-\phi(t_0)}\frac{\phi(t)-\phi(t_0)}{t-t_0}.$$

Note that there's no danger of zero denominators here as ϕ is strictly increasing so that $\phi(t) \neq \phi(t_0)$. Letting t tend to t_0 we have $\gamma(\phi(t)) \to \gamma(\phi(t_0))$ and so $\phi(t) \to \phi(t_0)$ since γ is one-one on (c, d). (If $\phi(t)$ had a "jump" discontinuity then $\lambda(t)$ would "miss out" some points through which γ passes). Thus we see that

$$\phi'(t_0) = \lim_{t \to t_0} \frac{\phi(t)-\phi(t_0)}{t-t_0} = \frac{\lambda'(t_0)}{\gamma'(\phi(t_0))}$$

which gives the expected formula for ϕ' (*and* shows that it's continuous).

(c) This is called the FUNDAMENTAL ESTIMATE; suppose that $|f(z)| \leqslant M$ on γ. Then we have

$$\left|\int_\gamma f(z)\ dz\right| \leqslant \int_a^b |f(\gamma(t))|\ |\gamma'(t)|\ dt \leqslant M\int_a^b |\gamma'(t)|dt = M. \text{ (length of } \gamma \text{).}$$

Example: let γ be the straight line from 2 to $3+i$, and let $I_n = \int_\gamma dz/(z^n+\bar{z})$, with n a positive integer. Show that $I_n \to 0$ as $n \to \infty$.

Some more definitions

By a PIECEWISE SMOOTH contour γ we mean finitely many smooth contours γ_k joined end to end, in which case we define

$$\int_\gamma f(z)\ dz = \sum_k \int_{\gamma_k} f(z)\ dz.$$

The standard example is a STEPWISE CURVE: a path made up of finitely many straight line segments, each parallel to either the real or imaginary axis, joined end to end. For example, to go from 0 to $1+i$ via 1 we can use $\gamma_1(t) = t$, $0 \leqslant t \leqslant 1$ followed by $\gamma_2(t) = 1+(1+i-1)t$, $0 \leqslant t \leqslant 1$.

Note that by 1.6(b) if you need $\int_\gamma f(z)dz$ it doesn't generally matter how you do the parametrization.

Suppose γ is a PSC made up of the smooth contours $\gamma_1,.....,\gamma_n$, in order. It is sometimes convenient to combine these n formulas into one. Assuming each γ_j is defined on $[0,1]$ (if not we can easily modify them) we can put

$$\gamma(t) = \gamma_j(t-j+1),\ j-1 \leqslant t \leqslant j. \tag{1}$$

The formula (1) then defines γ as a continuous function on $[0,n]$.

A piecewise smooth contour is SIMPLE if it never passes through the same point twice (i.e. γ as in (1) is one-one), CLOSED if it finishes where it started (i.e. $\gamma(n) = \gamma(0)$) and SIMPLE CLOSED if it finishes where it started but otherwise does not pass through any point twice (i.e. γ is one-one except that $\gamma(n) = \gamma(0)$). These are equivalent to:

γ is CLOSED if it finishes where it starts i.e. the last point of γ_n is the first point of γ_1.

γ is SIMPLE if it never passes through the same point twice (apart from the fact that γ_{k+1} starts where γ_k finishes).

γ is SIMPLE CLOSED if it finishes where it starts but otherwise doesn't pass through any point twice (apart again from the fact that γ_{k+1} starts where γ_k finishes).

Example

Let σ be the straight line from i to 1, and let γ be the stepwise curve from i to 1 via 0. Show that

$$\int_\gamma \bar{z}\,dz \neq \int_\sigma \bar{z}\,dz.$$

Thus the contour integral is not always independent of path (we will return to this important theme later).

1.7 Open Sets and Domains

Let $z \in \mathbb{C}$ and let $r > 0$. We define $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$. This is called the open disc of centre z and radius r. It consists of all points lying inside the circle of centre z and radius r, the circle itself being excluded.

Now let $U \subseteq \mathbb{C}$. We say that U is OPEN if it has the following property: for each $z \in U$ there exists $r_z > 0$ such that $B(z,r_z) \subseteq U$. Note that r_z will usually depend on z.

Examples

(i) An open disc $B(z,r)$ is itself an open set. Suppose w is in $B(z,r)$. Put $s = r - |w - z| > 0$. Then $B(w,s) \subseteq B(z,r)$. Why? Because if $|u - w| < s$ then $|u - z| \leqslant |u - w| + |w - z| < s + |w - z| = r$. What we've done is to inscribe a circle of radius s and centre w inside the circle of centre z and radius r.

(ii) Let $H = \{z : \mathrm{Re}(z) > 0\}$. Then H is open. Why? If $z \in H$, put $r_z = \mathrm{Re}(z) > 0$. Then $B(z,r_z) \subseteq H$, because if $w \in B(z,r_z)$ we have $w = z + te^{iv}$ for some real v and t with $0 \leqslant t < r_z$. So $\mathrm{Re}(w) = \mathrm{Re}(z) + t\cos v \geqslant \mathrm{Re}(z) - t > 0$.

(iii) Let $K = \{z = x + iy : x,y \in \mathbb{R} \setminus \mathbb{Q}\}$. Then K is not open. The point $u = \sqrt{2} + i\sqrt{2}$ is in K, but any open disc centred at u will contain a point with rational coordinates.

A domain is an open subset D of $\mathbb{C}$ which has the following additional property: any two points in D can be joined by a stepwise curve which does not leave D. An open disc is a domain, as is the half-plane $\text{Re}(z) > 0$, but the set $\{z : \text{Re}(z) \neq 0\}$ is not a domain, as any stepwise curve from -1 to 1 would have to pass through $\text{Re}(z) = 0$ (by the IVT).

We will say that a set E in $\mathbb{R}^2$ is open/a domain if the set in $\mathbb{C}$ corresponding to E, that is, $\{x + iy : (x, y) \in E\}$, is open/a domain.

A useful fact about domains

Let D be a domain in $\mathbb{R}^2$, and let u be a real-valued function such that $u_x \equiv 0$ and $u_y \equiv 0$ on D. Then u is constant on D.

Here the partials $u_x = \partial u / \partial x$, $u_y = \partial u / \partial y$, are defined by

$$u_x(a, b) = \lim_{x \to a} \frac{u(x, b) - u(a, b)}{x - a}, \quad u_y(a, b) = \lim_{y \to b} \frac{u(a, y) - u(a, b)}{y - b}.$$

Why is this fact true? Take any straight line segment S in D, parallel to the x axis, on which $y = y_0$, say. Then on S we can write $u(x, y) = u(x, y_0) = g(x)$, and we have $g'(x) = u_x(x, y_0) = 0$. So u is constant on S, and similarly constant on any line segment in D parallel to the y axis. Since any two points in D can be linked by finitely many such line segments joined end to end, u is constant on D.

2. Functions

2.1 Limits

If (z_n) is a sequence (i.e. non-terminating list) of complex numbers, we say that $z_n \to a \in \mathbb{C}$ if $|z_n - a| \to 0$ (i.e. the distance from z_n to a tends to 0).

As usual, if $E \subseteq \mathbb{C}$ a function f from E to $\mathbb{C}$ is a rule assigning to each $z \in E$ a unique value $f(z) \in \mathbb{C}$. Such functions can usually be expressed either in terms of $\text{Re}(z)$ and $\text{Im}(z)$ or in terms of z and $\bar{z}$.

For example, consider $f(z) = \bar{z}z^2$. If we put $x = \text{Re}(z)$, $y = \text{Im}(z)$ then we have $f(z) = z(\bar{z}z) = z(x^2 + y^2) = u(x,y) + iv(x,y)$, where $u(x,y) = x(x^2 + y^2)$ and $v(x,y) = y(x^2 + y^2)$. It is standard to write

$$f(x + iy) = u(x,y) + iv(x,y) , \qquad (1)$$

with x,y real and u,v real-valued functions (of x and y).

For any non-trivial study of functions you need limits. What do we mean by $\lim_{z \to a} f(z) = L \in \mathbb{C}$? We mean that as z approaches a, in any manner whatever, the value $f(z)$ approaches L. As usual, the value or existence of $f(a)$ makes no difference.

Definition

Let f be a complex-valued function defined near $a \in \mathbb{C}$ (but not necessarily at a itself).

We say that $\lim_{z \to a} f(z) = L \in \mathbb{C}$ if the following is true. For every sequence z_n which converges to a with $z_n \neq a$, we have $\lim_{n \to \infty} f(z_n) = L$.

This must hold for all sequences tending to, but not equal to, a, regardless of direction: the condition that $z_n \neq a$ is there because the existence or value of $f(a)$ makes no difference to the limit.

Using the decomposition (1) (with x,y,u,v real) it is easy to see that

$$\lim_{z \to a} f(z) = L \in \mathbb{C} \quad iff \quad \lim_{(x,y) \to (\mathrm{Re}(a),\mathrm{Im}(a))} u(x,y) = \mathrm{Re}(L) \ \text{ and } \lim_{(x,y) \to (\mathrm{Re}(a),\mathrm{Im}(a))} v(x,y) = \mathrm{Im}(L).$$

This is because

$$|u - \mathrm{Re}(L)| + |v - \mathrm{Im}(L)| \leqslant 2|f - L| \leqslant 2(|u - \mathrm{Re}(L)| + |v - \mathrm{Im}(L)|).$$

A standard Algebra of Limits result is also true, proved in exactly the same way as in the real analysis case.

Examples

(a) Let $g(x,y) = (x^3 + y^2 x^2)/(x^2 + 4y^2)$. Then $\lim_{(x,y) \to (0,0)} g(x,y) = 0$.

(b) Let $f(z) = |z|/(\pi + \mathrm{Arg}\, z)$ for $z \neq 0$. Does $\lim_{z \to 0} f(z)$ exist?

If we let $z \to 0$ along some ray $\arg z = t$ with t in $(-\pi,\pi]$, then the denominator is constant and $f(z) \to 0$. However, let $s > 0$ be small, and put $z = s e^{i(-\pi + s^2)}$. Then $\mathrm{Arg}\, z = -\pi + s^2$ and so $f(z) = s/s^2 = 1/s \to \infty$ if we let s tend to 0 through positive values. So the limit doesn't exist.

Continuity

This is easy to handle. We say f is continuous at a if $\lim_{z \to a} f(z)$ exists and is $f(a)$. Thus $f(z)$ is as close as desired to $f(a)$, for all z sufficiently close to a.

Note that $\mathrm{Arg}\, z$ is discontinuous on the negative real axis.

2.2 Complex differentiability

Now we can define our "good" functions. Let f be a complex-valued function defined on some open disc $B(a,r)$ and taking values in $\mathbb{C}$. We say that f is complex differentiable at a if there is a complex number $f'(a)$ such that

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Examples

1. Try $f(z) = \bar{z}$. Then we look at

$$\lim_{z \to a} \frac{\bar{z} - \bar{a}}{z - a} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

For $f'(a)$ to exist, the limit must be the same regardless of in what manner h approaches 0. If we let $h \to 0$ through real values, we see that $\bar{h}/h = 1$. But, If we let $h \to 0$ through imaginary values, say $h = ik$ with k real, we see that $\bar{h}/h = -ik/ik = -1$. So $\bar{z}$ is not complex differentiable anywhere. This is rather surprising, as $\bar{z}$ is a very well behaved function. It doesn't blow up anywhere and is in fact everywhere continuous. If you write it in the form $u(x,y) + iv(x,y)$ you get $u = x$ and $v = -y$, and these have partial derivatives everywhere. We'll see in a moment why $\bar{z}$ fails to be complex differentiable.

2. Try $f(z) = z^2$. Then, for any a,

$$\lim_{z \to a} \frac{z^2 - a^2}{z - a} = \lim_{z \to a} (z + a) = 2a,$$

and so the function z^2 *is* complex differentiable at every point, and $(d/dz)(z^2) = 2z$ as you'd expect. In fact, the chain rule, product rule and quotient rules all apply just as in the real case. So, for example, $(z^3 - 4)/(z^2 + 1)$ is complex differentiable at every point where $z^2 + 1 \neq 0$, and so everywhere except i and $-i$.

Meaning of the derivative

In real analysis we think of $f'(x_0)$ as the slope of the graph of f at x_0. In complex analysis it doesn't make sense to attempt to "draw a graph" but we can think of the derivative in terms of approximation. If f is complex differentiable at a then as $z \to a$ we have $\dfrac{f(z) - f(a)}{z - a} \to f'(a)$ and so $\dfrac{f(z) - f(a)}{z - a} = f'(a) + \rho(z)$, where $\rho(z) \to 0$, and we can write this as $f(z) - f(a) = (z - a)(f'(a) + \rho(z))$. In particular, f is continuous at a. We can use this to check the chain rule. Suppose g is complex differentiable at z_0 and f is complex differentiable at $w_0 = g(z_0)$. As $z \to z_0$ we have

$$\frac{g(z) - g(z_0)}{z - z_0} \to g'(z_0),$$

which we can write in the form

$$g(z) = g(z_0) + (z - z_0)(g'(z_0) + \rho(z))$$

where $\rho(z) \to 0$ as $z \to z_0$. Similarly

$$f(w) = f(w_0) + (w - w_0)(f'(w_0) + \tau(w))$$

where $\tau(w) \to 0$ as $w \to w_0$. Substitute $w = g(z)$, $w_0 = g(z_0)$. Then

$$f(g(z)) = f(g(z_0)) + (g(z) - g(z_0))(f'(g(z_0)) + \tau(g(z))) = (z - z_0)(g'(z_0) + \rho(z))(f'(g(z_0)) + \tau(g(z)))$$

and so

$$\frac{f(g(z)) - f(g(z_0))}{z - z_0} = (g'(z_0) + \rho(z))(f'(g(z_0)) + \tau(g(z))) \to g'(z_0) f'(g(z_0))$$

as $z \to z_0$, giving the rule $(f(g))' = f'(g)g'$ as expected.

2.3 Cauchy-Riemann equations, first encounter

Assume that the complex-valued function f is complex differentiable at $a = A + iB$, and as usual write

$$f(x + iy) = u(x,y) + iv(x,y) \qquad (1)$$

with A,B,x,y,u,v all real. Now, by assumption, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and the limit is the same regardless of how h approaches 0. So if we let h approach 0 through real values, putting $h = t$,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t} =$$

$$\lim_{t \to 0} (u(A+t,B) - u(A,B) + iv(A+t,B) - iv(A,B))/t = u_x(A,B) + iv_x(A,B).$$

(In particular, the partials u_x, v_x do exist.) Now put $h = it$ and again let $t \to 0$ through real values. We get

$$f'(a) = \lim_{t \to 0} (u(A,B+t) - u(A,B) + iv(A,B+t) - iv(A,B))/it = (1/i)(u_y(A,B) + iv_y(A,B)).$$

Equating real and imaginary parts we now see that, at (A,B), we have

$$u_x = v_y \quad , \quad u_y = -v_x .$$

These are called the Cauchy-Riemann equations. We also have (importantly) $f'(a) = u_x + iv_x$. These relations must hold if f is complex differentiable. Now we need a result in the other direction.

2.4 Cauchy-Riemann equations, second encounter

Theorem

Suppose that the functions f,u,v are as in (1) above, and that the following is true. The partial derivatives u_x, u_y, v_x, v_y all exist near (A,B), and are continuous at (A,B), and the Cauchy-Riemann equations are satisfied at (A,B). Then f is complex differentiable at $a = A + iB$, and $f'(a) = u_x + iv_x$.

Remark: the continuity of the partials won't usually be a problem in G12CAN: e.g. this is automatic if they are polynomials in x,y and (say) functions like $e^x, \cos y$. If there are denominators which are 0 at (A,B) some care is needed, though.

Proof of the theorem (optional) We can assume without loss of generality that $a = A = B = 0$, and that $f(a) = 0$ (if not look at $h(z) = f(z+a) - f(a)$: if $h'(0)$ exists then $f'(a)$ exists and is the same).

Suppose first that $u_x = v_y = 0$ and $u_y = -v_x = 0$ at $(0,0)$. We claim that $f'(0) = 0$. To prove this we have to show that $f(z)/z \to 0$ as $z \to 0$. Put $z = h + ik$, with h,k real. Look at

$$u(h,k) = u(h,k) - u(h,0) + u(h,0) - u(0,0).$$

Let $g(y) = u(h,y)$. Then $g'(y) = u_y(h,y)$ and the mean value theorem gives

$$u(h,k) - u(h,0) = g(k) - g(0) = kg'(c) = ku_y(h,c) = k\delta_1,$$

in which c lies between 0 and k and $\delta_1 \to 0$ as $h,k \to 0$ (because the partials are continuous at $(0,0)$). Now let $G(x) = u(x,0)$. Then the mean value theorem gives

$$u(h,0) - u(0,0) = G(h) - G(0) = hG'(d) = hu_x(d,0) = h\delta_2,$$

in which d lies between 0 and h and $\delta_2 \to 0$ as $h,k \to 0$. We get

$$\left| \frac{u(h,k)}{h+ik} \right| \leqslant \left| \frac{k\delta_1}{h+ik} \right| + \left| \frac{h\delta_2}{h+ik} \right| \leqslant |\delta_1| + |\delta_2| \to 0$$

as $h,k \to 0$. In the same way, $v(h,k)/(h+ik) \to 0$ as $h,k \to 0$ and we get $f(z)/z \to 0$ as required.

Now suppose that $u_x = v_y = \alpha, u_y = -v_x = \beta$ at $(0,0)$. Let $F(z) = f(z) - \alpha z + i\beta z = u - \alpha x - \beta y + i(v - \alpha y + \beta x) = U + iV$. Then $U_x = V_y = U_y = V_x = 0$ at $(0,0)$, and so $F'(0) = 0$, by the first part. Since $f(z) = F(z) + (\alpha - i\beta)z$ we get $f'(0) = \alpha - i\beta = u_x + iv_x$ as asserted.

Example
Where is $x^2 + iy^2$ complex differentiable?

2.5 Analytic Functions
We say that f is ANALYTIC at a point a (resp. analytic on a set X) if f is complex differentiable on an open set G which contains the point a (resp. the set X). Obviously, if f is comp. diffle on a domain D in $\mathbb{C}$ then f is analytic on D (take $G = D$). Other words for analytic are regular, holomorphic and uniform. A sufficient condition for analyticity at a is that the partial derivatives of u,v are continuous and satisfy the Cauchy-Riemann equations at all points near a.

Examples

1. The exponential function. We've already defined $e^{it} = \cos t + i \sin t$ for t real. We now define

$\exp(x+iy) = e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y$ for x,y real. We then have, using the standard decomposition,

$$u(x,y) = e^x \cos y \quad , \quad v(x,y) = e^x \sin y \quad ,$$

and

$$u_x = e^x \cos y \, , \, u_y = -e^x \sin y \, , \, v_x = e^x \sin y \, , \, v_y = e^x \cos y \, ,$$

and so the Cauchy-Riemann equations are satisfied. Obviously these partials are continuous. Thus $\exp(z)$ is complex differentiable at every point in $\mathbb{C}$, and so is analytic in $\mathbb{C}$, or ENTIRE. Further, the derivative of exp at z is $u_x + iv_x = \exp(z)$.

It is easy to check that $e^{z+w} = e^z e^w$ for all complex z,w. Also $|e^z| = e^{\mathrm{Re}(z)} \neq 0$, so $\exp(z)$ never takes the value zero. Since $e^0 = e^{2\pi i} = 1$ and $e^{\pi i} = -1$ this means that two famous theorems from real analysis are not true for functions of a complex variable!

2. sine and cosine. For $z \in \mathbb{C}$ we set

$$\sin z = (e^{iz} - e^{-iz})/2i \quad , \quad \cos z = (e^{iz} + e^{-iz})/2 \quad .$$

Exercise: put $z = x \in \mathbb{R}$ in these definitions and check that you just get $\sin x$, $\cos x$ on the RHS. With these definitions, the usual rules for derivatives tell us that sine and cosine are also entire, but it is important to note that they are not bounded in $\mathbb{C}$.

3. Some more elementary examples. What about $\exp(1/z)$? We've already observed that the chain rule holds for complex differentiability, as does the quotient rule. So this function is complex differentiable everywhere except at 0, and so analytic everywhere except at 0. Similarly $\sin(\exp(1/(z^4+1))$ is analytic everywhere except at the four roots of $z^4+1 = 0$.

4. At which points is

$$g(x+iy) = x^2 + 4y^2 + ixy \, , \qquad x,y \in \mathbb{R}$$

(i) complex differentiable (ii) analytic? We have

$$u = x^2 + 4y^2 \quad , \quad v = xy,$$

and so

$$u_x = 2x \quad , \quad u_y = 8y \quad , \quad v_x = y \quad , \quad v_y = x.$$

If g is complex differentiable at $x+iy$ then Cauchy-Riemann gives

$$2x = x \quad , \quad 8y = -y \, ,$$

and so $x = y = 0$. Thus g can only be complex differentiable at 0. Since the partial are continuous and the Cauchy-Riemann equations are satisfied at (0,0), our function g IS complex differentiable at 0. It is not, however, analytic anywhere.

5. Does there exist any function h analytic on a domain D in $\mathbb{C}$ such that $\mathrm{Re}(h)$ is x^2+4y^2 at each point $x+iy \in D$ (x,y real) ?

Suppose that $h = U+iV$ is such a function, with $U = x^2+4y^2$. Then we need

$$V_y = U_x = 2x \text{ and } V_x = -U_y = -8y.$$

The first relation tells us that the function $W = V-2xy$ is such that $W_y = 0$ throughout D. Let's fix some point $a = A+iB$ in D. Since $W_y = 0$, we see that near (A, B), the function $W(x,y)$ depends only on x. Thus we must have $W(x,y) = p(x)$, with p a function of x only, and so

$$V(x,y) = 2xy+p(x).$$

But this gives

$$-8y = V_x = 2y+p'(x),$$

which is plainly impossible. So no such function h can exist.

6. The logarithm. The aim is to find an analytic function $w = h(z)$ such that $\exp(h(z)) = z$. This is certainly NOT possible for $z = 0$, as $\exp(w)$ is never 0. Further,

$$\exp(h(z)) = e^{\mathrm{Re}(h(z))}e^{i\mathrm{Im}(h(z))} \quad \text{and} \quad z = |z|e^{i\arg z}.$$

So if such an h exists on some domain it follows that $\mathrm{Re}(h(z)) = \ln|z|$ and that $\mathrm{Im}(h(z))$ is an argument of z. Here we use $\ln x$ to denote the logarithm, base e , of a POSITIVE number x. A problem arises with this. If we start at $z = -1$, and fix some choice of the argument there, and if we then continue once clockwise around the origin, we find that on returning to -1 the argument has decreased by 2π and the value of the logarithm has changed by $-2\pi i$. Indeed, we've already seen that the argument of a complex number is discontinuous at the negative real axis. So to make our logarithm analytic we have to restrict the domain in which z can lie.

Let D_0 be the complex plane with the origin and the negative real axis both removed, and define, for z in D_0,

$$w = \mathrm{Log}\, z = \ln|z| + i\, \mathrm{Arg}\, z.$$

Remember that Arg will be taking values in $(-\pi,\pi)$. This choice for w gives

$$e^w = \exp(\,\mathrm{Log}\, z\,) = e^{\ln|z|}\exp(\,i\,\mathrm{Arg}\, z\,) = z$$

as required. Now for $z \in D_0$ we have $-\infty < \ln|z| < +\infty$ and $-\pi < \mathrm{Arg}\, z < \pi$ and so $w = \mathrm{Log}\, z$ maps D_0 one-one onto the strip $W = \{w \in \mathbb{C}: |\mathrm{Im}(w)| < \pi\}$. For $z, z_0 \in D$ and $w = \mathrm{Log}\, z$, $w_0 = \mathrm{Log}\, z_0$, we then have $z \to z_0$ if and only if $w \to w_0$. Hence

$$\lim_{z \to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} =$$

$$= \lim_{w \to w_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^w - e^{w_0}}.$$

The last limit is the reciprocal of the derivative of exp at w_0 and so is $1/\exp(w_0) = 1/z_0$. We conclude:

The PRINCIPAL LOGARITHM defined by $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ is analytic on the domain D_0 obtained by deleting from $\mathbb{C}$ the origin and the negative real axis, and its derivative is $1/z$. It satisfies $\exp(\operatorname{Log} z) = z$ for all $z \in D_0$.

Warning: It is not always true that $\operatorname{Log}(\exp(z)) = z$, nor that $\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w$. e.g. try $z = w = -1 + i$.

Powers of *z*

Suppose we want to define a complex square root $z^{1/2}$. A natural choice is

$$w = \sqrt{|z|}\, e^{\frac{1}{2} i \arg z},$$

because this gives $w^2 = |z| e^{i \arg z} = z$. If we do this, however, we encounter the same problem as with the logarithm. If we start at -1 and go once around the origin clockwise the argument decreases by 2π and the value we obtain on returning to -1 is the original value multiplied by $e^{-i\pi} = -1$. So we again have to restrict our domain of definition.

We first note that if n is a positive integer, then, on D_0,

$$\exp(n \operatorname{Log} z) = (\exp(\operatorname{Log} z))^n = z^n, \ \exp(-n \operatorname{Log} z) = (\exp(n \operatorname{Log} z))^{-1} = (\exp(\operatorname{Log} z))^{-n} = z^{-n}.$$

So, on D_0 , we can define, for each complex number α,

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

With this definition and properties of exp,

$$z^\alpha z^\beta = \exp(\alpha \operatorname{Log} z) \exp(\beta \operatorname{Log} z) = \exp((\alpha + \beta) \operatorname{Log} z) = z^{\alpha+\beta}.$$

However, it isn't always true that with this definition, $(z^\alpha)^\beta = z^{\alpha\beta}$. For example, take $z = i$, $\alpha = 3$, $\beta = 1/2$. Then $z^{\alpha\beta} = i^{3/2} = \exp((3/2) \operatorname{Log} i) = \exp((3/2) i\pi/2) = \exp(3\pi i/4)$. But $z^\alpha = i^3 = \exp(3 \operatorname{Log} i) = \exp(3\pi i/2)$, and this has principal logarithm equal to $-\pi i/2$, so that $(z^\alpha)^\beta = \exp((1/2)(-\pi i/2)) = \exp(-\pi i/4) \neq \exp(3\pi i/4)$.

To discover more about analytic functions and their derivatives it is necessary to integrate them.

Section 3 Integrals involving analytic functions

Theorem 3.1

Suppose that $\gamma:[a, b] \to D$ is a smooth contour in a domain $D \subseteq \mathbb{C}$, and that $F:D \to \mathbb{C}$ is analytic with continuous derivative f. Then $\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a))$ and so is 0 if γ is closed.

To prove this we just note that $H(s) = F(\gamma(s)) - \int_a^s f(\gamma(t))\, \gamma'(t)\, dt$ is such that $H'(s) = 0$ on (a, b) and so its real and imaginary parts are constant on $[a, b]$. So $H(b) = H(a) = F(\gamma(a))$.

If we do the same for a PSC γ, we find that the integral of f is the value of F at the finishing point of γ minus the value of F at the starting point of γ, and again if γ is closed we get 0.

Now we prove a very important theorem.

Theorem 3.2 (Cauchy-Goursat)

Let $D \subseteq \mathbb{C}$ be a domain and let T be a contour which describes once counter-clockwise the perimeter of a triangle whose perimeter and interior are contained in D. Let $f:D \to \mathbb{C}$ be analytic. Then $\int_T f(z)\, dz = 0$.

Proof

Let the length of T be L, and let $M = |\int_T f(z)\, dz|$. We bisect the sides of the triangle to form 4 new triangular contours, denoted Γ_j. In the subsequent proof, all integrals are understood to be taken in the positive (counter-clockwise) sense. Since the contributions from the interior sides cancel, we have

$$\int_T f(z)\, dz = \sum_{j=1}^{4} \int_{\Gamma_j} f(z)\, dz.$$

Therefore one of these triangles, T_1 say, must be such that $|\int_{T_1} f(z)\, dz| \geqslant M/4$. Now T_1 has perimeter length $L/2$. We repeat this procedure and get a sequence of triangles T_n such that:
(i) T_n has perimeter length $L/2^n$; (ii) T_{n+1} and its interior lie inside the union of T_n and its interior; (iii) $|\int_{T_n} f(z)\, dz| \geqslant M/4^n$.

Let V_n be the region consisting of T_n and its interior. Then we have $V_{n+1} \subseteq V_n$. It is not hard to see that there exists some point z^*, say, which lies in all of the V_n, and so on or inside EVERY T_n. (We could let z_n be the centre of V_n and note that z_n tends to a limit.) Since f is differentiable at z^* we can write (for $z \neq z^*$)

$$\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) = \eta(z), \quad \eta(z) \to 0 \ \ as \ \ z \to z^*.$$

Further,

$$\int_{T_n} f(z)\ dz = \int_{T_n} f(z^*) + (z-z^*)f'(z^*) + \eta(z)(z-z^*)\ dz = \int_{T_n} \eta(z)(z-z^*)\ dz.$$

This is using (3.1) and the fact that

$$f(z^*) = \frac{d}{dz}\Big((z-z^*)f(z^*)\Big)\ ,\quad (z-z^*)f'(z^*) = \frac{d}{dz}\Big(\tfrac{1}{2}(z-z^*)^2 f'(z^*)\Big)\ .$$

We therefore have $M/4^n \leqslant \left|\int_{T_n} f(z)dz\right| = \left|\int_{T_n} \eta(z)(z-z^*)\ dz\right| \leqslant$

$\leqslant$ (length of T_n) (sup of $|z-z^*|$ on T_n) (sup of $|\eta(z)|$ on T_n)
$\leqslant$ (length of T_n $)^2$ (sup of $|\eta(z)|$ on T_n) = $L^2 4^{-n}$ (sup of $|\eta(z)|$ on T_n).

But since $\eta(z) \to 0$ as $z \to z^*$ this now gives $ML^{-2} \leqslant$ (sup of $|\eta(z)|$ on T_n) $\to 0$ as $n \to \infty$. Therefore we must have $M = 0$.

Here's the proof that (z_n) converges. Since z_n and z_{n+1} both lie inside T_n, we have $|z_{n+1} - z_n| \leqslant$ length of T_n $= L/2^n$. Writing $z_n = x_n + iy_n$ (x_n, y_n real) we get

$$\sum_{k=1}^{\infty} |x_{k+1} - x_k| \leqslant \sum_{k=1}^{\infty} |z_{k+1} - z_k| \leqslant L \sum_{k=1}^{\infty} 2^{-k} < \infty.$$

Since every absolutely convergent real series converges (G1ALIM), the series $x_1 + \sum_{k=1}^{\infty}(x_{k+1} - x_k)$ converges, which means that $x_n = x_1 + \sum_{k=1}^{n-1}(x_{k+1} - x_k)$ tends to a finite limit x^*. The same works for y_n. Since all the z_k for $k \geqslant n$ lie in V_n, so must $z^* = x^* + iy^*$.

Example
Let T describe once counter-clockwise the triangle with vertices at 0, 1, i. We have already seen that $\int_T \bar{z}\, dz \neq 0$. Which of the following functions have integral around T equal to 0?

$(z-a)^{-2}$, $\exp(1/(z-a))$, $\exp(1/(z-10))$. Here $a = (1+i)/4$.

3.3 A special type of domain

A star domain D is a domain (in $\mathbb{C}$) which has a star centre, α say, with the following property. For every z in D the straight line segment from α to z is contained in D. Examples include an open disc, the interior of a rectangle or triangle, a half-plane. On star domains we can prove a more general theorem about contour integrals than Theorem 3.2.

Useful fact: if γ is a simple closed PSC in a star domain D, and w is a point inside γ, then w is in D. Why? Draw the straight line from the star centre α of D to w. Extend this line further. It must hit a point v on γ. Thus v is in D and so is w.

Theorem 3.4

Suppose that $f: D \to \mathbb{C}$ is continuous on the star domain $D \subseteq \mathbb{C}$ and is such that $\int_T f(z)\, dz = 0$ whenever T is a contour describing once counter-clockwise the boundary of a triangle which, together with its interior, is contained in D. Then the function F defined by $F(z) = \int_\alpha^z f(u) du$, in which the integration is along the straight line from α to z, is analytic on D and is such that $F'(z) = f(z)$ for all z in D.

Remark

An analytic function is continuous (see Section 2) and so Theorem 3.4 applies in particular when f is analytic in D.

Proof of 3.4

Let a be the star centre. We define $F(w) = \int_a^w f(z)\, dz$, where we integrate along the straight line from a to w. Let h be small, non-zero. Then the line segment from w to $w+h$ will lie in D, and so will the whole triangle T with vertices $a, w, w+h$, as well as its interior. Then $\int_T f(z)\, dz = 0$. This gives us

$$F(w+h) - F(w) = \int_w^{w+h} f(z)\, dz = \int_0^1 f(w+th)\, h\, dt$$

and so

$$\frac{F(w+h) - F(w)}{h} = \int_0^1 f(w+th)\, dt \to \int_0^1 f(w)\, dt = f(w)$$

as $h \to 0$. Thus $F'(w) = f(w)$. This leads at once to the following very important theorem.

Theorem 3.5 (stronger than 3.2)

Let $D \subseteq \mathbb{C}$ be a star domain and let $f: D \to \mathbb{C}$ be analytic. Let γ be any closed piecewise smooth contour in D. Then $\int_\gamma f(z)\, dz = 0$.

To see this, Theorem 3.4 gives us a function F such that $F'(z) = f(z)$ in D, and so the integral of f is just F evaluated at the final point of γ minus F evaluated at the initial point of γ. But these points are the same!

In fact, even more than this is true. We state without proof:

The general Cauchy theorem: let γ be a simple closed piecewise smooth contour, and let D be a domain containing γ and its interior. Let $f: D \to \mathbb{C}$ be analytic. Then $\int_\gamma f(z) dz = 0$.

For reasonably simple SCPSC γ, this can be seen by introducing cross-cuts and reducing to Theorem 3.5. For example, let f be analytic in $10 < |z| < 14$, and consider the SCPSC γ which describes once counter-clockwise the boundary of the region $11 < |z| < 12$, $|\arg z| < \pi/2$. We put in cross-cuts, each of which is a line segment $11 \leqslant |z| \leqslant 12$, $\arg z = c$. This gives us

$$\int_\gamma f(z)\,dz = \sum \int_{\gamma_j} f(z)\,dz$$

in which each γ_j is the boundary of a region $11 \leqslant |z| \leqslant 12, c_1 \leqslant \arg z \leqslant c_2$. We can do this so that each γ_j lies in a star domain on which f is analytic, and we deduce that $\int_\gamma f(z)\,dz = 0$.

The general case is, however, surprisingly difficult to prove, and is beyond the scope of G12CAN. We will, however, use the result, since the contours encountered in this module will be fairly simple geometrically.

Example 3.6

Suppose that f is analytic in the disc $|z| < S$. Suppose that $0 < s < S$, and that $w \in \mathbb{C}$, $|w| \neq s$. We compute

$$\frac{1}{2\pi i}\int_{|z| = s} \frac{f(z)}{z-w}\,dz,$$

in which the integral is taken once counter-clockwise.

First, if $|w| > s$ then $g(z) = f(z)/(z-w)$ is analytic on and inside $|z| = s$, so the integral is 0. Next, assume that $|w| < s$ and let δ be small and positive. Consider the domain D_1 given by

$$|z| < s,\ |z-w| > \delta.$$

Then $g(z) = f(z)/(z-w)$ is analytic on D_1. By cross-cuts, we see that the integral of $g(z)$ around the boundary of D_1, the direction of integration keeping D_1 to the left, is 0. Thus

$$\int_{|z| = s} g(z)\,dz = \int_{|z-w| = \delta} g(z)\,dz =$$

$$= \int_0^{2\pi} f(w+\delta e^{i\theta})\,i\,d\theta \to \int_0^{2\pi} f(w)\,i\,d\theta = 2\pi i f(w)$$

as $\delta \to 0$. Hence $\int_{|z| = s} f(z)/(z-w)\,dz = 2\pi i f(w)$ when $|w| < s$. This generalizes to:

3.7 Cauchy's integral formula

Suppose that f is analytic on a domain containing the simple closed piecewise smooth contour γ and its interior. Let $w \in \mathbb{C}$, with w not on γ. Then, integrating once counter-clockwise,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz$$

is $f(w)$ if w lies inside γ, and is 0 if w lies outside γ.

Note that we have to exclude the case where w lies on γ, as in this case the integral may fail to exist.

3.8 Liouville's theorem

Suppose that f is entire (= analytic in $\mathbb{C}$) and bounded as $|z|$ tends to ∞, i.e. there exist $M > 0$ and $R_0 > 0$ such that $|f(z)| \leqslant M$ for all z with $|z| \geqslant R_0$. Then f is constant.
The **Proof** is to take any u and v in $\mathbb{C}$. Take R very large. By Cauchy's integral formula, we have, integrating once counter-clockwise,

$$f(u)-f(v) = \frac{1}{2\pi i}\int_{|z| = R} \frac{f(z)(u-v)}{(z-u)(z-v)}\,dz.$$

If R is large enough then $|z-u| \geqslant R/2$ and $|z-v| \geqslant R/2$ for all z on $|z| = R$ and so the integral has modulus at most $(1/2\pi)(2\pi R)4MR^{-2}|u-v| \to 0$ as $R \to \infty$. Hence we must have $f(u) = f(v)$.

A corollary to this is the *fundamental theorem of algebra*: if $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial in z of positive degree n (i.e. $a_n \neq 0$) then there is at least one z in $\mathbb{C}$ with $P(z) = 0$. For otherwise $1/P$ is entire, and $1/P(z) \to 0$ as $|z| \to \infty$.

3.9 An application, and a physical interpretation of Cauchy's theorem

Analytic functions can be used to model fluid flow, as follows. Suppose that $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ is analytic in a star domain $D \subseteq \mathbb{C}$, and let γ be any simple closed piecewise smooth contour in D, parametrized with respect to arc length $s, 0 \leqslant s \leqslant L$. Then we have

$$0 = \int_\gamma f(z)dz = \int_\gamma (u+iv)(dx+idy) =$$

$$= \int_0^L \left(u\frac{dx}{ds}-v\frac{dy}{ds}\right)ds + i\int_0^L \left(v\frac{dx}{ds}+u\frac{dy}{ds}\right)ds.$$

So

$$\int_0^L \left(u\frac{dx}{ds}-v\frac{dy}{ds}\right)ds = 0 = \int_0^L \left(v\frac{dx}{ds}+u\frac{dy}{ds}\right)ds. \qquad (*)$$

Consider a fluid flow in D such that the velocity at the point (x,y) is given by $(u(x,y),-v(x,y))$. Now the vector $(x'(s),y'(s))$ is a unit vector tangent to the curve γ, and $udx/ds-vdy/ds$ is the component of velocity in the direction tangent to the curve. The first equation of (*) says that the average of this component, i.e. the *circulation* of the flow around the curve, is zero. Similarly, the vector $(y'(s), -x'(s))$ is normal to the curve γ, and $u dy/ds + v dx/ds$ may be interpreted as the component of velocity across the curve γ. The second equation in (*) says that the average flow of fluid across the curve γ is zero, i.e. there is no net *flux*. Both these conclusions are compatible with $(u, -v)$ representing an irrotational flow of incompressible fluid, with the velocity depending only on position and not on time.

Using the assumption that D is a star domain, take $F = P + iQ$ such that $F' = f = u + iv$. Then $u = P_x = Q_y$, $v = Q_x = -P_y$. On the path $(x(t), y(t))$ taken by a particle of fluid (t now time), $(x', y') = (u, -v)$ and $(u, -v).(Q_x, Q_y) = uv - uv = 0$. So Q is constant (streamline).

Note that the velocity vector is $(u, -v) = (P_x, P_y)$ and so is the gradient vector of P.

Example: in the quadrant $0 < \arg z < \pi/2$ take $u = x, v = y$. Then $F = z^2/2$ and $Q = xy$. Streamlines are arcs of hyperbolas.

Suppose that we have an incompressible irrotational fluid flow in the whole complex plane. The velocity vector at (x, y) is given by $(u, -v)$, where $u + iv$ is analytic in $\mathbb{C}$. Suppose now that the speed is bounded i.e. there exists $M > 0$ such that $|u + iv| \leqslant M$ throughout the plane. Then $u + iv$ is a bounded entire function and so by Liouville's theorem $u + iv$ is constant. Thus the velocity vector is constant, and we have a uniform flow across the plane.

Section 4 : Series and Analytic Functions

Example: for $r > 0$ let $I(r) = \int_{|z| = r} \sin 1/z \, dz$. Using cross-cuts, show that $I(r)$ is constant for $0 < r < \infty$.

By using series expansions, the methods to be justified in this chapter, we can calculate $I(r)$ and many other integrals directly.

We will use series to do two things. The first will be to construct new analytic functions. We will show that convergent power series, such as those that arise in applied mathematics and the solution of differential equations, are analytic. In the opposite direction, we will also show that analytic functions can be represented by series, and we will use these to compute the integral around a closed PSC, in cases more general than those we have met so far.

4.1 Complex Series

Let a_p, a_{p+1}, a_{p+2}, ... be a sequence (i.e. a non-terminating list) of complex numbers. For $n \geqslant p$, define

$$s_n = \sum_{k=p}^{n} a_k,$$

the *sequence of partial sums*. If the sequence s_n converges (i.e. tends to a finite limit as $n \to \infty$, with $S = \lim_{n \to \infty} s_n$, then we say that the series $\sum_{k=p}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=p}^{n} a_k$ converges, with sum S.

Example

Let $|t| < 1$. Then $s_n = \sum_{k=0}^{n} t^k = \frac{1-t^{n+1}}{1-t} \to \frac{1}{1-t}$ as $n \to \infty$. So $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$ if $|t| < 1$.

Fact 1

If $\sum_{k=p}^{\infty} a_k$ converges, with sum S, then $s_n \to S$ as $n \to \infty$ and so does s_{n-1}. Thus $a_n = s_n - s_{n-1} \to 0$ as $n \to \infty$. The converse is false: we have

$$\sum_{k=1}^{\infty} 1/k = 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + ... > 1 + 1/1 + 1/2 + 1/2 + ...$$

and the series diverges, because we can make the partial sums as large as we like.

Fact 2

Suppose that $\sum_{k=p}^{\infty} a_k$ and $\sum_{k=p}^{\infty} b_k$ both converge, and that α, β are complex numbers. Then $\sum_{k=p}^{\infty} (\alpha a_k + \beta b_k)$ converges, and equals $\alpha(\sum_{k=p}^{\infty} a_k) + \beta(\sum_{k=p}^{\infty} b_k)$.

So for example $\sum_{k=0}^{\infty} 2^{-k} + i3^{-k}$ converges, but $\sum_{k=1}^{\infty} 2^{-k} + i/k$ diverges.

Fact 3

Suppose that a_k is real and non-negative. Then $s_n = \sum_{k=p}^{n} a_k$ is a non-decreasing real sequence, and converges iff it is bounded above.

For example if $p > 1$ the series $\sum_{k=1}^{\infty} 1/k^p$ converges. This is proved in G1ALIM, but we can also note that every partial sum is

$$\leqslant 1 + (1/2^p + 1/3^p) + (1/4^p + 1/5^p + 1/6^p + 1/7^p) + ... <$$

$$< 1 + 2/2^p + 4/4^p + 8/8^p + ... = 1^{1-p} + 2^{1-p} + 4^{1-p} + \ldots = \sum_{k=0}^{\infty} (2^{1-p})^k = 1/(1 - 2^{1-p}).$$

Comparison test: if $0 \leqslant a_k \leqslant b_k$ and $\sum_{k=p}^{\infty} b_k$ converges then $\sum_{k=p}^{\infty} a_k$ converges (G1ALIM).

Fact 4

Suppose that $\sum_{k=p}^{\infty} |a_k|$ converges (in which case we say that $\sum_{k=p}^{\infty} a_k$ is absolutely convergent). Then $\sum_{k=p}^{\infty} a_k$ converges.

Proof

Write $a_k = b_k + ic_k$, with b_k, c_k real. Write $B_k = (|b_k| + b_k)/2$ and $C_k = (|b_k| - b_k)/2$. Then $0 \leqslant B_k \leqslant |b_k| \leqslant |a_k|$, and $0 \leqslant C_k \leqslant |b_k| \leqslant |a_k|$. So $\sum B_k, \sum C_k$ converge, and so does $\sum b_k = \sum (B_k - C_k)$. Similarly $\sum c_k$ converges, and $\sum a_k = \sum b_k + i \sum c_k$.

We will also need (from G1ALIM)

Fact 5 (The Ratio Test)

Suppose that a_k is real and positive for $k \in \mathbb{N}$ and that $L = \lim_{n \to \infty} a_{n+1}/a_n$ exists.

(i) If $L > 1$ then a_n does not tend to 0 (since $a_{n+1} > a_n$ for large n), and so $\sum_{k=1}^{\infty} a_n$ diverges.

(ii) If $0 \leqslant L < 1$ then $\sum_{k=1}^{\infty} a_n$ converges.

There is no conclusion if $L = 1$.

Example: if $0 < t < 1$ then $\sum_{k=1}^{\infty} kt^{k-1}$ converges.

4.2 Power series

Consider the *power series*

$$F(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k = c_0 + c_1(z-\alpha) + c_2(z-\alpha)^2 + \ldots\ldots,$$

in which the centre α and the coefficients c_k are complex numbers. Obviously $F(\alpha) = c_0$.

To investigate convergence for $z \neq \alpha$ we let T_F be the set of non-negative real t having the property that $|c_k| t^k \to 0$ as $k \to +\infty$. Then $0 \in T_F$.

The radius of convergence R_F is defined as follows. If T_F is bounded above (this means that T_F has an upper bound P, a real number P with $x \leqslant P$ for all x in T_F), we let R_F be its l.u.b., i.e. the least real number which is an upper bound for T_F (see G1ALIM notes, e.g. on *www.maths.nottingham.ac.uk/personal/jkl*). If T_F is not bounded above, then we set $R_F = \infty$.

In either case, the following is true. If $0 < r < R_F$ then r is not an upper bound for T_F and so there exists $s \in T_F$ with $s > r$.

Note that if $|z - \alpha| > R_F$ then $c_k(z-\alpha)^k$ cannot tend to 0 as $k \to \infty$ (since its modulus does not), and so $F(z)$ diverges.

We assume for the rest of this section that F has positive radius of convergence R_F. Set $D = B(\alpha, R_F)$ (if $R_F = \infty$ then $D = \mathbb{C}$). Then

(i) F converges absolutely (i.e. $\sum_{k=0}^{\infty} |c_k(z-\alpha)^k|$ converges) for z in D, and so does $f(z) = \sum_{k=1}^{\infty} kc_k(z-\alpha)^{k-1}$.

(ii) F is continuous on D.

(iii) if γ is a PSC in D and $\phi(z)$ is continuous on γ we have

$$\int_\gamma F(z)\phi(z)\,dz = \sum_{k=0}^{\infty}\int_\gamma c_k(z-\alpha)^k\phi(z)\,dz = \sum_{k=0}^{\infty} c_k\int_\gamma (z-\alpha)^k\phi(z)\,dz,$$

i.e. we can integrate term by term.

(iv) F is analytic on D, with derivative $f(z) = \sum_{k=1}^{\infty} kc_k(z-\alpha)^{k-1}$.

(v) If $F(u)$ converges, then $F(z)$ converges for every z with $|z-\alpha| < |u-\alpha|$.

Proof

We can assume WLOG that $\alpha = 0$ (if not we just put $G(z) = F(z+\alpha) = \sum_{k=0}^{\infty} c_k z^k$). Let $0 < r < R_F$ and let $E_r = \{z \in \mathbb{C} : |z| \leqslant r\}$. Since $r < R_F$ there exists a real number s with $r < s$ and $s \in T_F$, which means that $|c_k|s^k \to 0$ as $k \to \infty$. So there is some real $M > 0$ such that $|c_k|s^k \leqslant M$ for all integers $k \geqslant 0$. Hence $|c_k|r^k \leqslant M(r/s)^k$ for all integers $k \geqslant 0$. Therefore for $|z| \leqslant r$, and integer $N \geqslant 0$,

$$\left|\sum_{k=N}^{\infty} c_k z^k\right| \leqslant \sum_{k=N}^{\infty}|c_k z^k| \leqslant \sum_{k=N}^{\infty}|c_k|r^k \leqslant \sum_{k=N}^{\infty} M(r/s)^k = M(r/s)^N(1-r/s)^{-1}. \tag{3}$$

Taking $N = 0$ this proves that $F(z)$ converges absolutely for $|z| \leqslant r$, and hence for $|z| < R_F$ since r was arbitrary. Similarly,

$$\sum_{k=1}^{\infty}|kc_k z^{k-1}| \leqslant \sum_{k=1}^{\infty} k|c_k|r^{k-1} \leqslant \sum_{k=1}^{\infty} kMs^{-k}r^{k-1} = Ms^{-1}\sum_{k=1}^{\infty} k(r/s)^{k-1} < \infty,$$

using the last example of 4.1.

For the rest of the proof, retain M, r, s as in (3).

Now we prove (ii), that F is continuous on D. Take w in D and r with $|w| < r < R_F$. Let z be close to w, in particular so close that we also have $|z| \leqslant r$. Then

$$|F(z)-F(w)| = \left|\sum_{k=1}^{\infty} c_k(z^k - w^k)\right| \leqslant \sum_{k=1}^{\infty}|c_k(z^k - w^k)|.$$

But, for $k \in \mathbb{N}$,

$$z^k - w^k = \int_w^z ku^{k-1}\,du,$$

in which the integral is along the straight line L from w to z, which lies in $|u| \leqslant r$ and has length $|z-w|$. Applying the Fundamental Estimate, we get, since $|ku^{k-1}| \leqslant kr^{k-1}$ on L,

$$|z^k - w^k| \leqslant kr^{k-1}|z-w|.$$

Thus

$$|F(z)-F(w)| \leqslant \sum_{k=1}^{\infty} k|c_k|r^{k-1}|z-w| \leqslant \sum_{k=1}^{\infty} kMs^{-k}r^{k-1}|z-w| = (M/s)|z-w|\sum_{k=1}^{\infty} k(r/s)^{k-1}$$

which tends to 0 as $z \to w$, using again the fact that $\sum_{k=1}^{\infty} k(r/s)^{k-1}$ converges (see 4.1).

Now we prove (iii) i.e. that we can integrate term by term. To prove this, note that γ will lie in some E_r, with $0 < r < R_F$. But then, with L the length of γ and T the maximum of $|\phi(z)|$ on γ we get, using (3),

$$\left|\int_\gamma \phi(z)F(z)\,dz - \sum_{k=0}^{N}\int_\gamma \phi(z)c_k z^k dz\right| = \left|\int_\gamma \phi(z)F(z)\,dz - \int_\gamma \phi(z)\sum_{k=0}^{N} c_k z^k dz\right| =$$

$$= \left|\int_\gamma \phi(z)\sum_{k=N+1}^{\infty} c_k z^k dz\right| \leqslant L\,T\,M(r/s)^{N+1}(1-r/s)^{-1},$$

which tends to 0 as $N \to \infty$. So

$$\int_\gamma \phi(z)F(z)\,dz = \lim_{N\to\infty}\sum_{k=0}^{N}\int_\gamma \phi(z)c_k z^k dz = \sum_{k=0}^{\infty}\int_\gamma \phi(z)c_k z^k dz.$$

(iv) We now prove that F is analytic on D, with derivative

$$f(z) = \sum_{k=1}^{\infty} kc_k z^{k-1}.$$

To do this, let T be any triangular contour in D, described once counter-clockwise. Then, by (iii),

$$\int_T f(z)\,dz = \sum_{k=1}^{\infty}\int_T kc_k z^{k-1}dz = 0,$$

by Cauchy's theorem, since z^{k-1} is entire for $k \in \mathbb{N}$. Let

$$G(z) = \int_0^z f(u)\,du,$$

in which the integral is along the straight line from 0 to z. Then by Theorem 3.4 $G(z)$ is analytic on D with derivative f. But by (iii) we have

$$G(z) = \sum_{k=1}^{\infty}\int_0^z kc_k u^{k-1}du = \sum_{k=1}^{\infty} c_k z^k = F(z)-F(0).$$

Thus $F'(z) = f(z)$ on D.

(v) is obvious, since we must have $R_F \geqslant |u-\alpha|$.

We have thus shown that if $F(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ has radius of convergence $R_F > 0$ then

$$F'(z) = f(z) = \sum_{k=1}^{\infty} kc_k(z-\alpha)^{k-1}$$

for $|z-\alpha| < R_F$. In particular, $F(\alpha) = c_0, F'(\alpha) = c_1$. But the new series f converges for $|z-\alpha| < R_F$, and so $R_f \geqslant R_F$. This means that we can repeat the argument, and so differentiate F as many times as we like on $|z-\alpha| < R_F$, and we get $c_k = F^{(k)}(\alpha)/k!$. We have proved:

Theorem 4.3 The main theorem on power series

Suppose that $F(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ has positive radius of convergence R_F. Set $D = B(\alpha, R_F)$ (if $R_F = \infty$ then $D = \mathbb{C}$). Then

(i) F converges absolutely for z in D, and $F:D \to \mathbb{C}$ is a continuous function;

(ii) if γ is a PSC in D and $\phi(z)$ is continuous on γ we have

$$\int_\gamma F(z)\phi(z)dz = \sum_{k=0}^{\infty}\int_\gamma c_k(z-\alpha)^k\phi(z)dz,$$

i.e. we can integrate term by term.

(iii) F is analytic on D and can be differentiated as many times as we like on D. Also, $c_k = F^{(k)}(\alpha)/k!$ and, in D,

$$F'(z) = \sum_{k=1}^{\infty} kc_k(z-\alpha)^{k-1} = \sum_{k=0}^{\infty}(k+1)c_{k+1}(z-\alpha)^k.$$

Further, all derivatives $F^{(k)}$ exist on D, and $F^{(k)}(\alpha) = k!c_k$.

4.4 Series in negative powers

We need a similar result for series of form $G(z) = \sum_{k=1}^{\infty} c_k(z-a)^{-k}$. We can regard this as a power series in $1/(z-a)$, and the following facts can be proved by setting $u = 1/(z-a)$ and $F(u) = \sum_{k=1}^{\infty} c_k u^k$. Obviously, $G(z) = F(1/(z-a))$.

Case 1: suppose that $R_F = 0$.
Then $F(u)$ converges only for $u = 0$, and so $G(z)$ diverges for every z in $\mathbb{C}$.

Case 2: suppose that $R_F > 0$.
Then $F(u)$ converges absolutely for $|u| < R_F$ and diverges for $|u| > R_F$. Thus $G(z)$ converges absolutely for $|z-a| > S_G = 1/R_F$, and diverges for $|z-a| < S_G$. Also, G is analytic, and can be differentiated term by term with

$$G'(z) = -(z-a)^{-2}F'(u) = -(z-a)^{-2}\sum_{k=1}^{\infty} kc_k u^{k-1} = \sum_{k=1}^{\infty} -kc_k(z-a)^{-k-1}$$

on the domain $D = \{z \in \mathbb{C}: S_G < |z-a| < \infty\}$.

Next, the same argument as in 4.3(iii) shows that if γ is a PSC in D and $\phi(z)$ is continuous on γ, then

$$\int_\gamma \phi(z)G(z)dz = \sum_{k=1}^{\infty}\int_\gamma \phi(z)c_k(z-a)^{-k}dz.$$

Finally, if $G(v)$ converges then $F(1/(v-a))$ converges, and so $F(u)$ converges for $|u| < |1/(v-a)|$ so $G(z)$ converges for $|z-a| > |v-a|$.

Example

Let w be a complex number. Then $\sum_{k=0}^{\infty}(w/z)^k = 1+(w/z)+(w/z)^2+.... = 1/(1-w/z)$ for $|z| > |w|$.

4.5 Laurent's theorem

Let $0 \leqslant R < S \leqslant \infty$, and let f be analytic in the annulus A given by $R < |z-a| < S$. Then there are constants $a_k, k \in \mathbb{Z}$, such that for all z in A we have

$$f(z) = \sum_{k\in\mathbb{Z}} a_k(z-a)^k = \sum_{k=0}^{\infty} a_k(z-a)^k + \sum_{j=1}^{\infty} a_{-j}(z-a)^{-j}. \qquad (1)$$

The series (i.e. both series) converge absolutely for z in A, and integrals $\int_\gamma \phi(z)f(z)dz$ can be computed by integrating term by term, for any PSC contour γ in A, i.e.

$$\int_\gamma \phi(z)f(z)dz = \sum_{k\in\mathbb{Z}} a_k \int_\gamma \phi(z)(z-a)^k dz$$

provided $\phi(z)$ is continuous on γ. In particular, if $R < T < S$ then integrating once counter-clockwise gives

$$a_k = \frac{1}{2\pi i}\int_{|z-a| = T} f(z)(z-a)^{-k-1}dz, \qquad (2)$$

so that there is just one Laurent series (1) representing $f(z)$ in A. Finally, the series (1) can be differentiated term by term in A.

Proof

We may assume that $a = 0$, for otherwise we just look at $g(z) = f(z+a)$. Fix T_1 with $R < T_1 < S$, and assume that $R < r < T_1 < s < S$.

Claim I:

There are constants a_k such that $f(z)$ is given by a series (1) in $r < |z| < s$.

Using cross-cuts we see that Cauchy's integral formula (Theorem 3.7) gives, with all integrals once counter-clockwise,

$$f(w) = \frac{1}{2\pi i}\int_{|z| = s} \frac{f(z)}{z-w}\, dz - \frac{1}{2\pi i}\int_{|z| = r} \frac{f(z)}{z-w}\, dz.$$

Now set $H(z) = \dfrac{1}{1-z/w}$. Then

$$H(z) = \frac{1}{1-z/w} = \sum_{k=0}^{\infty}(z/w)^k$$

for $|z| < |w|$ and in particular on $|z| = r$, and the series for $H(z)$ can be integrated term by term on $|z| = r$. Thus

$$-\frac{1}{2\pi i}\int_{|z| = r}\frac{f(z)}{z-w}\,dz = \frac{1}{2\pi i}\int_{|z| = r}\frac{f(z)H(z)}{w}\,dz =$$

$$= \frac{1}{2\pi i}\int_{|z| = r}\frac{f(z)}{w}\sum_{k=0}^{\infty}(z/w)^k\,dz = \sum_{k=0}^{\infty}w^{-k-1}\frac{1}{2\pi i}\int_{|z| = r}f(z)z^k\,dz = \sum_{k=0}^{\infty}d_k w^{-k-1},$$

which is a sum in negative powers of w, in which d_k is independent of w, for $r < |w| < s$.

Also on $|z| = s$ we have $|w/z| < 1$ and so

$$\frac{1}{2\pi i}\int_{|z| = s}\frac{f(z)}{z-w}dz = \frac{1}{2\pi i}\int_{|z| = s}\frac{f(z)}{z(1-w/z)}dz =$$

$$= \frac{1}{2\pi i}\int_{|z| = s}\frac{f(z)}{z}\sum_{k=0}^{\infty}(w/z)^k dz = \sum_{k=0}^{\infty}c_k w^k,$$

with

$$c_k = \frac{1}{2\pi i}\int_{|z| = s}\frac{f(z)}{z^{k+1}}dz$$

again independent of w, for w with $r < |w| < s$.

We have thus proved Claim I, and the rest follows easily. In the region $r < |z| < s$ we have $f(z)$ given in A by a convergent series (1), which we can write as

$$f(z) = F(z) + G(1/z). \tag{3}$$

Here $F(z) = \sum_{k=0}^{\infty}a_k z^k$ is a power series which must have radius of convergence at least s, while $G(w) = \sum_{j=1}^{\infty}a_{-j}w^j$ is a power series with radius of convergence at least $1/r$. Integrating term by term now gives (2) for $r < T < s$, and taking $T = T_1$ we see that the coefficients do not depend on the particular choice of r, s, as long as $R < r < T_1 < s < S$. Since r, s are arbitrary, we have (1) for all z in A. Finally, term by term differentiability follows from (3) and the properties of power series.

Example: find the Laurent series of $f(z) = 1/z(z-i)^2$ in (i) $0 < |z| < 1$ (ii) $1 < |z| < \infty$ (iii) $1 < |z-i| < \infty$.

Find the Laurent series of $1/(z+1)(z+2)$ in $1 < |z| < 2$.

Using the geometric series $1/(1-u) = 1+u+u^2+........$ and its differentiated versions, we can get Laurent series for rational functions fairly straightforwardly. The next problem is to obtain series for functions like e^z. We begin by considering what happens when f is in fact analytic in $|z-a| < S$.

Theorem 4.6 (Taylor's theorem)
Suppose that f is analytic in $|z-a| < S \leqslant \infty$. Then f can be differentiated as many times as we like on $|z-a| < S$ and for $|z-a| < S$ we have (Taylor series)

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k.$$

In particular, all these derivatives $f^{(k)}(a)$ exist.

Proof of Taylor's theorem
We derive Taylor's theorem from Laurent's theorem, with $R = 0$. We get a series (1) valid for $0 < |z-a| < S$, and the coefficients a_k are give by (2). But, if $k \in \mathbb{Z}, k < 0$, it follows that $-k-1 \geqslant 0$ and so $f(z)(z-a)^{-k-1}$ is analytic in $|z-a| < S$. Hence $a_k = 0$ for $k < 0$ by Cauchy's theorem. This gives $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$ for $|z-a| < S$, and this is a power series with radius of convergence at least S. In particular, we see from 4.3 that $f^{(k)}(a)/k!$ exists and equals a_k.

4.7 Remarks and Examples
1. It follows from Taylor's theorem that if $f{:}D \to \mathbb{C}$ is analytic on the domain $D \subseteq \mathbb{C}$ then so is f'. To see this, take any w in D and just note that Taylor's theorem shows that $f^{(k)}(w)$ exists for each non-negative integer k.

2. There are functions $g{:}\mathbb{R} \to \mathbb{R}$ for which $d^k g/dx^k$ exists for every k, but which do not always equal their Taylor series. For example, let $g(0) = 0$, with $g(x) = \exp(-1/x^2)$ for $x \neq 0$. Then the *real* derivative $g^{(k)}(0)$ exists, and can be shown to be 0, for every k, and so the Taylor series about 0 is 0, whereas $g(x)$ is non-zero for real $x \neq 0$. Note that $g(z)$ blows up as $z \to 0$ with z imaginary, so that this g is certainly not analytic at 0.

3. Taylor's theorem tells us that, for all z,

$$e^z = 1+z+z^2/2!+...., \sin z = z-z^3/3!+z^5/5!-.....$$

Also, if $F(z) = \sum_{k=0}^{\infty} c_{(z} - \alpha)^k$ has $R_F > 0$ then F is its own Taylor series about α. In particular, the standard series

$$1/(1-z) = 1+z+z^2+....., \quad 1/(1-z)^2 = 1+2z+3z^2+.....$$

are valid for $|z| < 1$ and very useful.

4. The binomial theorem. Suppose that b is a complex number, and consider $(1+z)^b$ for $|z| < 1$.

It is not immediately clear how to *define* this. However, for $|z| < 1$, the number $1+z$ will lie in the domain of definition of the principal logarithm Log, and so we define

$$(1+z)^b = h(z) = \exp(\ b\ \mathrm{Log}\,(1+z)\ ),\ |z| < 1.$$

This function h is then analytic in $|z| < 1$ by the chain rule. We also have

$$h'(z) = h(z)b/z = h(z)b\exp(-\mathrm{Log}\,(1+z)) = b(1+z)^{b-1}.$$

Thus $h'(0) = b$, $h''(0) = b(b-1)$, and $h^{(k)}(0) = b(b-1)\ldots(b-k+1)$ for every positive integer k. Thus Taylor's theorem gives

$$(1+z)^b = 1+bz+z^2b(b-1)/2+z^3b(b-1)(b-2)/3!+.....,$$

which is the binomial theorem. If b is a positive integer, the series terminates and the expansion is valid for all z.

5. The Cauchy product. Suppose that $F(z),G(z)$ are both analytic in $|z-a| < S$, with Taylor series

$$F(z) = a_0+a_1(z-a)+.....\ ,\ G(z) = b_0+b_1(z-a)+.....\ ,$$

there. Then $H(z) = F(z)G(z)$ is analytic in the same disc. If we multiply the Taylor series of F by that of G we get

$$(a_0+a_1(z-a)+.....)(b_0+b_1(z-a)+.....) =$$

$$= a_0b_0+(a_1b_0+a_0b_1)(z-a)+\ldots+(a_kb_0+\ldots+a_0b_k)(z-a)^k+\ldots$$

Is this the Taylor series of H? We know that in $|z-a| < S$ we have

$$H(z) = \sum_{k=0}^{\infty}(z-a)^kH^{(k)}(a)/k!$$

and, by Leibnitz' rule,

$$H^{(k)}(a) = \sum_{j=0}^{k}F^{(j)}(a)G^{(k-j)}(a)k!/j!(k-j)! = k!\sum_{j=0}^{k}a_jb_{k-j}.$$

So the answer to our question is yes.

6. Find the Taylor series of $(\cos z)/(1-z^2)$ in $|z| < 1$.

7. Evaluate $\int\exp(z^2)z^{-17}dz$ with the integral once counter-clockwise around $|z| = 1$.

8. Function of a function. Suppose that $f(u)$ is analytic in $|u-b| < r$ and that $g(z)$ is analytic in $|z-a| < s$, with $g(a) = b$. Then if z is close enough to a we have $|g(z)-b| < r$, and so $f(g(z)) = h(z)$ is analytic in $|z-a| < t$, for some $t > 0$. Suppose that

$$f(u) = a_0+a_1(u-b)+....,\ \ |u-b| < r,$$

and

$$g(z) = b + b_1(z-a) + ..., \quad |z-a| < s.$$

If we set $u = g(z)$ and substitute the series for g into that for $f(u)$, we get

$$a_0 + a_1(b_1(z-a) + b_2(z-a)^2 + ...) + a_2(b_1(z-a) + b_2(z-a)^2 + \ldots)^2 +$$

$$+ a_3(b_1(z-a) + b_2(z-a)^2 + \ldots)^3 + ... =$$

$$= a_0 + a_1 b_1(z-a) + (z-a)^2(a_1 b_2 + a_2 b_1^2) + (z-a)^3(a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) + ...$$

when we gather up powers of $z-a$. Is this the Taylor series of h? Again, yes. For $|z-a| < t$ we have

$$h(z) = \sum_{k=0}^{\infty} (z-a)^k h^{(k)}(a)/k!$$

and the chain rule gives

$$h'(a) = f'(b)g'(a) = a_1 b_1, \;\; h''(a)/2! = (f''(b)g'(a)^2 + f'(b)g''(a))/2! = a_2 b_1^2 + a_1 b_2.$$

A theorem on the Taylor series of a composition (optional): Suppose that g is analytic at a and f is analytic at $b = g(a)$. Then the composition h defined by $h(z) = f(g(z))$ is analytic at a, by the chain rule, and the Taylor series of h about a is obtained by substituting the Taylor series of g about a into the Taylor series of f about b and gathering up powers of $z-a$.

Warning: this only works if $g(a) = b$.

Proof of the theorem (optional!)
First we need the following.

Claim: suppose that G is analytic at a and F is analytic at $b = G(a)$ and H is the composition $H(z) = F(G(z))$. If $m \in \mathbb{N}$ then

$$H^{(m)}(z) = \sum_{k=1}^{m} A_k(z) F^{(k)}(G(z)), \tag{1}$$

in which the A_k are analytic at a (the A_k depend on m and k).

(1) is obviously true for $m = 1$, with $A_1 = G'$, because $(F(G))' = F'(G)G'$. Now suppose that (1) is true for m. Then

$$H^{(m+1)}(z) = \sum_{k=1}^{m} (A_k'(z)F^{(k)}(G(z)) + A_k(z)F^{(k+1)}(G(z))G'(z))$$

which we can write in the form $\sum_{k=1}^{m+1} B_k(z) F^{(k)}(G(z))$, with the B_k analytic at a. Thus the Claim is proved by induction on m.

Completion of the proof of the composition rule for Taylor series: there is no loss of generality in assuming that $b = 0$. For otherwise we can put $f_1(w) = f(w+b)$ and $g_1(z) = g(z)-b$. Then $f_1(g_1(z)) = f(g(z))$ and $f^{(k)}(b) = f_1^{(k)}(0)$ and so substituting the Taylor series of g about a into the series of f about b gives the same result as substituting the series of g_1 about a into the series of f_1 about 0.

Let $c_k = \dfrac{g^{(k)}(a)}{k!}$ and let $d_k = \dfrac{f^{(k)}(0)}{k!}$. Let n be a positive integer. Near a we can write

$$g(z) = P(z)+r(z), \quad P(z) = \sum_{k=1}^{n} c_k(z-a)^k, \quad r(z) = \sum_{k=n+1}^{\infty} c_k(z-a)^k,$$

noting that $c_0 = g(a) = 0$. Near $b = 0 = g(a)$ we can write

$$f(w) = Q(w)+s(w), \quad Q(w) = \sum_{k=0}^{n} d_k w^k, \quad s(w) = \sum_{k=n+1}^{\infty} d_k w^k.$$

Here P and Q are polynomials and r is analytic near a, while s is analytic near 0.

For z close to a we know that $g(z)$ is close to 0. If we substitute $w = g(z)$ then for z close to a we have

$$h(z) = f(g(z)) = \left(\sum_{k=0}^{n} d_k(P(z)+r(z))^k\right)+s(g(z)) = R(z)+s(g(z)).$$

Now $s(0) = s'(0) = ... = s^{(n)}(0)$, because s is a power series with first term $d_{n+1}w^{n+1}$. We calculate the p'th derivative of $s(g)$ at a, where $1 \leqslant p \leqslant n$, using the Claim above, with $F = s$ and $G = g$ and $m = p$. Putting $z = a$ we now see that $(s(g))^{(p)}(a) = 0$ for $1 \leqslant p \leqslant n$, and this is also true for $p = 0$, since $s(0) = 0$. This means that $h^{(p)}(a) = R^{(p)}(a)$ for $0 \leqslant p \leqslant n$.

But we can expand out and write

$$R(z) = \left(\sum_{k=0}^{n} d_k P(z)^k\right)+r(z)S(z) = T(z)+r(z)S(z),$$

in which $S(z)$ is analytic at a. Since $r^{(p)}(a) = 0$ for $0 \leqslant p \leqslant n$ (since $r(z)$ is a power series with first term $c_{n+1}(z-a)^{n+1}$) we see that $(rS)^{(p)}(a) = 0$ and $R^{(p)}(a) = T^{(p)}(a)$ for $0 \leqslant p \leqslant n$.

To summarize: for $0 \leqslant p \leqslant n$ it is the case that $h^{(p)}(a) = R^{(p)}(a)$, and this is the p'th derivative at a of $\sum_{k=0}^{n} d_k P(z)^k$, in which $P(z) = \sum_{k=1}^{n} c_k(z-a)^k$. This means that to calculate $h^{(p)}(a)$, for $0 \leqslant p \leqslant n$, we can substitute $w = \sum_{k=1}^{n} c_k(z-a)^k$ into $\sum_{k=0}^{n} d_k w^k$ and expand out, collecting up powers of $z-a$, and the coefficient of $(z-a)^p$ is $h^{(p)}(a)/p!$. But we get the same coefficient of $(z-a)^p$ if we substitute the Taylor series of g about a into that of f about 0, because the powers greater than n make no difference to the coefficient of $(z-a)^p$.

Having now proved this difficult but important theorem on the Taylor series of a composition we return to our consideration of examples.

9. Find the integral once counter-clockwise around $|z| = 1$ of $1/z^4(1-\sin z)$.

10. The same, for $z^{-4}(1+\cos z)^{-1}$.

11. The same, for $\exp(1/z)$.

12. Same again, for $z^{-8}\sin(z^3)$.

13. Find the integral once counter-clockwise around $|z| = 4$ of $z^{-6}(1-z)^{-1}$.

14. Find the Laurent series of $1/(z^2-4)$ in (a) $|z| < 1$ (b) $0 < |z-2| < 4$ (c) $4 < |z+2| < \infty$ (d) $10 < |z| < \infty$.

15. Calculate $\int_{|z| = 3} e^{1/z} z^4 \, dz$, the integral being once counter-clockwise.

Section 5 : Singularities and the residue calculus

5.1 Singularities

We say that the complex-valued function f has an isolated singularity at a if f is not defined at a but there is some $s > 0$ such that f is analytic in the punctured disc $\{ z \in \mathbb{C} : 0 < |z-a| < s \}$. The singularity can be classified according to how f behaves as z approaches a.

Examples

1. $f(z) = \dfrac{\cos z}{z}$. Clearly, 0 is a problem point for this function. As $z \to 0$, we easily see that $|f(z)| \to \infty$. We say that f has a pole at 0. A pole is an isolated singularity a with the property that $|f(z)| \to \infty$ as $z \to a$.

2. $g(z) = \dfrac{\sin z}{z}$. Here the behaviour is not so obvious. However, for $z \neq 0$ we can write

$$g(z) = z^{-1}(z-z^3/6+....) = 1-z^2/6+.....$$

The RHS is now a power series, converging for all $z \neq 0$, and so for all z. If we set $g(0) = 1$, then g becomes analytic at 0 as well, and we have removed the singularity.

A removable singularity a of a function h is an isolated singularity with the property that $\lim_{z \to a} h(z)$ exists and is finite.

3. Now try $H(z) = \frac{\sin z}{z^2}$. For $z \neq 0$ we have

$$H(z) = z^{-1} - z/6 + z^3/5! - .....$$

As $z \to 0$ the term $|z^{-1}| \to \infty$, while the rest of the series tends to 0. This is again a pole.

4. $F(z) = e^{1/z}$. This is an altogether worse kind of singularity, called essential. $F(z)$ has no limit of any kind as $z \to 0$. It is interesting to look at the behaviour as z tends to zero along the real and imaginary axes.

5. $T(z) = \operatorname{cosec}(1/z)$. Here 0 is not an isolated singularity at all, but a much bigger problem. The function has singularities at all the points where $1/z$ is an integer multiple of π.

Residues

If f has an isolated singularity at a, we compute the Laurent series $\sum_{k=-\infty}^{\infty} c_k(z-a)^k$ which represents f on some annulus A_ρ given by $0 < |z-a| < \rho$ on which f is analytic. Provided $\rho > 0$ and f is analytic on A_ρ, the coefficients don't depend on ρ (since the Laurent series for a given function and annulus is unique). The residue of f at a is c_{-1}. Note that if $0 < t < \rho$ then $\int_{|z-a|=t} f(z)dz = 2\pi i c_{-1}$, when we integrate once counter-clockwise.

Examples

1. Let γ be the circle $|z| = 2$ described once counter-clockwise. Determine $\int_\gamma \frac{\sin z}{(z-1)^2}\, dz$.

2. Let Γ be the contour which describes once counter-clockwise the square with vertices at $\pm 10 \pm 10i$. Determine $\int_\Gamma \frac{1}{z^2(z+1)}\, dz$.

5.2 Cauchy's residue theorem

Suppose that γ is a simple closed piecewise smooth contour described in the positive (i.e. counter-clockwise) sense. This means that as we move around γ the region interior to γ always lies to our left.

Suppose that $D \subseteq \mathbb{C}$ is a domain which contains γ and its interior. Suppose that f is analytic in D apart from a finite set of isolated singularities $\alpha_1, \ldots, \alpha_n$, which lie inside (and not on) γ. Then

$$\int_\gamma f(z)\, dz = 2\pi i \left(\sum_{j=1}^{n} \operatorname{Res}(f, \alpha_j) \right).$$

We stress that the integral must be once around γ in the positive sense.

To compute the residue $\mathrm{Res}(f, \alpha_j)$, we look at the Laurent series of f which represents f in an annulus $0 < |z-\alpha_j| < \rho$, for some $\rho > 0$. The residue is just the coefficient of $(z-\alpha_j)^{-1}$ in this series.

5.3 Examples

1. Keeping the notation of 5.2, suppose there are no singularities α_j. Then the integral is 0 (this is an even stronger form of Cauchy's theorem than 3.5).

2. The Cauchy integral formula.

3. Consider $\int_{|z| = 300} \frac{z-17}{(z-2)(z-4)}\, dz$.

4. Consider $\int_{|z| = 1} \frac{1}{e^z - 1}\, dz$.

5. Consider $\int_{|z| = 1} \frac{1}{(e^z - 1)^2}\, dz$.

6. Let γ be the semicircular contour through $-R, R$ and iR, and calculate $\int_\gamma e^{iz}/(z^2+1)\, dz$.

7. Determine $\lim_{R \to \infty} \int_0^R (\cos x)/(x^2+1)\, dx$.

8. Evaluate $\int_{-\infty}^{\infty} 1/(x^2+2x+6)\, dx$.