1 G14CAN Functions of a Complex Variable

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Lectures: Wed at 10.00 in A22, Th at 1.00 in C2, Fri at 11.00 in A21 (all in Pope Building).

1.1 Assessment

2.5 hour written examination: five questions, best four count. Coursework (non-assessed): a short assignment will be set each Friday, starting in the second week.

1.2 Pre-requisites:

a first course in complex analysis (such as G12CAN). It is slightly advantageous to have some knowledge of metric spaces (as in G13MTS), but all concepts needed will be covered in the lectures.

1.3 Recommended books

Complex Analysis, by LV Ahlfors, Functions of One Complex Variable, by JB Conway. (in Short/One Week Loan Collection).

1.4 Aims and Objectives

Aims: to teach elements of the advanced theory of functions of a complex variable.

Objectives: a successful student will:

1. be acquainted with some aspects of the advanced theory of functions of a complex variable;
2. have sufficient grounding in the subject to be able to read and understand some research texts on functions of a complex variable;
3. be acquainted with the principal theorems as treated and their proofs and able to use them in the investigation of examples;
4. be able to prove basic unseen propositions concerning those aspects of functions of a complex variable treated in the module.

1.5 Outline of the module

We will review some basic complex analysis and discuss some elementary topological ideas which are needed, including a new metric on $\mathbb{C}$. We will then prove a strong version of Cauchy's theorem, and discuss the topological properties of analytic functions. The famous Riemann mapping theorem and Picard theorem will be proved.

Note that this is a 'theory' module like G12RAN/G1CMIN and not a 'calculations' module like G12CAN.

2 The Riemann sphere

Consider a sphere S^* in $\mathbb{R}^3$ of radius $\frac{1}{2}$, tangent to the plane $Z = 0$ at the origin (we use (X, Y, Z) to denote points in space). Then S^* is given by $X^2+Y^2+(Z-\frac{1}{2})^2 = \frac{1}{4}$ and its centre is $(0, 0, \frac{1}{2})$.

Identify the plane $Z = 0$ with the complex plane (so that $(X, Y, 0)$ is identified with $X+iY$). Denote the north pole $(0, 0, 1)$ of the sphere by N. Let $z = x + iy$ be a complex number, and draw a straight line from N to z (i.e. to the point $(x, y, 0)$). Let the point where this straight line intersects the sphere be z^*.

It is easy to see that as $|z| \to +\infty$ the point z^* approaches N. We introduce a new point, the 'point at infinity', corresponding to N, and we denote it simply by ∞. There is only one such point, there being no distinction between $\pm\infty$ etc. We then consider the set $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

We introduce a distance on this set, as follows. We have $\infty^* = N$ and, for any two points z and w, we define $q(z, w)$ to be the distance between the points z^* and w^* in $\mathbb{R}^3$, and so the length of the chord joining z^* to w^*. For this reason, q is called the 'chordal' or 'spherical' metric. We calculate z^* and q.

The general point on the straight line from N in the direction of z has coordinates $(0, 0, 1) + t(x, y, -1) = (tx, ty, 1-t)$ with $t \geq 0$. For z^* we need this point to lie on the sphere S^* and so

$$\frac{1}{4} = t^2x^2 + t^2y^2 + (1 - t - \frac{1}{2})^2,$$

which solves to give

$$t = \frac{1}{1 + |z|^2}.$$

Thus

$$z^* = \left(\frac{x}{1 + |z|^2}, \frac{y}{1 + |z|^2}, \frac{|z|^2}{|z|^2 + 1}\right).$$

Now $q(z, \infty)$ is the length of the vector $t(x, y, -1)$ and so

$$q(z, \infty)^2 = |Nz^*|^2 = t^2(x^2 + y^2 + 1) = t^2(1 + |z|^2),$$

which gives

$$|Nz^*| = q(z, \infty) = (1 + |z|^2)^{-1/2} = 1/|Nz|. \tag{1}$$

To calculate q for z and w both finite, we can use the cosine rule. We form a triangle with vertices N, z^*, w^* and get

$$|z^* - w^*|^2 = |Nz^*|^2 + |Nw^*|^2 - 2|Nz^*||Nw^*| \cos\gamma.$$

Here γ is the angle subtended at N by the line segments Nz^*, Nw^*. Using (1) this gives

$$q(z, w)^2 = \frac{1}{|Nz|^2|Nw|^2}(|Nz|^2 + |Nw|^2 - 2|Nz||Nw| \cos\gamma) = \frac{1}{|Nz|^2|Nw|^2}(|z - w|^2)$$

by the cosine rule again, using the fact that N, z, z^* are collinear. Thus

$$q(z, w) = |z - w| \left[(1 + |z|^2)(1 + |w|^2)\right]^{-1/2} = |z - w|q(z, \infty)q(w, \infty).$$

Note that this tends to $q(z, \infty)$ if you let $|w| \to +\infty$. Note also that $q(z,u) \leq q(z,w)+q(w,u)$, because q is really the distance between points in $\mathbb{R}^3$. Thus q is a DISTANCE, or metric, on $\mathbb{C}^*$.

Note also that there is another formulation of the spherical metric, in which we use a sphere of radius 1, with centre $(0,0,0)$. This gives the same metric $q(z,w)$, but with an extra factor 2. In some books (and in old exam G14CAN papers) the "doubled" q is used, but I have changed to the definition here because it is used in most advanced texts.

2.1 Circles on the Riemann sphere

A plane in $\mathbb{R}^3$ is given by $((X,Y,Z)-(A,B,C)).(u,v,w) = 0$, where (X,Y,Z) is a general point in the plane, (A,B,C) is some pre-determined point in the plane, and (u,v,w) is a vector perpendicular to the plane. We can write this in the form $uX + vY + wZ = T$.

Now let C be a circle on the Riemann sphere S^*. Then C is the intersection of S^* with some plane P given by $aX + bY + cZ = d$, with a, b, c not all 0.

If $(X,Y,Z) \neq (0,0,1) = N$, which is the same as $z = x + iy \in \mathbb{C}$, then using the formula for z^* in terms of $z = x + iy$ we get z^* on P if and only if

$$ax + by + c|z|^2 = d(|z|^2 + 1)$$

or

$$(c-d)(x^2+y^2) + ax + by = d.$$

The last equation represents either a point, a circle or a straight line. For a straight line we have $c = d$, and so $(0,0,1)$ on P, which corresponds to C passing through N. We thus think of ∞ as lying on every straight line.

Conversely, given a circle or straight line in the plane, write it as

$$\alpha(x^2+y^2) + ax + by = \beta,$$

and solve for c, d via $c - d = \alpha, d = \beta$, to get the equation of a plane P, which must intersect S^* and does so in a circle.

Provided our plane P intersects S^* more than once, it divides $S^* \setminus P$ into two "complementary domains" (given by $aX + bY + cZ > d, aX + bY + cZ < d$), and working through the equations we see that these corresponds to different "sides" (complementary domains) of our circle/line in $\mathbb{C} \cup \{\infty\}$.

2.2 Metric spaces

A metric space (X, d) consists of a non-empty set X with a distance function d defined on $X \times X$ such that

(i) $d(x,y) \geq 0$.
(ii) $d(x,y) = 0$ iff $x = y$.
(iii) $d(x,y) = d(y,x)$.
(iv) $d(x,z) \leq d(x,y) + d(y,z)$.

An open ball centred at $x \in X$ is a set $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

A subset U of X is called open if for every $x \in U$ there exists $r_x > 0$ such that $B_d(x, r_x) \subseteq U$. We say $A \subseteq X$ is closed if $X \setminus A$ is open. The following properties are easy to verify:

(a) The union of any family of open sets in X is open. To see this, suppose that V_t is open for each t in some set T, and let $W = \bigcup_{t \in T} V_t$, so that W is the set of all x in X such that x belongs to at least one V_t. If x is in W, choose t such that $x \in V_t$ and (because V_t is open) take $r > 0$ such that $B_d(x, r) \subseteq V_t$. Then $B_d(x, r) \subset W$ and this proves that W is open.

(b) The intersection of finitely many open sets of X is open. For if $V_1, \ldots\ldots, V_n$ are open and x lies in each of them, take $r_j > 0$ with $B_d(x, r_j) \subseteq V_j$. Then with s the minimum of the r_j, the ball $B_d(x, s)$ is contained in each V_j and so in the intersection.

(c) Each $B_d(x, r)$ is itself open. For if $d(x, y) = s < r$, put $t = r - s$. Then $B_d(y, t) \subseteq B_d(x, r)$, because $d(y, z) < t$ implies that $d(x, z) \leq d(x, y) + d(y, z) < s + t = r$.

An open ball, or disc, in $\mathbb{C}$ is a set $B(a, s) = \{z : |z - a| < s\}$ (interior of a circle of radius s, centre a). We can also define, for w in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, a spherical disc $Q(w, r) = \{z \in \mathbb{C}^* : q(z, w) < r\}$. We have the following easy:

2.3 Fact

A subset H of $\mathbb{C}$ is open in $\mathbb{C}$ iff it is open as a subset of $(\mathbb{C}^*, q)$.

This can be seen geometrically. A small circle in the plane of centre w corresponds to a circle on the sphere surrounding w^* (though not necessarily with centre w^*), and vice versa.

Alternatively, take w in $\mathbb{C}$. Let $\varepsilon > 0$. Since $q(z, w) \leq |z - w|$ we have $B(w, \varepsilon) \subseteq Q(w, \varepsilon)$. In the other direction, let δ be small and positive, in particular with $2\delta < q(w, \infty)$. Then $q(z, w) < \delta$ gives $q(z, \infty) > q(w, \infty) - \delta > \frac{1}{2} q(w, \infty)$ by the triangle inequality and so

$$|z - w| = \frac{q(z, w)}{q(z, \infty) q(w, \infty)} < 2\delta q(w, \infty)^{-2},$$

so that $Q(w, \delta) \subseteq B(w, \varepsilon)$ if δ is small enough.

Thus we get the following simple lemma.

2.4 Lemma

If $\varepsilon > 0$ and $w \in \mathbb{C}$ there exists $\delta > 0$ such that:

$q(z, w) < \delta$ *implies* $|z - w| < \varepsilon$;

$|z - w| < \varepsilon$ *implies* $q(z, w) < \varepsilon$.

Hence if $w \in H \subseteq \mathbb{C}$ and H is open with respect to q then we get a $Q(w, \varepsilon) \subseteq H$, from which $B(w, \varepsilon) \subseteq H$. On the other hand if H is open in the standard metric we get some $B(w, \varepsilon) \subseteq H$, and then there exists $\delta > 0$ such that $Q(w, \delta) \subseteq B(w, \varepsilon) \subseteq H$.

2.5 Continuity, paths and sequences in a metric space

Let $f : E \to Y$ be a function, where $E \subseteq X$ and (X, d) and (Y, ρ) are metric spaces. We say that f is continuous on E if the following is true. To each x in E and $\varepsilon > 0$ corresponds a $\delta > 0$ such that $\rho(f(y), f(x)) < \varepsilon$ for all y in E with $d(x, y) < \delta$. Note that it is possible to characterize continuity in terms of open sets, but we will not use this in G14CAN.

A path γ in X is just a continuous function from a closed interval in $\mathbb{R}$ to X.

We say that the sequence (x_n) in X converges to $a \in X$ if $d(x_n, a) \to 0$ as $n \to \infty$.

If (x_n) is a convergent sequence in a closed set A then its limit x must be in A. For if not then x is in the open set $B = X \setminus A$ and there exists $r > 0$ with $B_d(x, r) \subseteq B$. But then $x_n \in B_d(x, r) \subseteq B$ for all large n, which contradicts the fact that $x_n \in A$.

It's easy to check that the composition of continuous functions is continuous. Specifically, let g be continuous on E, and f continuous on $g(E)$. Let the metrics on $E, g(E), f(g(E))$ be d, ρ, σ. Take $x_0 \in E$ and $\varepsilon > 0$. Set $y_0 = g(x_0)$. Then there exists $\eta > 0$ such that $\sigma(f(y), f(y_0)) < \varepsilon$ for all $y \in g(E)$ with $\rho(y, y_0) < \eta$, since f is continuous. But since g is continuous there exists $\delta > 0$ such that $\rho(g(x), g(x_0)) < \eta$ for all $x \in E$ with $d(x, x_0) < \delta$, and for these x we have $\sigma(f(g(x)), f(g(x_0))) < \varepsilon$.

Also, if (x_n) converges to α and f is continuous at α then $f(x_n)$ converges to $f(\alpha)$. Why? Take $\varepsilon > 0$. There exists $\delta > 0$ such that $d(x, \alpha) < \delta$ implies $\rho(f(x), f(\alpha)) < \varepsilon$. But then for all large enough n we have $d(x_n, \alpha) < \delta$ and so $\rho(f(x_n), f(\alpha)) < \varepsilon$.

Also the map $z \to z$ from $(\mathbb{C}, q)$ to "standard" $\mathbb{C}$ is continuous in both directions, by Lemma 2.4. So if considering functions on or into $\mathbb{C}$, or sequences in $\mathbb{C}$ with limit in $\mathbb{C}$, it doesn't matter which metric we use.

2.6 The Bolzano-Weierstrass theorem

If (z_n) is a bounded sequence in $\mathbb{C}$ then (z_n) has a convergent subsequence.

To prove this, write $z_n = x_n + iy_n$ and first take a subsequence (z_{n_k}) such that the real part (x_{n_k}) converges, then a further subsequence $(z_{n_{k_r}})$ such that the imaginary part also converges. It's then clear that every sequence in $\mathbb{C}^*$ has a convergent subsequence (with limit possibly ∞).

2.7 Corollary

If E is a closed and bounded subset of $\mathbb{C}$, or is all of $\mathbb{C}^*$, and if $f : E \to \mathbb{R}$ is continuous, then f has a maximum on E i.e. there exists $z_0 \in E$ with $f(z) \le f(z_0)$ for all z in E.

Proof. Let $s = \sup A$, where $A = \{f(z) : z \in E\}$, with the convention that the sup is $+\infty$ if A is not bounded above. There then exists a sequence $y_n \in A$ with $y_n \to s$, and we can write $y_n = f(x_n), x_n \in E$. Taking a subsequence we may then assume that x_n converges, to a limit z_0, and we get $z_0 \in E$. Thus $f(z_0) = s$ by continuity.

It follows from this that a path $\gamma : [a, b] \to \mathbb{C}$ is a bounded set (since $|\gamma(t)|$ has a maximum on $[0, 1]$). It also forms a closed set, since if w does not lie on γ the function $|1/(\gamma(t) - w)|$ has a maximum on $[0, 1]$ and $|\gamma(t) - w|$ has a positive minimum r, so that $B(w, r)$ does not meet γ and $\mathbb{C} \setminus \gamma$ is open.

2.8 Example

The following example shows the advantage of the chordal metric and $\mathbb{C}^*$. Let $f(z) = 1/\sin z$. As $z \to 0$, we have $|f(z)| \to +\infty$ and $q(f(z), \infty) \to 0$. So setting $f(0) = \infty$ we find that f is continuous on $B(0, 1)$ with respect to the chordal metric. Note that f takes real *and* imaginary values as $z \to 0$.

Warning for those who've done measure theory. The convention $0.\infty = 0$ generally used there does not always apply here. For example, $g(z) = z^{-2} \sin z$ is such that $g(z) \to \infty$ as $z \to 0$, so that it makes sense to put $g(0) = \infty$.

2.9 Limit points

Let E be any subset of a metric space X. We say that ζ is a LIMIT POINT of E if the following is true.

(a) for every open set U with $\zeta \in U$, there exists $x \neq \zeta$ with $x \in E \cap U$ i.e. there are points of E, not equal to ζ, which are in U.

Since every open U with $\zeta \in U$ contains some $B_d(\zeta, r)$, and since open balls are themselves open, this is equivalent to:

(b) for every $r > 0$ there exists $x \in E$ with $0 < d(x, \zeta) < r$.

By taking smaller and smaller values of r, we see that this is equivalent to:

(c) there is a sequence (z_n) such that $z_n \in E, z_n \neq \zeta, z_n \to \zeta$.

Note that we do not require that ζ be in E, and that only infinite sets can have limit points.

Example: what are the limit points of $\mathbb{N}$ in $\mathbb{C}$ and $\mathbb{C} \cup \{\infty\}$?

The CLOSURE of E is $Cl(E)$, which is the set of x such that either x is in E or x is a limit point of E. Thus $x \in Cl(E)$ if and only if there is a sequence $z_n \in E$ with $z_n \to x$.

The FRONTIER, or boundary, of a set E is $Cl(E) \cap Cl(X \backslash E) = \partial E$.

Warning: If H is the set $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ then if we consider H as a subset of $\mathbb{C}$ its closure is the closed upper half plane $L = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$. If we consider H as a subset of $\mathbb{C}^*$ its closure is $L \cup \{\infty\}$.

2.10 Theorem

In a metric space (X, d):

(a) A closed set A contains all its limit points (if any), and so equals its closure, and contains its frontier.

(b) If U is open, then $U \cap \partial U$ is empty.

Proof: (a) If ζ is a limit point of A, then ζ is the limit of a sequence in A, and so is in A, since A is closed. This proves that the limit points of A are all in A, so that $A = Cl(A)$. (b) is now obvious, as $\partial U = \partial(X \backslash U)$.

2.11 Path-connected sets

A subset E of a metric space X is called path-connected if the following is true. To each pair of points a and b in E corresponds a path γ in E joining a to b. Note that if E is path-connected and f is continuous on E then $f(E)$ is path-connected (just use $f(\gamma)$).

By a domain D we mean a path-connected open subset of $\mathbb{C}^*$. Suppose that D is a domain and that $D = A \cup B$, where A and B are open, and $A \cap B = \emptyset$. Then we assert that either $A = \emptyset$ or $B = \emptyset$. Suppose not. Then we can find a path $\gamma : [0, 1] \to D$ with $\gamma(0) \in A, \gamma(1) \in B$. Let $v(z) = 1$ for $z \in B$, with $v(z) = 0$ for $z \in A$. Then v is continuous on D, and $u(t) = v(\gamma(t))$ is a continuous integer-valued function on $[0, 1]$. But $u(0) = 0, u(1) = 1$, which is a contradiction.

3 Möbius transformations

3.1 Definition

By a Möbius transformation we mean a map $f : \mathbb{C}^* \to \mathbb{C}^*$ given by

$$f(z) = \frac{az+b}{cz+d}$$

in which $a, b, c, d \in \mathbb{C}$, with $ad-bc \neq 0$. The last condition is required since otherwise $f(-d/c) = 0/0$ is not defined.

Note that $f'(z)$ exists for all z in $\mathbb{C}$ with $cz+d \neq 0$, with $f'(z) = (ad-bc)/(cz+d)^2$ (this is another reason for $ad-bc \neq 0$: otherwise f would be constant where defined).

If $c = 0$ then f maps $\mathbb{C}$ into $\mathbb{C}$ and $f(z) \to \infty$ iff $z \to \infty$. It makes sense to define $f(\infty) = \infty$. If $c \neq 0$, then $f(z) \to \infty$ as $z \to -d/c$, and $f(z) \to a/c$ as $z \to \infty$. We thus define $f(-d/c) = \infty, f(\infty) = a/c$.

Writing

$$w = f(z), \quad z = g(w) = \frac{-dw+b}{cw-a}$$

we see that g is a Möbius transformation (as $(-d)(-a) - bc = ad - bc \neq 0$). Also $g(\infty) = -d/c, g(a/c) = \infty$, and g is the inverse function of f, so that f is automatically a one-one mapping of $\mathbb{C}^*$ onto itself.

The Möbius transformations form a group under composition.

3.2 Theorem

Möbius transformations map circles and straight lines in $\mathbb{C}^*$ onto circles or straight lines. Also the complementary domains are mapped onto complementary domains.

Proof: note first that if L is a straight line in $\mathbb{C}$ then we regard $L \cup \{\infty\}$ as a straight line in $\mathbb{C}^*$. Recall also that a circle or straight line h in $\mathbb{C}^*$ corresponds to a circle h^* on the Riemann sphere S^*.

Complementary domains: for a straight line these are two open half-planes (in $\mathbb{C}$). For a circle in $\mathbb{C}$ they are the interior and the exterior (which contains ∞).

Now obviously the maps $z \to az, z \to z+b$ send circles to circles and straight lines to straight lines. Next consider $z \to 1/z$. Since

$$\frac{1}{z} = \frac{x-iy}{|z|^2}$$

a simple calculation gives

$$z^* = (X, Y, Z), \quad \left(\frac{1}{z}\right)^* = (X, -Y, 1-Z),$$

which is a rotation of S^*. We can also note that a circle or line is given by

$$A(x^2+y^2) + Bx + Cy = D.$$

If we write $z = 1/(u + iv)$ this gives

$$A + Bu - Cv = D(u^2 + v^2)$$

which is again a circle or line. So $z \to 1/z$ sends circles/lines to circles or lines.

Since

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - ad/c}{c(z + d/c)}$$

every Möbius map is a composition of maps of the three basic types above.

Since Möbius maps T are one-one and continuous, a complementary domain of C is mapped into a complementary domain of $T(C)$ (take the image of one point and use path connectedness), and by looking at T^{-1} we see that the mapping is onto.

3.3 Examples

(i) $w = (1 + z)/(1 - z)$. This maps $|z| = 1$ to $i\mathbb{R} \cup \{\infty\}$. Since $w(0) = 1$, we get that w maps $B(0, 1)$ onto $\{z : \mathrm{Re}(z) > 0\}$.

(ii) Let $|a| < 1$ and let

$$f(z) = \frac{z - a}{1 - \overline{a}z}.$$

If $|z| = 1$ then

$$z - a = z(1 - a\overline{z}) = z\overline{(1 - \overline{a}z)}$$

so $|f(z)| = 1$. Since a Möbius transformation maps a circle onto a circle or straight line, f maps the unit circle $|z| = 1$ onto itself.

Also $f(a) = 0$ and so $f(B(0, 1)) = B(0, 1)$.

3.4 Fixpoints and dynamics of Möbius maps

We say that x is a fixpoint of a map F if $F(x) = x$. We consider the fixpoints of Möbius maps $f(z) = \frac{az+b}{cz+d}$. Now if $c = 0$ then $f(\infty) = \infty$ and solving $(az + b)/d = z$ gives either one fixpoint in $\mathbb{C}$ or none. If $c \neq 0$ then ∞ is not a fixpoint and setting $f(z) = z$ gives either one or two roots in $\mathbb{C}$.

We investigate what happens to the sequence $z, f(z), f(f(z)), \ldots$. For brevity write $f_1 = f, f_{n+1} = f(f_n)$. Notice that if $f_n(z) \to h \in \mathbb{C}^*$ then (since f is continuous on $\mathbb{C}^*$) we have $f_{n+1}(z) \to f(h)$ and so $h = f(h)$ i.e. h is a fixpoint.

To determine whether or not $f_n(z)$ does converge, we write $w = T(z)$, in which T is a Möbius map chosen so that it maps one fixpoint of f to ∞ and the other (if it exists) to 0. This is easy: if ∞ is a fixpoint just use $w = T(z) = z$ (one fixpoint) or use $w = T(z) = z - B$ (if f has another fixpoint at B). If f has just one (finite) fixpoint at A, use $w = T(z) = 1/(z - A)$. If f has distinct finite fixpoints A, B, use $w = T(z) = (z - B)/(z - A)$.

Let g be the composition $g = TfT^{-1}$ (conjugation). If D is one fixpoint of f and $T(D) = C$, then $g(C) = T(f(D)) = T(D) = C$. So T maps fixpoints of f to fixpoints of g. Also $g_n(w) = Tf_nT^{-1}(w) = Tf_n(z)$. In particular, since T^{-1} is continuous,

$$g_n(w) \to \alpha \iff f_n(z) \to T^{-1}(\alpha).$$

Thus if f has just one fixpoint D, the only fixpoint of g is ∞, so that $g(w) = w + b$ with $b \neq 0$. In this case, for every w in $\mathbb{C}$ we have $g_n(w) = w + nb \to \infty$. So for every $z \in \mathbb{C}^*$, we have $g_n(w) \to \infty$ and so $f_n(z) \to T^{-1}(\infty) = D$.

If there are two fixpoints A, B, then we have $T(A) = \infty, T(B) = 0$ and $g(\infty) = \infty$ and $g(0) = 0$. Thus $g(w) = \beta w$ for some non-zero constant β.

If $|\beta| < 1$ then $g_n(w) \to 0$ for all $w \in \mathbb{C}$. So for $z \neq A$ we get $w = T(z) \neq \infty$ and $g_n(w) \to 0$ so $f_n(z) \to B$.

If $|\beta| > 1$ then $g_n(w) \to \infty$ for $w \neq 0$. So if $z \neq B$ then $w = T(z) \neq 0$ and $g_n(w) \to \infty$, so $f_n(z) \to A$.

Finally, if $|\beta| = 1, \beta \neq 1$, then g acts like a 'rotation' on $\mathbb{C}\backslash\{0\}$. In this case $\lim g_n(w)$ exists only for $w = 0, \infty$, and so $f_n(z)$ only tends to a limit if $z = A, B$.

Example:

Let $f(z) = -2/(z + 3)$. The fixpoints are $-2, -1$. Let $w = T(z) = (z + 2)/(z + 1)$ and $g = TfT^{-1}$. We have $z = T^{-1}(w) = (2 - w)/(w - 1)$ and $f(T^{-1}(w)) = (2 - 2w)/(2w - 1)$. Thus $g(w) = 2w$ and $g_n(w) \to \infty$ for $w \neq 0$. So for $z \neq -2$ we have $w = T(z) \neq 0$ and $g_n(w) = T(f_n(z)) \to \infty$ and so $f_n(z) \to T^{-1}(\infty) = -1$.

4 Review of basic Complex Analysis

Some of this section is review of G12CAN.

4.1 Definition

Let f be a complex-valued function defined on an open disc $B(a,s)$ in $\mathbb{C}$. We say f is complex differentiable at a if there is some complex number $f'(a)$ such that the following holds. To each $\varepsilon > 0$ corresponds $\delta > 0$ such that $0 < |z-a| < \delta$ implies that

$$\left| \frac{f(z)-f(a)}{z-a} - f'(a) \right| < \varepsilon,$$

which is the same as saying that

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a).$$

Note that if f is complex differentiable at a then $f(z)-f(a) \to f'(a).0 = 0$ as $z \to a$, so that f is continuous at a.

f is analytic at a if there is some open disc centred at a on which f is complex differentiable.

If we say that a function f is analytic on a domain D it should be taken as read that f takes values in $\mathbb{C}$.

4.2 The Cauchy-Riemann equations

If $f(x+iy) = u(x,y) + iv(x,y)$ (x, y, u, v all real) and f is complex differentiable at $a = b+ic$, then at (b,c), we have $u_x = v_y$ and $u_y = -v_x$. Conversely, if the partial derivatives of u and v are continuous and satisfy these Cauchy-Riemann equations at every point (b,c) such that $b+ic$ belongs to some domain D in $\mathbb{C}$, then f is analytic on D.

4.3 Contours

A smooth contour is a curve $\gamma : [a,b] \to \mathbb{C}$ such that the derivative γ' exists and is continuous and never 0 on $[a,b]$. The length of γ is $\int_a^b |\gamma'(t)|dt$, and is denoted by $|\gamma|$. If f is a function taking values in $\mathbb{C}$ and continuous on the curve γ (strictly speaking, on the set $\gamma([a,b])$) then $f(\gamma(t))$ is continuous on $[a,b]$ and we set

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

We have the following properties:

(a) If λ is given by $\lambda(t) = \gamma(b+a-t)$ (so that λ is like γ 'backwards') then

$$\int_\lambda f(z)dz = \int_a^b f(\gamma(b+a-t))(-\gamma'(b+a-t))dt = -\int_\gamma f(z)dz.$$

This λ is sometimes referred to as γ^{-1}.

(b) We have

$$|\int_\gamma f(z)dz| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq (\max\{|f(\gamma(t))| : a \leq t \leq b\}) .|\gamma|.$$

Here we have used the fact that if $g : [a,b] \to \mathbb{C}$ is continuous, then for some real s,

$$|\int_a^b g(t)dt| = e^{is}\int_a^b g(t)dt = \int_a^b e^{is}g(t)dt = \int_a^b \mathrm{Re}(e^{is}g(t))dt,$$

and this is, by real analysis, at most

$$\int_a^b |\mathrm{Re}(e^{is}g(t))|dt \leq \int_a^b |e^{is}g(t)|dt = \int_a^b |g(t)|dt.$$

By a PIECEWISE SMOOTH contour γ we mean finitely many smooth contours γ_k joined end to end. Here the integral over γ is the sum of the integrals over the γ_k.

A piecewise smooth contour is SIMPLE if it never passes through the same point twice, CLOSED if it finishes where it started, and SIMPLE CLOSED if it finishes where it started but otherwise does not pass through any point twice (i.e. γ is one-one except that $\gamma(a) = \gamma(b)$).

(c) Suppose that γ is simple or simple closed. Suppose that $\lambda : [c,d] \to \mathbb{C}$ describes the same set of points as γ, in the same direction. Then $\lambda(t) = \gamma(\phi(t))$ for some function $\phi : [c,d] \to [a,b]$ which is strictly increasing. For t close to, but not equal to, t_0, we have $\phi(t) \neq \phi(t_0)$ and

$$\frac{\lambda(t) - \lambda(t_0)}{t - t_0} = \frac{\gamma(\phi(t)) - \gamma(\phi(t_0))}{\phi(t) - \phi(t_0)}\frac{\phi(t) - \phi(t_0)}{t - t_0}.$$

If $t \to t_0$ then $\phi(t) \to \phi(t_0)$ since ϕ is onto and strictly increasing and so continuous. This gives $\phi'(t_0) = \lambda'(t_0)/\gamma'(\phi(t_0))$. Thus

$$\int_\lambda f(z)dz = \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)dt = \int_a^b f(\gamma(s))\gamma'(s)ds = \int_\gamma f(z)dz.$$

4.4 Theorem

Suppose that $\gamma : [a,b] \to D$ is a piecewise smooth contour in a domain D in $\mathbb{C}$, on which F is analytic and has continuous derivative f. Then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$ and so is 0 if γ is closed.

To prove this we just note that $H(s) = F(\gamma(s)) - \int_a^s f(\gamma(t))\gamma'(t)dt$ is such that $H'(s) = 0$ on (a,b) and so its real and imaginary parts are constant on $[a,b]$. So $H(b) = H(a) = F(\gamma(a))$.

4.5 Cauchy-Goursat Theorem

Let D be a domain in $\mathbb{C}$ and let T be a contour which describes once counter-clockwise the perimeter of a triangle whose perimeter and interior are contained in D. Let f be analytic on D. Then $\int_T f(z)dz = 0$.

Proof: Let the length of T be L, and let $M = |\int_T f(z)dz|$, and take $\varepsilon > 0$. We bisect the sides of the triangle to form 4 new triangular contours. In the subsequent proof, all integrals are understood to be taken in the positive (counter-clockwise) sense.

Since the contributions from the interior sides cancel, one of these triangles, T_1 say, must be such that $|\int_{T_1} f(z)dz| \geq M/4$. Now T_1 has perimeter length $L/2$. We repeat this procedure and get a sequence of triangles T_n such that:

(i) T_n has perimeter length $L/2^n$;

(ii) Let V_n be the union of T_n and its interior: then $V_{n+1} \subseteq V_n$;

(iii) $|\int_{T_n} f(z)dz| \geq M/4^n$.

Let z_n be the centre of T_n. Then z_m lies on or inside T_n for all $m \geq n$. By the Bolzano-Weierstrass theorem a subsequence z_{n_k} converges to z^*, say. Thus $z^* \in V_{n_k}$ (closed set). By (ii), z^* must lie on or inside EVERY T_n.

Since f is differentiable at z^* there must be some open disc U with centre z^* such that for each z in U we can write

$$\frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) = \eta(z)$$

with $|\eta(z)| \leq \varepsilon$. But if n is large enough, T_n lies in U and we get

$$\int_{T_n} f(z)dz = \int_{T_n} f(z^*) + (z - z^*)f'(z^*) + \eta(z)(z - z^*)dz = \int_{T_n} \eta(z)(z - z^*)dz$$

using the previous theorem, and the last integral has absolute value at most $\varepsilon L^2/4^n$, since $|z - z^*| \leq |T_n|$. But this gives, using (iii), $M/4^n \leq \varepsilon L^2/4^n$, and since ε is arbitrary we must have $M = 0$.

We need the following (technical) result.

4.6 Theorem

Let D and T be as in the Cauchy-Goursat theorem, and let f be continuous in D and analytic in $D\backslash\{w\}$, for some w in D. Then we still have $\int_T f(z)dz = 0$.

Proof: If w lies outside T then this is obvious (apply Cauchy-Goursat with $D \setminus \{w\}$.

If w is a vertex of T, then consider the following, in which T is the triangle wAB, and C, D are close to w.

On triangle wCD we have, if C, D are close enough to w, an estimate $|f(z) - f(w)| < 1$ and so $|f(z)| \leq M = |f(w)| + 1$. Since the integrals of f around triangles CAB and CBD both vanish, we get

$$|\int_T f(z)dz| = |\int_{wCD} f(z)dz| \leq M|wCD|,$$

which we can make as small as we like.

If w lies on an edge or inside T, just divide up T.

In this module, we don't make much use of the star domains which were so important in G12CAN. We first work just in discs, and will later prove a very general form of Cauchy's theorem, which will apply, in particular, to star domains.

4.7 Theorem

Suppose that f is continuous (taking values in $\mathbb{C}$) in the disc $U = B(a, R)$ and is such that $\int_T f(z)dz = 0$ whenever T is a contour describing once counter-clockwise the boundary of a triangle contained in U (N.B. in particular this will be the case if f is continuous in the disc $U = B(a, R)$ and analytic in $U\backslash\{w\}$, for some $w \in U$). Then there is a function F analytic on U such that $F'(z) = f(z)$ for all z in U.

Note that it then follows that if γ, σ are PSCs from A to B in U then $\int_\gamma f(z)dz = \int_\sigma f(z)dz = F(B) - F(A)$, so that the integration is independent of path in U.

Proof: we define $F(z) = \int_a^z f(u)du$, where we integrate along the straight line from a to z. Then if h is small, non-zero, the hypotheses tell us that

$$F(z+h) - F(z) = \int_z^{z+h} f(u)du = h \int_0^1 f(z+ht)dt.$$

Here we integrate along the straight line from z to $z + h$. Now dividing by h we see that

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f(z+ht)dt \to f(z)$$

as $h \to 0$.

Integrals of the next type are an important source of analytic functions and are useful in many proofs.

4.8 Theorem: Rule AFFI

Suppose that γ is a piecewise smooth contour and that f is a complex-valued function such that $f(\gamma(t))$ is continuous. Let $w \in \mathbb{C}$ with w not on γ, and for $n \in \mathbb{N}$ define

$$F_n(w) = \int_\gamma \frac{f(u)}{(u-w)^n} du.$$

Then each F_n is analytic at w, with derivative $F_n'(w) = nF_{n+1}(w)$.

Proof: we show first that F_1 is continuous. Let $w \in \mathbb{C}$ not lie on γ. We saw that $\mathbb{C} \setminus \gamma$ is an open set (Chapter 2) so we can take $s > 0$ such that $B(w, 2s) \cap \gamma = \emptyset$. Let $z \in B(w, s), z \neq w$. Then $|u - w| \geq 2s$ for all u on γ and $|u - z| > s$ for all u on γ. Now

$$F_1(z) - F_1(w) = \int_\gamma \frac{f(u)(z-w)}{(u-z)(u-w)} du$$

has (with L the length of γ and M the max of $|f|$ on γ) modulus at most $LM|z-w|s^{-1}(2s)^{-1} \to 0$ as $z \to w$.

Now we look at the general case. For w and z as before we have, with $c_k = \frac{n!}{k!(n-k)!}$, the following:

$$F_n(z) - F_n(w) = \int_\gamma f(u) \left(\frac{1}{(u-z)^n} - \frac{1}{(u-w)^n} \right) du =$$

$$= \int_\gamma \frac{f(u)}{(u-w)^n} \left(\frac{(u-w)^n}{(u-z)^n} - 1 \right) du = \int_\gamma \frac{f(u)}{(u-w)^n} \left[\left(1 + \frac{z-w}{u-z} \right)^n - 1 \right] du$$

which equals

$$\int_\gamma \frac{f(u)}{(u-w)^n} \left[\sum_{k=1}^n c_k \left(\frac{z-w}{u-z} \right)^k \right] du = \sum_{k=1}^n c_k (z-w)^k \left[\int_\gamma \frac{f(u)}{(u-w)^n (u-z)^k} du \right].$$

Now divide by $z - w$ to get

$$\frac{F_n(z) - F_n(w)}{z-w} = \sum_{k=1}^n c_k (z-w)^{k-1} \left[\int_\gamma \frac{f(u)}{(u-w)^n (u-z)^k} du \right].$$

Let $z \to w$. By the first part, with $f(u)$ replaced by $f(u)(u-w)^{-n}$, we have

$$\int_\gamma \frac{f(u)}{(u-w)^n (u-z)} du \to \int_\gamma \frac{f(u)}{(u-w)^{n+1}} du.$$

Also for $2 \leq k \leq n$,

$$\int_\gamma \frac{f(u)}{(u-w)^n (u-z)^k} du$$

has modulus at most $LM(2s)^{-n} s^{-k}$ and the $(z-w)^{k-1}$ term tends to 0 as $z \to w$. Thus

$$F_n'(w) = \int_\gamma \frac{nf(u)}{(u-w)^{n+1}} du = nF_{n+1}(w).$$

4.9 Definitions

A CYCLE Γ consists of finitely many closed piecewise smooth contours $\gamma_1, ..., \gamma_n$, and the integral over Γ is the sum of the integrals over the γ_k.

If a is a point NOT lying on Γ (i.e. not on any of the γ_k) we define

$$n(\Gamma, a) = \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{\gamma_k} \frac{1}{z-a} dz,$$

which will be called the WINDING NUMBER of Γ about a.

4.10 Properties of the winding number

We have:

(i) $n(\Gamma, a)$ is an integer.

(ii) If $\sigma : [c,d] \to \mathbb{C}$ is a path not intersecting Γ, then $n(\Gamma, \sigma(t))$ is constant, and hence if D is a domain in $\mathbb{C}$ not intersecting Γ, then $n(\Gamma, a)$ is constant on D.

(iii) If $|w|$ is large enough, $n(\Gamma, w) = 0$.

(iv) If Γ is the circle $|z-a| = s > 0$, described once counter-clockwise, then $n(\Gamma, a) = 1$.

Proof: (i) We only need to prove this when Γ is a closed piecewise smooth contour, which we can assume is defined on $[A, B]$. We then set

$$h(t) = \int_A^t \frac{\Gamma'(u)}{\Gamma(u) - a} du, \quad H(t) = (\Gamma(t) - a) \exp(-h(t)).$$

We then have, on (A, B), the formula

$$H'(t) = \Gamma'(t) \exp(-h(t)) - (\Gamma(t) - a)h'(t) \exp(-h(t)) = 0,$$

so that $H(B) = H(A) = \Gamma(A) - a = \Gamma(B) - a$, which means that $\exp(-h(B))$ must be 1.

(ii) The function $\phi(t) = n(\Gamma, \sigma(t))$ is a continuous integer-valued function on $[c, d]$ and so constant there, by the intermediate value theorem. Since any two points in a domain may be joined by a path in the same domain, the second assertion follows.

It is also worth noting that, again with Γ a closed PSC and a a point not on Γ,

$$\frac{d}{dw} \left(n(\Gamma, w) \right) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{(z-w)^2} dz = 0,$$

for w close to a, since $1/(z-w)^2$ is the derivative of $-1/(z-w)$, which is analytic on a domain in $\mathbb{C}$ containing Γ. Hence $n(\Gamma, w)$ is constant near a.

(iii) Since Γ is bounded we see that if $|w|$ is large enough then $|z - w| \geq |w|/2$ for all z on Γ. This gives $|n(\Gamma, w)| \leq (1/2\pi)(2/|w|)(|\Gamma|)$, which tends to 0 as $|w|$ tends to $+\infty$.

(iv) Parametrize by $z = a + se^{it}$, $0 \leq t \leq 2\pi$ and evaluate.

4.11 Cauchy's integral formula for a disc

Let γ be a cycle in $U = \{z \in \mathbb{C} : |z-a| < R\}$, $0 < R \leq +\infty$, and let $f : U \to \mathbb{C}$ be analytic. Then for all w in U not lying on γ, and for all integers $k \geq 0$, we have

$$n(\gamma, w) f^{(k)}(w) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{k+1}} dz.$$

Proof: We define $g(z)$ as follows. For $z \neq w$ we put $g(z) = \frac{f(z)-f(w)}{z-w}$, and we set $g(w) = f'(w)$. Then g is continuous on U and analytic on $U \backslash \{w\}$, and so we have $\int_T g(z)dz = 0$ for every triangular contour T in U. Hence there is a function G analytic on U with $G' = g$, and so we have

$$0 = \int_\gamma g(z)dz = \int_\gamma \frac{f(z)}{z-w} dz - \int_\gamma \frac{f(w)}{z-w} dz.$$

But $\int_\gamma \frac{f(w)}{z-w} dz = f(w) 2\pi i n(\gamma, w)$.

This gives the formula for $k = 0$, and it can be differentiated using Rule AFFI. Since $n(\gamma, w)$ is constant on a nbd of w, we get

$$n(\gamma, w) f^{(k)}(w) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{k+1}} dz.$$

4.12 Corollary

The derivative of an analytic function is analytic.

For, suppose that f is analytic on the domain D in $\mathbb{C}$. Take a in D and $s > 0$ such that $B(a, 2s) \subseteq D$. Then for $|w-a| < s$, we take γ to be the circle $|z-a| = s$, so $n(\gamma, w) = 1$, and we have a formula for $f'' = (f')'$. Thus f' has a derivative on $B(a, s)$ and so is analytic at a.

The following theorem will be necessary for our generalization of the C.I.F. to other domains.

4.13 Liouville's theorem

Suppose that f is entire (= analytic in $\mathbb{C}$) and bounded as $z \to \infty$, i.e. there exist real $T > 0$ and $M > 0$ such that $|f(z)| \leq M$ for $T \leq |z| < +\infty$. Then f is constant.

The proof is to take any u and v in $\mathbb{C}$ and take R very large. We have, integrating once counter-clockwise,

$$f(u) - f(v) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)(u-v)}{(z-u)(z-v)} dz.$$

If R is large enough then $|z| = R$ gives $|z-u| \geq |z| - |u| \geq R/2$ and $|z-v| \geq R/2$. Thus we get $|f(u) - f(v)| \leq (1/2\pi)(2\pi R) 4MR^{-2}|u-v|$. Since R can be chosen arbitrarily large we must have $f(u) = f(v)$.

5 A General Version of Cauchy's Theorem

The name Cauchy's theorem is given to a general class of criteria which tell you when the integral of an analytic function around a closed curve is 0. This is not always the case, as

$$\int_{|z|=1} dz/z = 2\pi i,$$

integrating once counter-clockwise. On the other hand, if f is analytic in a disc U in $\mathbb{C}$ then f has an antiderivative F on U, by Theorem 4.7, and so the integral around any closed curve in U is 0. Further, if γ_j are paths in U from α to β then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$ i.e. the integral is independent of path in U.

The following proof of a quite strong version of Cauchy's theorem is due to Peter Loeb (University of Illinois) ca. 1991. We work in terms of winding number.

5.1 Lemma

Let D be a domain in $\mathbb{C}$ and let γ be a cycle in D and let w be in D. Then there is a cycle σ in D such that w does not lie on σ and $\int_\sigma f(z)dz = \int_\gamma f(z)dz$ for every function f which is analytic on D.

Proof Obviously if w is not on γ we set $\sigma = \gamma$. Assume henceforth that w lies on γ.

Assume first that $\gamma : [a, b] \to D$ is a closed PSC (so $\gamma(a) = \gamma(b)$) and $w \neq \gamma(a) = \zeta$.

We take a small $s > 0$ such that $B(w, 2s) \subseteq D$ and such that $|\zeta - w| > 2s$. Suppose we have some u such that $\gamma(u) = w$. We know that $a < u < b$ and so we can move away from u in either direction till we hit a point at which $|\gamma(v) - w| = s$. Going as far as possible subject to $|\gamma(v) - w| \leq s$, this gives us a sub-interval $J_u = [c_u, d_u]$ of $[a, b]$, defined to be the largest closed sub-interval of $[a, b]$ containing u and such that $|\gamma(v) - w| \leq s$ for all v in J_u.

We form σ as follows. We replace the part of γ for $c_u \leq v \leq d_u$ by an arc of the circle $|z - w| = s$ from $\gamma(c_u)$ to $\gamma(d_u)$, parametrized over $[c_u, d_u]$. Since each f which is analytic on D is analytic on $B(w, 2s)$, doing this does not change the value of $\int_\gamma f(z)dz$. We do this for each of the intervals J_u.

It is perhaps worth pointing out that there are only finitely many u. For suppose that a *smooth* contour $\Gamma : [A, B] \to \mathbb{C}$ passes through a point w infinitely often i.e. there exists a sequence $u_n \in [A, B]$, the u_n distinct, such that $\Gamma(u_n) = w$. By the Bolzano-Weierstrass theorem we can assume without loss of generality that (u_n) converges, say to u^*. But then $u^* \in [A, B]$, and $\Gamma(u^*) = w$ by continuity. But this gives

$$\Gamma'(u^*) = \lim_{n\to\infty} \frac{\Gamma(u_n) - \Gamma(u^*)}{u_n - u^*} = 0,$$

contradicting the fact that Γ' is assumed to be non-zero. Since γ is made up of finitely many smooth contours, the same is true of γ.

Now suppose that $\gamma : [a, b] \to D$ is a closed PSC but $\gamma(a) = w$. Then we choose ζ in D, very close to w, but not equal to w, and replace γ by the closed PSC λ which is the line from ζ

to w followed by γ followed by the line from w to ζ. Then $\int_\lambda f(z)dz = \int_\gamma f(z)dz$ and we need only choose σ with w not on σ and $\int_\sigma f(z)dz = \int_\lambda f(z)dz$ for every f analytic on D.

Finally, suppose γ is a cycle made up of finitely many closed PSC $\gamma_k, k = 1, \ldots, n$ in D. For each k we form a closed PSC σ_k in D, with w not on σ_k and $\int_{\sigma_k} f(z)dz = \int_{\gamma_k} f(z)dz$ for every f analytic on D, and σ is the cycle made up of the σ_k.

5.2 Lemma

Let f be analytic on the domain D in $\mathbb{C}$. For z, w in D, we define $G(z,w) = \frac{f(z)-f(w)}{z-w}$ if $z \neq w$, and $G(w,w) = f'(w)$.

Then for each fixed w in D, $q(z) = G(z,w)$ is an analytic function of z on D. Further, for any cycle γ in D, $g(w) = \int_\gamma G(z,w)dz$ is an analytic function of w on D.

Proof: First part: keep w fixed, so that $q(z) = G(z,w)$ is clearly analytic on $D \backslash \{w\}$ and continuous on D. Take a small disc D_1 of centre w contained in D. We have $\int_T q(z)dz = 0$ for every triangular contour T in D_1. So q is the derivative of an analytic function on D_1 and so q is analytic on D_1.

Second part. Fix u in D and consider $g(w)$ for w near u. It may be that u lies on γ, in which case we choose a cycle σ as in the previous lemma such that u does not lie on σ and $\int_\sigma F(z)dz = \int_\gamma F(z)dz$ for every F analytic in D. If u does not lie on γ, just put $\sigma = \gamma$.

Now for w near u, w does not lie on σ and

$$g(w) = \int_\gamma G(z,w)dz = \int_\sigma G(z,w)dz = \int_\sigma \frac{f(z)}{z-w}dz - f(w)\int_\sigma \frac{1}{z-w}dz.$$

However, since u does not lie on σ, both the integrals in the last expression are analytic functions of w for w near u. This is by Rule AFFI. This proves that g is analytic near u.

5.3 A general form of Cauchy's theorem and Cauchy's integral formula

Let f be analytic on the domain D in $\mathbb{C}$, and let γ be a cycle in D such that $n(\gamma, a) = 0$ for all a not in D. Then $\int_\gamma f(z)dz = 0$, and for all w in D but not on γ,

$$n(\gamma, w)f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}dz.$$

Proof: we define $G(z,w)$ and $g(w) = \int_\gamma G(z,w)dz$ as in the previous lemma, and recall that $g(w)$ is analytic on D. We also set $h(w) = \int_\gamma \frac{f(z)}{z-w}dz$. Then by Rule AFFI, h is an analytic function of w in the open set $\mathbb{C} \backslash \{\gamma\}$ (i.e. at all points not on γ). We define a set

$$H = \{w \notin \gamma : n(\gamma, w) = 0\}.$$

This H is open, and we know from the hypothesis that $\mathbb{C} \backslash D \subseteq H$.

Now if w is in $H \cap D$, then

$$g(w) = \int_\gamma \frac{f(z) - f(w)}{z - w} dz = h(w) - f(w) \int_\gamma \frac{dz}{z - w} = h(w).$$

So we define a function $K(w)$ by $K(w) = g(w)$ if w is in D and $K(w) = h(w)$ if w is in H. This function is well-defined, by the previous fact, and is defined for all complex w, since $\mathbb{C} \backslash D \subseteq H$. Moreover, K is an analytic function in $\mathbb{C}$, i.e. K is entire.

We claim that $K(w) \equiv 0$. To see this, note that for $|w|$ large enough, we have $K(w) = h(w)$, because $n(\gamma, w)$ will be 0. Also, suppose that $|z| \leq M$ for all z on γ. Then, if $|w|$ is large enough, we have

$$|h(w)| \leq |\gamma|. \max\{|f(z)| : z \in \gamma\}(|w| - M)^{-1} \to 0$$

as $|w|$ tends to $+\infty$. Thus $K(w)$ is bounded as $w \to \infty$, and so constant, and so identically 0.

So for all w in D but not on γ, we have $g(w) = K(w) = 0$, so that

$$0 = \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f(w)}{z - w} dz = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz - n(\gamma, w)\ f(w).$$

To obtain $\int_\gamma f(z)dz = 0$, choose any point b in D but not on γ, and apply the previous formula to $F(z) = (z - b)f(z)$. We have

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \frac{1}{2\pi i} \int_\gamma \frac{F(z)}{z - b} dz = n(\gamma, b)F(b) = 0.$$

5.4 Simply connected domains

The example $f(z) = 1/z$, $D = C \backslash \{0\}$, γ the circle of radius 1, centre 0, shows that you cannot delete the hypothesis that $n(\gamma, a) = 0$ for all a not in D (here $\int_\gamma f(z)dz = 2\pi i \neq 0$).

A domain D in $\mathbb{C}$ is called SIMPLY CONNECTED if it has the property that for every a not in D and for every closed piecewise smooth contour γ in D, we have $n(\gamma, a) = 0$ (in which case the same is true for all cycles in D).

If D is simply connected and $f : D \to \mathbb{C}$ is analytic and $a, b \in \mathbb{C}$ then the integral of f from a to b through D is independent of which PSC we choose.

We have the following simple criterion for a domain to be simply-connected.

5.5 Theorem

The domain D in $\mathbb{C}$ is simply-connected if the following is true. For all a not in D and for all $R > 0$, there is a point b with $|b| \geq R$ and a path from a to b through $\mathbb{C} \backslash D$. (That is, every point not in D can be joined to points of arbitrarily large modulus by paths which do not pass through D).

It is fairly easy to show that star domains have this property.

Proof: let a lie outside D and let γ be a closed piecewise smooth contour in D. We choose R so large that $n(\gamma, w) = 0$ for all w with $|w| \geq R$ and we join a to such a w by a path $\beta : [A, B] \to \mathbb{C} \setminus D$. Then $n(\gamma, z)$ is constant on β (Chapter 4) and so $n(\gamma, a) = 0$.

5.6 Theorem

Let D be a simply-connected domain, and let f be analytic and non-zero on D. Then there exists an analytic branch of $\log f$ on D, i.e. there exists an analytic function h on D with $e^h \equiv f$ there. Further, if $k \in \mathbb{N}$, there exists g analytic on D such that $g(z)^k = f(z)$ on D.

Proof:

As $f \neq 0$ on D, the function f'/f is analytic on D. We choose a in D, and take any value w such that $e^w = f(a)$, and we set

$$h(z) = w + \int_a^z (f'(u)/f(u))du.$$

Here the integral is along any piecewise smooth contour from a to z through D, and the value of h is not affected by which such contour we choose, by Cauchy's theorem. Now set $H(z) = f(z)\exp(-h(z))$. Then we have $H(a) = 1$, and

$$H'(z) = f'(z)\exp(-h(z)) - f(z)(f'(z)/f(z))\exp(-h(z)) = 0,$$

so $H(z) \equiv 1$ on D.

To form g just set $g(z) = \exp(h(z)/k)$.

Note that if $D = \mathbb{C}\backslash\{x \in \mathbb{R} : x \leq 0\}$ then we defined $L(z) = \mathrm{Log}z = \log|z| + i\mathrm{Arg}z$ in G12CAN. If we take $f(z) = z$ and form h as above, with $a = 1, w = 0$, then $L(z) - h(z)$ is constant on D, since the derivative is 0, and so $L = h$.

6 Sequences and Series of Analytic Functions

6.1 Definition

The functions $f_n(x) = x^n, n \in \mathbb{N}$ are continuous on $[0,1]$. Their (pointwise) limit is $f(x) = \lim_{n\to\infty} f_n(x)$ which is not continuous at 1. We need a stronger condition which will make the limit of continuous functions continuous.

Let (X,d) and (Y,ρ) be metric spaces. Let E be any subset of X and let (f_n) be a sequence of functions and let f be a function, in both cases from E to Y. We say that the sequence (f_n) converges uniformly to f on E if the following is true.

To each $\varepsilon > 0$ corresponds an integer N such that $\rho(f_n(z), f(z)) < \varepsilon$ for all $n \geq N$ and for all z in E.

Now suppose that D is an open set in X. We say that the functions f_n converge locally uniformly to f on D if for each w in D there is an open disc $B_d(w,p) \subseteq D$ with $p > 0$ such that (f_n) converges uniformly to f on $B_d(w,p)$.

6.2 Examples

(i) The functions $f_n(z) = z/n$ on $\mathbb{C}$.

(ii) Let $F(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ be a power series with positive radius of convergence

$$R = \sup\{t \geq 0 : |c_k|t^k \to 0\}.$$

Then the partial sums $F_n(z) = \sum_{k=0}^{n} c_k(z-\alpha)^k$ converge locally uniformly to F on $|z-\alpha| < R$. To see this, take any w with $|w-\alpha| < R$. We can find s,t with $|w-\alpha| < s < t < R$ such that $|c_k|t^k \to 0$ as $k \to \infty$. Since $|c_k|s^k = |c_k|t^k(s/t)^k$ we see that $\sum_{k=0}^{\infty} |c_k|s^k$ converges. If we are given $\varepsilon > 0$, choose N such that $\sum_{k=N+1}^{\infty} |c_k|s^k < \varepsilon$. Then for $|z-\alpha| < s$, and for $n \geq N$, we have

$$|F_n(z) - F(z)| = \left| \sum_{k=n+1}^{\infty} c_k(z-\alpha)^k \right| \leq \sum_{k=n+1}^{\infty} |c_k||z-\alpha|^k \leq \sum_{k=n+1}^{\infty} |c_k|s^k \leq \sum_{k=N+1}^{\infty} |c_k|s^k < \varepsilon.$$

This region $|z-\alpha| < s$ contains an open disc $B(w,q)$ with centre w and $q > 0$.

Note that F will always diverge for $|z-\alpha| > R$ (terms don't tend to 0).

Hence if F converges for $|z-\alpha| < r$ then $R \geq r$ and F converges LU for $|z-\alpha| < r$.
(iii) Unfortunately, in the previous example we cannot always get uniform convergence on $|z-\alpha| < R$, as the example $F_n(z) = \sum_{k=0}^{n} z^k, F(z) = 1/(1-z)$ shows. Here $R = 1$ and $|F_n(z) - F(z)| = \frac{|z|^{n+1}}{|1-z|}$. No matter how large you choose n, this will be very large for z close enough to 1.

(iv) We handle series in negative powers of z as follows: if $G(z) = \sum_{k=1}^{\infty} a_k z^{-k}$ let $F(z) = \sum_{k=1}^{\infty} a_k z^k = G(1/z)$.

If we can find r, s such that $G(z)$ converges for $r < |z| < s$ then $F(z) = G(1/z)$ converges for $r < |1/z| < s$ and so for $1/s < |z| < 1/r$. So F has radius of convergence at least $1/r$, and $F(z)$ converges LU for $|z| < 1/r$, so that $G(z) = F(1/z)$ converges LU for $|1/z| < 1/r$ i.e. $|z| > r$.

(v) Note that if $f_n : E \to \mathbb{C}$ converges uniformly to $f : E \to C$ with respect to the Euclidean metric, then the convergence is also uniform wrt the spherical metric. This is obvious, because $q(z, w) \leq |z - w|$. The converse is false. Let $E = \mathbb{N}$ and for $m, n \in E$ define $f(m) = m, f_n(m) = (1 + 1/n)m$. Then $f_n(n) - f(n) = 1$ so f_n does not converge uniformly to f on E wrt the standard metric. However, $q(f_n(m), f(m)) \leq n^{-1}mm^{-2} \leq n^{-1}$.

6.3 Lemma

Suppose that the complex valued functions f_n converge locally uniformly to f on the open set D in $\mathbb{C}$, with respect to the standard metric.

(a) For every closed and bounded subset E of D, f_n converges uniformly to f on E.

(b) If the f_n are continuous on D, then so is f.

(c) If γ is a piecewise smooth contour in D, and $\phi(z)$ and the $f_n(z)$ are continuous on γ, then $\int_\gamma \phi(z)f(z)dz = \lim_{n\to\infty} \int_\gamma \phi(z)f_n(z)dz$.

Proof: (a) Let E be a closed and bounded subset of D. Suppose that f_n does not tend to f uniformly on E. Then there exist $\varepsilon > 0$ and $n_k \to \infty$ and $z_{n_k} \in E$ such that

$$|f_{n_k}(z_{n_k}) - f(z_{n_k})| \geq \varepsilon.$$

But WLOG z_{n_k} converges, to z^*, and $z^* \in E \subseteq D$ since E is closed. This gives a nbd U of z^* on which $f_n \to f$ uniformly, so that $|f_n(z) - f(z)| < \varepsilon$ for all $n \geq n_0$ and for all z in U. This is a contradiction since $z_{n_k} \in U$ for large k.

(b) Take w in D and an open disc $B_w = B(w, s)$ on which $f_n \to f$ uniformly. Take $\varepsilon > 0$. Then there is some Q such that $|f_n(z) - f(z)| < \varepsilon/3$ for all z in B_w and for all $n \geq Q$. But f_Q is continuous at w so there is some $\delta, 0 < \delta \leq s$, such that $|f_Q(z) - f_Q(w)| < \varepsilon/3$ for $|z - w| < \delta$. Therefore, for such z,

$$|f(z) - f(w)| \leq |f(z) - f_Q(z)| + |f_Q(z) - f_Q(w)| + |f_Q(w) - f(w)| < \varepsilon.$$

To prove part (c), just note that the set of points on γ is a closed and bounded subset of D. Thus

$$\left| \int_\gamma \phi(z)(f_n(z) - f(z))dz \right| \leq |\gamma| \max\{|\phi(z)| : z \in \gamma\} \sup\{|f_n(z) - f(z)| : z \in \gamma\},$$

and the last term tends to 0 as $n \to \infty$, as $f_n \to f$ uniformly on γ.

6.4 Weierstrass' theorem

Suppose that f_n are functions analytic on a domain D in $\mathbb{C}$, converging locally uniformly to a function f on D, with respect to the standard metric. Then f is analytic on D, and f_n' converges locally uniformly to f' on D.

Proof: We know that f is continuous on D. Take w in D and $s > 0$ such that $B(w, 4s) \subseteq D$. With all integrations done once counter-clockwise, we have, in $|u - w| < 2s$,

$$f(u) = \lim_{n\to\infty} f_n(u) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{|z-w|=2s} \frac{f_n(z)}{z-u} dz = \frac{1}{2\pi i} \int_{|z-w|=2s} \frac{f(z)}{z-u} dz.$$

But Rule AFFI tells us that the last integral defines an analytic function of u in $|u - w| < 2s$, and this proves that f is analytic on D. Now, for $|u - w| < s$, we have

$$|f'(u) - f_n'(u)| = \left| \frac{1}{2\pi i} \int_{|z-w|=2s} \frac{f(z) - f_n(z)}{(z-u)^2} dz \right|.$$

Since $|z - u| \geq s$, this is at most

$$2s^{-1} \sup\{|f_n(z) - f(z)| : |z - w| = 2s\} \to 0$$

as $n \to \infty$.

In particular, a power series F of centre α with radius of convergence $R > 0$ is analytic in $|z - \alpha| < R$, because the partials sums are polynomials and so analytic, and converge LU to F.

Simple examples show that there is no analogue of Weierstrass' theorem for differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

We now give a fairly straightforward proof of Laurent's theorem, and will deduce from it Taylor's theorem.

6.5 Laurent's theorem

Suppose that $f(z)$ *is analytic (taking values in* $\mathbb{C}$*) in the annulus* $A = \{z \in \mathbb{C} : R < |z-\alpha| < S\}$*, where* $0 \le R < S \le \infty$*. Then there are constants* a_j *such that for all* z *in* A*,*

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-\alpha)^j = \sum_{j=0}^{\infty} a_j (z-\alpha)^j + \sum_{j=1}^{\infty} a_{-j} (z-\alpha)^{-j}. \tag{2}$$

Both series converge locally uniformly on A *(i.e. the partial sums converge locally uniformly on* A*). Moreover, for any* T *with* $R < T < S$*, we have*

$$a_j = \frac{1}{2\pi i} \int f(z)(z-\alpha)^{-j-1} dz, \tag{3}$$

where the integration is once counter-clockwise around $|z-\alpha| = T$*.*

For the proof, we may assume that $\alpha = 0$, by otherwise considering the function $g(z) = f(z+\alpha)$ for $R < |z| < S$. If $g(z) = \sum a_k z^k$ then $f(z) = \sum a_k (z-\alpha)^k$.

Fix $z_0 \in A$ and T with $R < T < S$, and take r and s with $R < r < |z_0| < s < S$ and $R < r < T < s < S$. We shall prove that there exists a series representation for $f(z)$ on $r < |z| < s$. To do this let γ_1 be the circle $|u| = r$ described once CLOCKWISE, with γ_2 the circle $|u| = s$ described once counter-clockwise. If Γ is the cycle made up of γ_1 and γ_2, then we have $n(\Gamma, z) = 0 + 1 = 1$ for $r < |z| < s$, while if a lies outside the annulus A, then $n(\Gamma, a) = 0$, since $n(\Gamma, a) = 0 + 0$ if $|a| \ge S$ and $n(\Gamma, a) = 1 - 1$ if $|a| \le R$. So Cauchy's integral formula gives

$$2\pi i f(z) = \int_{\gamma_1} \frac{f(u)}{u-z} du + \int_{\gamma_2} \frac{f(u)}{u-z} du$$

for $r < |z| < s$. Now on γ_1 we have $|u| = r < |z|$ and, setting $w = u/z$,

$$1/(u-z) = -1/z(1-w) = \lim_{n\to\infty} \left[\sum_{k=0}^{n} -w^k z^{-1} \right],$$

with $|w| = r/|z| < 1$. The convergence is uniform so

$$\int_{\gamma_1} \frac{f(u)}{u-z} du = \lim_{n\to\infty} \left(\sum_{k=0}^{n} \int_{\gamma_1} \frac{-f(u)u^k}{z^{k+1}} du \right)$$

which equals (now integrating once counter-clockwise)

$$\sum_{k=0}^{\infty} z^{-k-1} \int_{|u|=r} f(u) u^k du = \sum_{j=1}^{\infty} z^{-j} \int_{|u|=r} f(u) u^{j-1} du = G(z),$$

where

$$G(z) := \sum_{j=1}^{\infty} z^{-j} 2\pi i A_{-j}.$$

This series $G(z)$ converges for $r < |z| < s$, and so converges LU for $|z| > r$ (by Examples 6.2, (iv)).

Similarly, on $|u| = s > |z|$ we can write $1/(u-z) = 1/u(1-z/u) = \sum_{k=0}^{\infty} z^k/u^{k+1}$ with the partial sums converging uniformly on $|u| = s$. We thus get

$$\int_{\gamma_2} \frac{f(u)}{u-z} du = \int_{\gamma_2} f(u) \sum_{k=0}^{\infty} z^k/u^{k+1} du =$$

$$= \sum_{k=0}^{\infty} \left(z^k \int_{\gamma_2} \frac{f(u)}{u^{k+1}} du \right) = F(z), \quad F(z) := \sum_{k=0}^{\infty} z^k 2\pi i A_k.$$

Here $F(z)$ has radius of convergence at least s and so converges LU for $|z| < s$.

This gives us a series

$$f(z) = (G(z) + F(z))/2\pi i = \sum A_k z^k, \quad r < |z| < s.$$

The series converges LU, and in particular uniformly on a neighbourhood of z_0.

Finally taking this series and integrating $f(z) z^{-j-1}$ once counter-clockwise around $|z| = T$ gives $A_j = a_j$, where a_j is given by (3). This is because the integral of z^{k-j-1} around $|z| = T$ is 0 unless $k - j - 1 = -1$ i.e. $k = j$, in which case it is $2\pi i$. Thus $f(z)$ is given on A by the LU convergent expansion (2), in which a_j satisfies (3).

6.6 Some Special Cases

We keep the same notation as in Laurent's theorem.

(a) Suppose that f is in fact analytic in $|z-\alpha| < S$, and we form the Laurent series of f on $R < |z-\alpha| < S$. Then for $j < 0$, the function $f(z)(z-\alpha)^{-j-1}$ is analytic in $|z-\alpha| < S$, and so $a_j = 0$. Further, if $j \geq 0$,

$$a_j = \frac{1}{2\pi i} \int_{|z-\alpha|=T} \frac{f(z)}{(z-\alpha)^{j+1}} dz = \frac{f^{(j)}(\alpha)}{j!}.$$

Thus the Laurent series of f is the Taylor series about α, and equals $f(z)$ for $|z-\alpha| < S$. This is Taylor's theorem.

(b) Conversely, suppose that you calculate the Laurent series of f in $R < |z-\alpha| < S$ and it turns out that $a_j = 0$ for $j < 0$. Then $f(z)$ is equal to a power series in A, and this power series has radius of convergence at least S, which means that f extends to be analytic in $|z-\alpha| < S$. If, in this case, $R = 0$ we say that α is a REMOVABLE SINGULARITY of f.

(c) Suppose that $R = 0$ and that f is bounded as $z \to \alpha$ say $|f(z)| \leq M$ for $0 < |z-\alpha| < T_0$. Then for $j < 0$ and $0 < T < T_0$ we have $|a_j| \leq (1/2\pi)(2\pi T)MT^{-j-1} = MT^{-j} \to 0$ as $T \to 0$. So $a_j = 0$ for $j < 0$ and we have case (b). This is sometimes called Riemann's extension theorem.

(d) Suppose that $R = 0$ and that $a_j \neq 0$ for some, but only finitely many, negative j. Let Q be the greatest integer such that $a_{-Q} \neq 0$. Then $Q > 0$ and we can write, in A,

$$f(z) = (z-\alpha)^{-Q}(a_{-Q} + a_{-Q+1}(z-\alpha) + \ldots) = (z-\alpha)^{-Q}h(z).$$

Here $h(z)$ is a power series, and converges for $0 < |z-\alpha| < S$ and so is analytic in $|z-\alpha| < S$. Further, $h(\alpha) = a_{-Q} \neq 0$, and as $|z-\alpha| \to 0$, we see that $|f(z)| \to +\infty$. We say that f has a POLE at α, of order (also called multiplicity) Q.

(e) Suppose that f is analytic in $|z-\alpha| < S$, with $f(\alpha) = 0$, and that f is not identically zero in $|z-\alpha| < S$. Then there must be a first non-zero coefficient in the Taylor series of f about α, which we denote by a_p. For $0 < |z-\alpha| < S$, we can write

$$f(z) = (z-\alpha)^p(a_p + a_{p+1}(z-\alpha) + ...) = (z-\alpha)^p h(z)$$

with h analytic in $|z-\alpha| < S$. We say that f has a ZERO OF ORDER p at α. Now for z close to, but not equal to, α we have $h(z) \neq 0$ and

$1/f(z) = (z-\alpha)^{-p}h(z)^{-1} = (z-\alpha)^{-p}$ (Taylor series of $1/h$ about α) $=$
$(z-\alpha)^{-p}(1/h(\alpha) + \ldots)$.

We see that $1/f$ has a pole of order p at α.

(f) If $R = 0$ and $|f(z)| \to +\infty$ as $z \to \alpha$ it is easy to see that $1/f$ is analytic in some annulus $0 < |z-\alpha| < s$ and tends to 0 as $z \to \alpha$. Thus $1/f$ has a removable singularity at α which must be a zero of $1/f$, so f has a pole.

(g) If $R = 0$ and $a_j \neq 0$ for infinitely many negative j, we say that f has an essential singularity at α. In this case f cannot have a finite limit as $z \to \alpha$, and $|f|$ cannot have limit $+\infty$ as $z \to \alpha$.

(h) Finally, suppose that $\alpha = 0$ and $S = +\infty$. We can describe the behaviour of f 'at infinity' as follows. We just look at the function $f(1/z) = g(z)$, which is analytic for $0 < |z| < 1/R$. We say that f has a pole, removable or essential singularity at infinity according to what kind of singularity g has at 0.

6.7 Definition

We say that a complex-valued function f has an isolated singularity at $w \in \mathbb{C}^*$ if f is analytic in a punctured disc $\{z : 0 < |z - w| < s\}$. If $w = \infty$ we replace this by $\{z : s < |z| < +\infty\}$. The singularity is then removable/ a pole/essential according to whether f has a finite/infinite/non-existent limit as $z \to w$.

We also say that a function f is analytic apart from isolated singularities in a domain $D \subseteq \mathbb{C}$ if for every $a \in D$ either f is analytic at a or a is an isolated singularity. So in either case there is some $r > 0$ such that f is analytic on $0 < |z - a| < r$ so there cannot exist a sequence z_n of isolated singularities tending to a. There can exist, however, isolated singularities tending to $b \in \partial D$: this is the case for $1/\sin(1/z)$ on $\mathbb{C} \setminus \{0\}$ (poles at $1/n\pi \to 0$).

6.8 The residue theorem

Let D be a domain in $\mathbb{C}$ and let f be analytic on D, apart from isolated singularities z_j. Let γ be a cycle in D such that no z_j lies on γ and such that $n(\gamma, a) = 0$ for all $a \in \mathbb{C} \backslash D$ (this is the same condition as in the general Cauchy theorem 5.3). Then there are just finitely many z_j for which $n(\gamma, z_j) \neq 0$, and

$$\int_\gamma f(z)dz = 2\pi i \sum n(\gamma, z_j)\mathrm{Res}(f, z_j).$$

Here the residue is the coefficient of $(z - z_j)^{-1}$ in the Laurent series of f valid near z_j.

Proof. Suppose we have $n(\gamma, z_j) \neq 0$ for infinitely many distinct z_j, WLOG $j = 1, 2, 3, \ldots$. The sequence (z_j) is bounded, since $n(\gamma, w) = 0$ for $|w|$ sufficiently large. So WLOG $z_j \to \alpha$. Then $\alpha \in D$, because otherwise $n(\gamma, \alpha) = 0$ and $n(\gamma, z_j) = 0$ for all sufficiently large j (the winding number is constant on a disc centred at a). But $\alpha \in D$ is also impossible, because neither a point at which f is analytic nor an isolated singularity can be a limit point of z_j.

For those finitely many z_j with $n(\gamma, z_j) \neq 0$, choose r positive but so small that the discs $B(z_j, 2r)$ all lie in D, do not intersect each other or γ, and each contain no singularity of f other than z_j itself. We then have, integrating once counter-clockwise,

$$\int_{|z-z_j|=r} f(z)dz = 2\pi i\mathrm{Res}(f, z_j).$$

Let ρ_j be the circle $|z - z_j| = r$, described $n(\gamma, z_j)$ times clockwise, and let Γ be the cycle made up of γ and these (finitely many) ρ_j.

Let D_1 be D with all the singularities of f deleted. Then D_1 is a domain. To see this, D_1 is open since the z_j are each isolated, and D_1 is a domain, because if a path in D passes through a z_j we can deform it slightly to miss z_j. Also, we have $n(\Gamma, a) = 0$ for all $a \in \mathbb{C} \backslash D_1$. This is because if $a \notin D$ then $n(\gamma, a) = 0$ and a lies outside each ρ_j. Also if z_m is one of the isolated singularities then either $n(\gamma, z_m) = 0$ and z_m lies outside all the ρ_j, or $n(\gamma, z_m)$ and $n(\rho_m, z_m)$ cancel each other out.

Since f is analytic on D_1, Cauchy's theorem gives

$$0 = \int_\Gamma f(z)dz = \int_\gamma f(z)dz - \sum n(\gamma, z_j)2\pi i\mathrm{Res}(f, z_j).$$

6.9 Meromorphic functions

Let D be a domain in $\mathbb{C}$. A function $f : D \to \mathbb{C} \cup \{\infty\}$ is called MEROMORPHIC on D if it has the following property. For every a in D, either f is analytic at a (and so analytic, taking values in $\mathbb{C}$, on an open disc containing a), or a is an isolated singularity of f which is a pole, and $f(a) = \infty$. Meromorphic functions are continuous with respect to the chordal q metric.

Obviously, if f is analytic on D, then f is meromorphic on D.

For example, from the function $z/(e^z - 1)$ we can make a function meromorphic on $\mathbb{C}$. Note that for this function ∞ is a non-isolated singularity (limit of poles) and may be thought of as essential, since $f(z)$ has no limit as $z \to \infty$.

On the other hand

$$\sin(1/z) = 1/z - 1/3!z^3 + \ldots, \quad 0 < |z| < +\infty,$$

is not meromorphic on $\mathbb{C}$, because the singularity at 0 is essential.

7 The local behaviour of meromorphic functions

If $f : D \to \mathbb{C}^*$ is meromorphic, the set $\{z \in D : f(z) = \infty\}$ can't have a limit point in D. This is because each $a \in D$ has $r > 0$ with f analytic on $0 < |z - a| < r$, this open annulus containing no poles.

7.1 The identity theorem

Suppose that f is meromorphic on a domain D in $\mathbb{C}$. Suppose that $b \in \mathbb{C}$. Then either $f(z) \equiv b$ on D, or the set $E = \{z \in D : f(z) = b\}$ has no limit point in D.

Proof: WLOG $b = 0$. Let F be the set of points in D which are limit points of E, and let $G = D \backslash F$. Then for any w in G there is some $\rho > 0$ such that $f(z) \neq 0$ on $U_w = \{z : 0 < |z - w| < \rho\}$ (by definition of not being a limit point). But then no point in U_w is a limit point of E, and we see that G is open.

Now suppose $\alpha \in F$. Then there are points $z_n \to \alpha$ as $n \to \infty$ with $z_n \neq \alpha$ and $f(z_n) = 0$; so $f(\alpha) = 0$, and $F \subseteq E$. We look at the Taylor series of f about α, which equals f on a neighbourhood V of α. If this series is not identically zero, we can write, near α, the identity $f(z) = (z - \alpha)^p h(z)$ with p a positive integer and h analytic on V such that $h(\alpha) \neq 0$. But then $h(z) \neq 0$ for z close enough to α, which contradicts $f(z_n) = 0$. So $f(z) \equiv 0$ on V, and $V \subseteq F$. So F is open.

Since D is a domain, either F or G must be empty (see Chapter 1). If $G = \emptyset$ then $D = F$ and we've already seen that $F \subseteq E$.

Note that:

(i) Limit points on ∂D are allowed: e.g. $f(z) = \sin(1/z)$ is analytic on $H = \mathbb{C} \setminus \{0\}$, and 0 is a limit point of zeros $z = 1/\pi n$ of f. Similarly 0 is a limit point of poles of $1/f$.

(ii) Let $E \subseteq D$ be a closed and bounded set. Then a meromorphic function f on D can't have infinitely many poles in E, and if $f \not\equiv 0$ then f cannot be 0 at infinitely many points of E. This is because if E is a closed and bounded subset of $\mathbb{C}$ then any infinite subset H of E has a limit point in E. To see this, take distinct z_j in H. Then Bolzano-Weierstrass tells us that this bounded sequence (z_j) has a convergent subsequence, and the limit α is in E, since E is closed. Thus α is a limit point of H.

(iii) Examples show that the conclusion of the identity theorem isn't true without the analyticity: consider $z - 1/\overline{z}$.

7.2 The functions meromorphic on a given domain form a *differential field*

More precisely: let f and g be meromorphic on a domain D in $\mathbb{C}$. Then so are $f + g, fg, f'$. If $f \not\equiv 0$ then $1/f$ is also meromorphic on D.

Proof. Take any $a \in D$. Then both f and g are analytic in some annulus (punctured disc) U given by $0 < |z-a| < r$. Each has a Laurent series in U and these have at most finitely many negative powers of $z - a$.

Suppose first that $f \equiv 0$ on U. Then $f \equiv 0$ on D by the identity theorem. In particular $f + g \equiv g$ on D, and $fg \equiv 0$ on U, and setting $(fg)(a) = 0$ extends fg to be analytic at a, and doing this for every a shows that $fg \equiv 0$ on D.

The same works if $g \equiv 0$ on U.

Assume henceforth that neither f nor g is identically 0 on D. Then neither f nor g is identically 0 on U. Thus the Laurent series of each in U has at least one term. Adding these Laurent series shows that $f + g$, which is analytic in U, can have at worst a pole at a.

To examine fg, factor out the lowest power of $z - a$ occurring in each Laurent series to get $f(z) = (z-a)^m h(z)$ and $g(z) = (z-a)^n H(z)$ with m, n integers and h, H analytic and non-zero near a. Then near a we have $f(z)g(z)$ equal to $(z-a)^{m+n}$ multiplied by the Taylor series about a of hH. Thus if $m+n > 0$, then $(fg)(a) = 0$, while if $m+n < 0$, then $(fg)(a) = \infty$.

Next, f' is meromorphic because we can differentiate the Laurent series of f term by term in U, and this is valid by Weierstrass' theorem. We get a series for f' with at most finitely many negative powers of $z - a$.

Finally, write $1/f(z) = (z-a)^{-m} h(z)^{-1}$, and expand out the series for $1/h$. This shows that $1/f$ has a pole at a if $m > 0$ and is analytic at a if $m \leq 0$.

In particular, if f_1, f_2 are analytic on D and $f_2 \not\equiv 0$ then $f = f_1/f_2$ becomes meromorphic on D. Mittag-Leffler's theorem (not proved or applied in G14CAN) tells us that every function f meromorphic on D has such a representation.

7.3 Theorem

Let $f : \mathbb{C} \to \mathbb{C}^*$ be meromorphic. Then $\lim_{z\to\infty} f(z)$ exists iff f is a rational function..

Proof. The 'if' part is easy. Every rational function $f(z) = P(z)/Q(z)$ (with P, Q polys in z, $Q \not\equiv 0$) is meromorphic in $\mathbb{C}$ and has a limit as $z \to \infty$.

Now suppose the limit exists. Assume first that $\lim_{z\to\infty} f(z)$ is finite. Then WLOG this limit is 0. So there exists $R > 0$ such that $f(z) \neq \infty$ for $|z| > R$. Now f has only finitely many poles in $|z| \leq R$. If b is a pole of multiplicity q, let $R_b(z) = \sum_{j=1}^{q} b_{-j}(z-b)^{-j}$ be the negative powers in the Laurent series of f valid in some $0 < |z-b| < S_b$. Do this for all such poles $b_1, \ldots, b_n$. Then $g(z) = f(z) - \sum_{k=1}^{n} R_{b_k}(z)$ is analytic in $\mathbb{C}$ and $\to 0$ as $z \to \infty$. So $g \equiv 0$, by Liouville's theorem.

Finally, if $f(z) \to \infty$ as $z \to \infty$ then $f \not\equiv 0$, and so $1/f$ is meromorphic in $\mathbb{C}$, and $1/f$ is rational.

7.4 The argument principle

Let D be a domain in $\mathbb{C}$ and let γ be a cycle in D such that $n(\gamma, a) = 0$ for all a in $\mathbb{C}\backslash D$. Let f be meromorphic on D and satisfy $f(z) \neq 0, \infty$ on γ. Then we have $n(\gamma, u) = 0$ for all but finitely many zeros and poles u of f, and

$$\frac{1}{2\pi i}\int_\gamma f'(z)/f(z)dz = \sum_{j=1}^{m} \mu(z_j)n(\gamma, z_j) - \sum_{k=1}^{n} \mu(w_k)n(\gamma, w_k),$$

in which the sum is over those finitely many zeros $z_1, \ldots, z_m$ and poles $w_1, \ldots, w_n$ for which the winding number is non-zero, and $\mu(u)$ denotes the multiplicity (order) of a zero or pole u.

Proof. f'/f is meromorphic, since f' is, and $f \not\equiv 0$. Also, f'/f can have singularities only at the zeros and poles of f. Suppose a is a zero or pole of f. Laurent's theorem lets us write, in $0 < |z - a < r$, say, $f(z) = (z - a)^p h(z)$, with p an integer and h analytic at a, having $h(a) \neq 0$. We get

$f'(z)/f(z) = p/(z-a) + h'(z)/h(z) = p/(z-a)+$ Taylor series about a of h'/h.

Hence $\mathrm{Res}(f'/f, a) = p$.

The result now follows by the residue theorem.

The name arises because f'/f is (locally) the derivative of $\log f = \log|f| + i\arg f$. As we go around γ, the real part $\log|f|$ is continuous, but $\arg f$ can change: for example $\arg z^n$ on $|z| = 1$.

7.5 A special case

Suppose that γ is a circle, described once counter-clockwise, and f is meromorphic on a plane domain containing γ and its interior, with no zeros or poles of f on γ. Then

$$\frac{1}{2\pi i}\int_\gamma f'(z)/f(z)dz$$

is equal to the number of zeros minus the number of poles of f inside γ, these zeros and poles counted according to multiplicity.

This has a nice application to zeros of polynomials.

7.6 Rouché's theorem

Suppose that f and g are meromorphic in a domain D in $\mathbb{C}$ containing the circle C and its interior, and suppose that on C we have $|g(z)| < |f(z)| < \infty$. Then the number of zeros minus the number of poles inside C, counting multiplicity, is the same for $f + g$ as it is for f.

Proof It suffices to show that

$$\int_\gamma f'(z)/f(z)dz = \int_\gamma (f'(z)+g'(z))/(f(z)+g(z))dz$$

and so that

$$\int_\gamma h'(z)/h(z)dz = 0,$$

in which $h(z) = (f(z)+g(z))/f(z)$. Now h is analytic on C, and since

$$h(z) = 1 + g(z)/f(z)$$

we see that h has positive real part on and near C. So the principal logarithm Log is such that $H(z) = \text{Log } h(z)$ is defined on and near C, so that

$$\int_\gamma h'(z)/h(z)dz = \int_\gamma H'(z)dz = 0.$$

This is useful in applications e.g. $4z^2 = e^{iz}$ has a solution in $|z| < 1$.

7.7 Open mapping theorem

Suppose that f is meromorphic and non-constant on the domain G in $\mathbb{C}$, and suppose D is an open subset of G. Then $f(D)$ is an open subset of $\mathbb{C}^*$.

Proof: take $a \in D$ and suppose first that $f(a) \in \mathbb{C}$. We know that a is not a limit point of zeros of $f(z) - f(a)$ and so there is some $\rho > 0$ such that $B(a, 2\rho) \subseteq D$ and $f(z) \neq f(a)$ in $0 < |z-a| < 2\rho$. Let t be the (positive) minimum of $|f(z) - f(a)|$ on the circle C given by $|z-a| = \rho$.

Suppose that $|w - f(a)| < t$. Then on C we have $|f(z) - f(a)| > |f(a) - w|$ and so the previous theorem tells us that $f(z) - w = f(z) - f(a) + f(a) - w$ has a zero inside C. Thus $B(f(a), t) \subseteq f(D)$ as required.

If a is a pole, look at $1/f$, which is analytic near a. Thus $(1/f)(D)$ contains a neighbourhood of 0 and so $f(D)$ contains a neighbourhood of ∞.

7.8 Corollary

If f is analytic and one-one on the domain D in $\mathbb{C}$, the inverse function f^{-1} is continuous on $f(D)$.

Proof: let $g = f^{-1}$, and let $g(b) = c$ and $\varepsilon > 0$. Then $U = D \cap B(c, \varepsilon)$ is open, so, as we have just seen, $g^{-1}(U)$ is open. But $b \in g^{-1}(U)$, so there is some $\delta > 0$ such that $B(b, \delta) \subseteq g^{-1}(U)$. So $|w - b| < \delta$ gives $w \in g^{-1}(U)$ and so $g(w) \in U$ and so $|g(w) - g(b)| < \varepsilon$.

7.9 Theorem

Let f be analytic at a. Then f is one-one on some neighbourhood of a iff $f'(a) \neq 0$. If f is meromorphic with a pole at a, then f is one-one on a neighbourhood of a iff a is a simple pole (multiplicity 1).

Proof: suppose first that $f(a) = 0$ and $a = 0$. Suppose that $f'(0) \neq 0$. Put $g(z) = f(z)/f'(0)$. Then for $|z|$ small we have $g'(z) = 1 + h(z)$, where h is analytic near 0 and $h(0) = 0$. This means that, for z, w close enough to 0, with $z \neq w$, we have $|h(u)| \leq 1/4$ on the line segment from z to w and so

$$|g(z) - g(w) - z - w| = |\int_w^z h(u)du| \leq (1/4)|z - w|$$

which forces $g(z) \neq g(w)$.

Now suppose $f'(0) = 0$, and suppose that f is one-one on a neighbourhood U of 0. By Taylor's theorem there is some $k > 0$ such that we can write

$$f(z) = z^k h(z),$$

where h is analytic and non-zero at 0. Thus, near 0, we have $h(z) = V(z)^k$, with V analytic at 0. Hence $f(z) = (zV(z))^k$. Let t be small and positive. By the open mapping theorem, $zV(z)$ takes all k values

$$t\exp(2\pi ij/k), j = 0, 1, \ldots, k - 1$$

in U, say at $z_1, \ldots, z_k$. But then the k values $f(z_j)$ are the same, contradicting the assumption that f is one-one near 0.

This handles the case where f is analytic at a and, in the case of a pole, we look at $1/f$.

7.10 Theorem

Suppose that f is analytic and one-one on the domain D in $\mathbb{C}$. Then the inverse function $g = f^{-1}$ is analytic on $f(D)$.

Proof: note that $f(D)$ is a domain (why?). We also know that $f' \neq 0$ on D. Let $a \in D$ and set $b = f(a)$. If w_n is any sequence tending to b, but never equal to b, then setting $z_n = g(w_n)$, we see that $z_n \to a$, as g is continuous, and

$$\frac{g(w_n) - g(b)}{w_n - b} = \frac{z_n - a}{f(z_n) - f(a)} \to 1/f'(a)$$

as $n \to \infty$, so that $g'(b) = 1/f'(a)$.

7.11 The maximum principle

We have:

(i) If f is a non-constant analytic function on the domain D in $\mathbb{C}$, then $|f(z)|$ has no maximum in D.

(ii) If g is analytic on the bounded domain G and continuous on $E = G \cup \partial G$ then there exists $w \in \partial G$ such that $|g(z)| \leq |g(w)|$ for all $z \in E$.

Proof: (i) Suppose that $a \in D$. Then $f(D)$, being open, contains a neighbourhood of $f(a)$ and so a point of larger modulus.

(ii) Since $|g|$ is continuous on E, which is closed and bounded, $|g|$ has a maximum at some point w in E. To find it, take a sequence $w_n \in E$ such that

$$|g(w_n)| \to Y = \sup\{|g(z)| : z \in E\}$$

and a convergent subsequence (still called w_n) tending to $w \in E$. Then we get $|g(w)| = Y$. If $w \in G$ then g is constant on G and so constant on E.

Note that we now get a second version of Rouché's theorem as follows. If f and g are analytic on a bounded domain D and on its boundary, and $|f(z)| > 2|g(z)|$ on ∂D, and if f has a zero u in D, then so has $f + g$. To see this, assume that $f + g \neq 0$ on D, and look at $v(z) = g(z)/(f(z) + g(z))$. On the boundary of D we have $|v(z)| < 1$, and so $\max\{|v(z)| : z \in \partial D\} < 1$. Thus $|v(z)| < 1$ in D, which contradicts the fact that $v(u) = 1$.

7.12 Hurwitz' Theorem

Suppose that as $n \to \infty$ the analytic functions f_n converge locally uniformly on a domain D in $\mathbb{C}$ to the analytic function f. Then either f is constant on D or the following is true. If $a \in D$ and $\rho > 0$ then for all sufficiently large n, the function f_n takes the value $f(a)$ at some point in $B(a, \rho)$.

Remark: it follows that if the f_n are one-one then f is either one-one or constant, for otherwise if $f(a) = f(b)$ we could take disjoint neighbourhoods $B(a, \rho), B(b, \rho)$ and f_n would have to take the value $f(a)$ in both, for n large enough.

Proof of the Theorem: assuming that f is non-constant, we can take $s < \rho$ such that $|f(z) - f(a)| \geq t > 0$ on $|z - a| = s$. If n is large enough, we have $2|f_n(z) - f(z)| < t \leq |f(z) - f(a)|$ on $|z - a| = s$, and Rouché's theorem now says that $f - f(a) + (f_n - f) = f_n - f(a)$ has a zero inside $|z - a| = s$.

8 The Riemann Mapping Theorem

This is one of the classic theorems of complex analysis - it says that any simply-connected domain D in $\mathbb{C}$ with $D \neq \mathbb{C}$ is the image of the unit disc $B(0,1)$ under some one-one analytic function. We start with a simple but very useful result.

8.1 Schwarz' lemma

Let $\Delta = B(0,1)$. Let $f : \Delta \to \Delta$ be analytic, with $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \Delta$. If $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in \Delta \setminus \{0\}$ then $f(z) \equiv e^{i\alpha} z$ for some real α.

Proof: since $f(0) = 0$ the function $g(z) = f(z)/z, g(0) = f'(0)$, has a removable singularity at 0 and is analytic in Δ. We claim that $|g(z)| \leq 1$ for all $z \in \Delta$.

For, suppose that $|g(\beta)| = t > 1$ for some $\beta \in \Delta$. Put $r_n = 1 - 1/n$. Then for n large enough, part (ii) of the maximum principle, applied to $G = B(0, r_n)$, gives points z_n with $|z_n| = r_n$ such that $|g(z_n)| \geq t$. But this gives $|f(z_n)| \geq r_n t > 1$ if n is large enough.

This proves the claim, and now there are two possibilities. Either g is a constant of modulus 1, or by (i) of the maximum principle we have $|g(z)| < 1$ for all $z \in \Delta$.

Next we need a sort of 'Bolzano-Weierstrass theorem' for sequences of analytic functions.

8.2 Arzela-Ascoli theorem

Let D be any domain in $\mathbb{C}$, and let $(f_n), n \geq 1$ be a sequence of functions analytic on D, such that these functions are uniformly bounded on D. This means that there exists $M > 0$ such that $|f_n(z)| \leq M$ for all z in D and for all $n \geq 1$. Then there exist a subsequence f_{n_k} and an analytic function f such that, as $k \to \infty$, the functions f_{n_k} tend to f locally uniformly (LU) on D.

Proof: we take all the points in D with rational real and imaginary parts. These are countable, so we denote them by $\zeta_1, \zeta_2, \ldots$. Now the sequence $f_n(\zeta_1), n = 1, 2, \ldots$ is bounded, and so there is a subsequence $f_{11}, f_{12}, f_{13}, \ldots$ of f_n such that as $n \to \infty$, the sequence $f_{1n}(\zeta_1)$ converges.

Now this sequence f_{1n} is such that $f_{1n}(\zeta_2)$ is a bounded sequence, and so we can find a further subsequence $f_{2n}, n \geq 1$, such that $f_{2n}(\zeta_2)$ converges (as $n \to \infty$). Also $f_{2n}(\zeta_1)$ converges, because it is a subsequence of $f_{1n}(\zeta_1)$. We keep repeating this.

We form subsequences $f_{kn}, k \geq 1, n \geq 1$, such that, for each k, the sequence $f_{k1}, f_{k2}, f_{k3}, \ldots$ is a subsequence of $f_{(k-1)1}, f_{(k-1)2}, f_{(k-1)3}, \ldots$, and such that as $n \to \infty$, the sequence $f_{kn}(\zeta_k)$ converges. Now if $p \leq k$, we find that $f_{kn}(\zeta_p)$ converges as $n \to \infty$, because it is a subsequence of $f_{pn}(\zeta_p)$. We get a list of lists:

$f_{11}, f_{12}, f_{13}, f_{14}, \ldots$

$f_{21}, f_{22}, f_{23}, f_{24}, \ldots$

$f_{31}, f_{32}, f_{33}, f_{34}, \ldots$

$f_{41}, f_{42}, f_{43}, f_{44}, \ldots$

Here each row is a subsequence of each previous one. Now we put $g_n = f_{nn}$, thus 'going down the diagonal'.

For each k, we claim that $f_{kk}, f_{(k+1)(k+1)}, f_{(k+2)(k+2)}, \ldots$ is a subsequence of $f_{k1}, f_{k2}, f_{k3}, \ldots$. To see this, let $k \leq r < s$. Then f_{ss} is element s of row s, which is a subsequence of row r. Thus row s is obtained by deleting some elements of row r and this forces $f_{ss} = f_{rx}$ for some $x \geq s > r$. But row r is a subsequence of row k, and so $f_{rr} = f_{kv}, f_{ss} = f_{rx} = f_{kw}$, with $r \leq v < w$. Thus f_{rr} and f_{ss} both occur in row k, and f_{rr} occurs first. This proves our claim.

Thus for each k, the sequence $g_n(\zeta_k) = f_{nn}(\zeta_k)$ converges as $n \to \infty$. We set $f(\zeta_k) = \lim_{n\to\infty} g_n(\zeta_k)$. Note that $|f(\zeta_k)| \leq M$ for all k.

So we have defined f at all the points with rational real and imaginary parts, and we need to extend f to all of D. With this aim in mind, we take any w in D, and any $s > 0$ such that $B(w, s) \subseteq D$.

Claim 1: for $z, z' \in B(w, s/4)$ we have $|g_n(z') - g_n(z)| \leq (6M/s)|z - z'| \quad \forall n \in \mathbb{N}$.

To see this, take such a z, and set $h(u) = (g_n(z + us/2) - g_n(z))/3M$. Setting $z' = z + us/2$, we find that $|u| < 1$ iff $|z' - z| < s/2$, and the latter is true if $z' \in B(w, s/4)$.In particular, since

$|g_n| \leq M$, the function h is analytic for $|u| < 1$ with $|h(u)| < 1$ and $h(0) = 0$. So Schwarz' lemma gives $|h(u)| \leq |u|$, and so $|g_n(z') - g_n(z)| \leq 3M|u| \leq (6M/s)|z - z'|$ for $z' \in B(w, s/4)$.

Claim 2: for $\zeta_k, \zeta_j \in B(w, s/4)$ we have $|f(\zeta_k) - f(\zeta_j)| \leq (6M/s)|\zeta_k - \zeta_j|$.

To see this, choose n so large that $g_n(\zeta_k) - f(\zeta_k)$ and $g_n(\zeta_j) - f(\zeta_j)$ are both very small, and apply Claim 1 to g_n.

Claim 3: the function f can be defined on $B(w, s/4)$ (and hence on all of D, since w is arbitrary) as follows. Let (ϕ_p) be any sequence of ζ_j points which converges to z. The sequence $f(\phi_p)$ is bounded, and so has a convergent subsequence, and we define $f(z)$ to be the limit of this subsequence.

It is clear that there is such a subsequence, but we have to be careful that different choices of (ϕ_p) and different choices of subsequence give the same limit for $f(z)$. But this is easy, by Claim 2. If $\chi_p \to z$ and $\psi_p \to z$ and these are both sequences of ζ_j points then we have $|f(\chi_p) - f(\psi_p)| \leq (6M/s)|\chi_p - \psi_p| \to 0$.

Claim 4: for z, z' in $B(w, s/4)$ we have $|f(z) - f(z')| \leq (6M/s)|z - z'|$, and in particular f is continuous.

To see this, just choose ζ_k close to z and ζ_j close to z', such that $f(\zeta_k)$ is close to $f(z)$ and $f(\zeta_j)$ is close to $f(z')$ (we can do this by Claim 3). Now apply Claim 2 again.

Claim 5: let $\delta > 0$. Then there exists n_0 such that for all $n \geq n_0$ and for all z in the set C_1 given by $|z - w| \leq s/8$, we have $|g_n(z) - f(z)| < \delta$.

To do this, let $t > 0$ be very small, in particular so small that $6Mt/s < \delta/3$. Let $|z-w| \leq s/8$. Then z lies in some disc $B(\zeta_j, t)$ and since t is very small, this disc lies in $B(w, s/4)$.

We claim that the set $\{z : |z - w| \leq s/8\}$ lies in the union of finitely many discs $B(\zeta_j, t)$. To see this, take t' positive and rational, small compared to t, and take all points of form $kt' + ik't'$, with k, k' integer. These points are in $\mathbb{Q} + i\mathbb{Q}$, finitely many of them lie in $B(w, s/4)$, and each z with $|z - w| \leq s/8$ lies within $2t' < t$ of one of them.

So we need only choose n_0 so large that for all $n \geq n_0$ we have $|g_n(\zeta_j) - f(\zeta_j)| < \delta/3$ for all these finitely many ζ_j. For $z \in B(\zeta_j, t)$ we then have,using Claims 1 and 4,

$$|g_n(z)-f(z)| \leq |g_n(z)-g_n(\zeta_j)|+|g_n(\zeta_j)-f(\zeta_j)|+|f(\zeta_j)-f(z)| < (12M/s)|z-\zeta_j|+\delta/3 < \delta.$$

Claim 5 tells us that $g_n \to f$ uniformly on $|z-w| \leq s/8$, and so LU on D, since w is arbitrary. Hence the function f is analytic on D, and $g_n' \to f'$ as $n \to \infty$, locally uniformly in D, both of these by Weierstrass' theorem.

In addition, if the g_n are one-one, then f is one-one or constant, by Hurwitz' theorem.

8.3 The Riemann mapping theorem

Let $D \neq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$, and let $z_0 \in D$. Let $\Delta = B(0,1)$. Then there exists an analytic function f which maps D one-one onto Δ, with $f(z_0) = 0$ and $f'(z_0) > 0$. This function f is unique.

Notes:

(i) not true for $D = \mathbb{C}$ (Liouville).

(ii) To motivate the proof, assume that a one-one analytic f maps D onto Δ with $f(z_0) = 0$, and suppose that $h : D \to \Delta$ is analytic with $h(z_0) = 0$. Then $p(w) = h(f^{-1}(w))$ maps Δ into itself and sends 0 to 0. By Schwarz, $|p'(0)| \leq 1$. But $p(f) = h$, so $h' = p'(f)f'$, so $|h'(z_0)| \leq |f'(z_0)|$. So the f we look for has the largest possible derivative at 0, the normalization that $f'(z_0)$ is real and positive being to make f unique.

Proof: let H be the family of functions h which are analytic and one-one on D, with $h(D) \subseteq \Delta$, $h(z_0) = 0$ and $h'(z_0) > 0$.

Claim 1: H is not empty. (Note that if D were $\mathbb{C}$, H would be empty.)

To prove this, we recall first that if $|A| < 1$ the function $T(z) = (z - A)/(1 - \overline{A}z)$ maps Δ one-one analytically onto itself. Now we know that there is some $a \notin D$, and the function $z - a$ is analytic and non-zero on the simply connected domain D. So we can form an analytic function p on D such that $p(z)^2 = z - a$, and p is one-one, since p^2 is. Indeed, suppose that z_1, z_2 are in D, and $p(z_1) = \pm p(z_2)$. Then $z_1 - a = z_2 - a$ and so $z_1 = z_2$. Thus if $z \in D$ then p does not take the value $-p(z)$ in D.

We want to use this to find a whole set of values not taken by p in D. The open mapping theorem gives us some r with $0 < r < |p(z_0)|$ such that $B(p(z_0), r) \subseteq p(D)$, and so this implies that $B(-p(z_0), r) \cap p(D)$ is empty, since each w in $B(-p(z_0), r)$ is $-u$ for some u in $B(p(z_0), r)$ and so is $-p(z)$ for some $z \in D$. So $|p(z) + p(z_0)| \geq r > 0$ for all z in D.

Set $q(z) = r/2(p(z) + p(z_0))$, so that q is analytic on D, with $|q(z)| \leq 1/2$ there. Further, q is one-one, because p is.

Now we need a function such that z_0 is mapped to 0, and we just need to set

$$Q(z) = \frac{q(z) - q(z_0)}{1 - \overline{q(z_0)}q(z)}.$$

This Q is analytic and one-one on D and maps D into Δ, with $Q(z_0) = 0$. Multiplying by a constant of modulus 1 to make the derivative at 0 real and positive, we get a function in H.

Having proved Claim 1, we set

$$\sigma = \sup\{h'(z_0) : h \in H\}.$$

In principle, this σ could be infinite, but we'll see in a moment that it's not. In fact we can choose a sequence $h_n \in H$ such that as $n \to \infty$ we have $h_n'(z_0) \to \sigma$. But h_n is a sequence of analytic functions on D all satisfying $|h_n(z)| \leq 1$ on D, and so there is a subsequence, which we may as well assume is the whole sequence h_n, converging locally uniformly on D to an analytic function f. But then Weierstrass' theorem tells us that $h_n' \to f'$ locally uniformly on D, which gives us $f'(z_0) = \sigma$. Further, the last assertion in the proof of the Arzela-Ascoli theorem tells us that since the h_n are one-one on D, then either f is one-one on D or f is constant there.

The latter is impossible since $f'(z_0) \neq 0$. So f is one-one and this function f is our mapping function, and there are two more things to check.

Claim 2: f is onto, from D to Δ.

To prove this, we use the formula

$$\frac{d}{dz}\left(\frac{z-c}{1-\overline{c}z}\right) = \frac{1-|c|^2}{(1-\overline{c}z)^2}.$$

Suppose that $w \in \Delta \backslash f(D)$. Then $g(z) = \frac{f(z)-w}{1-\overline{w}f(z)}$ is analytic and one-one and non-zero from D to Δ, as is the function k defined by $k(z) = g(z)^{1/2}$ i.e. $k(z)^2 = g(z)$. Now we set

$$G(z) = \left(\frac{k(z)-k(z_0)}{1-\overline{k(z_0)}k(z)}\right), \quad F(z) = \frac{|k'(z_0)|}{k'(z_0)}G(z).$$

Thus F is one-one on D, mapping D analytically to Δ, with $F(z_0) = 0$. Now, since $f(z_0) = 0$, we get

$$g'(z_0) = (1-|w|^2)f'(z_0),$$

and

$$k'(z_0) = \frac{1}{2}g(z_0)^{-1/2}g'(z_0), \quad |k'(z_0)| = \frac{1-|w|^2}{2|w|^{1/2}}f'(z_0).$$

Finally,

$$G'(z_0) = k'(z_0)(1-|k(z_0)|^2)^{-1} = k'(z_0)(1-|w|)^{-1},$$

so that

$$F'(z_0) = |k'(z_0)|(1-|w|)^{-1} = \frac{1+|w|}{2|w|^{1/2}}f'(z_0) > f'(z_0).$$

Thus F is in H, but $F'(z_0)$ is greater than σ. This contradiction proves Claim 2.

Claim 3: f is unique.

To see this, suppose we have another such function T, and look at $\phi = T(f^{-1})$, and $\psi = f(T^{-1})$, where the -1 denotes the inverse function. Both are analytic functions from Δ onto itself, mapping 0 to 0, with positive derivative at 0. Schwarz' lemma tells us that $\psi'(0) \leq 1$ and $\phi'(0) \leq 1$, so since $\phi^{-1} = \psi$ we must have $\phi'(0) = \psi'(0) = 1$. But now Schwarz' lemma tells us that $\phi(z) \equiv z \equiv \psi(z)$ in Δ.

8.4 Applications (optional!)

Riemann's mapping theorem has many applications, mainly because of the following: a function $u : D \to \mathbb{R}$, where D is a domain in $\mathbb{C}$, is called harmonic if its partial derivatives of up to second order are continuous and u satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ in D (with, as usual, $z = x + iy$ and x, y real). If u is harmonic and h is analytic it follows from the chain rule and Cauchy-Riemann that $u(h)$ is harmonic.

Now suppose that we have a domain G bounded by a simple closed PSC contour Γ, and a continuous real-valued function $F(w)$ on Γ, and we want to find the steady state temperature in G, given that it is $F(w)$ on the boundary Γ. By the heat equation, the solution is a function $v(w)$ continuous on $G \cup \{\Gamma\}$ and harmonic on G, which equals F on Γ.

Let g map $B(0,1)$ one-one analytically onto G (thus g is the inverse function of the mapping coming from the RMT). It can be shown that g extends continuously and one-one to $|z| \leq 1$. Define ϕ for $|z| = 1$ by $\phi(z) = F(g(z))$. Then ϕ is continuous on $|z| = 1$ and there is an explicit formula for a function U harmonic on $|z| \leq 1$ and equal to ϕ on $|z| = 1$: it is Poisson's formula (for $0 \leq r < 1, t \in \mathbb{R}$)

$$U(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(s - t)} \phi(e^{is}) ds.$$

So we define v on G by $v(g(z)) = U(z)$ and so $v(w) = U(g^{-1}(w))$ is harmonic on G. Also, on Γ, $v(w) = U(g^{-1}(w)) = \phi(g^{-1}(w)) = F(w)$.

A second application of the RMT is to complex dynamics. Suppose that $D \neq \mathbb{C}$ is a simply connected plane domain, and that $f : D \to D$ is analytic, with a fixpoint $f(w_0) = w_0 \in D$. We wish to investigate the sequence $f_n(w), f_{n+1} = f \circ f_n$, for $w \in D$.

Let ϕ map D one-one analytically onto $B(0,1)$, with $\phi(w_0) = 0$, and set $F = \phi \circ f \circ \phi^{-1}$. Then F maps $B(0,1)$ analytically into itself, with $F(0) = 0$. Applying Schwarz' lemma, we see that there are two possibilities. The first is that $F(z) \equiv e^{i\alpha} z$ for some real constant α, so that F acts as a rotation on $B(0,1)$. The other possibility is that $G(z) = F(z)/z$ has $|G(z)| < 1$ on $B(0,1)$. In this case, let $0 < s < 1$. Then $\max\{|G(z)| : |z| \leq s\} = t < 1$, and $|z| \leq s$ gives $|F(z)| \leq t|z| \leq s, |F_2(z)| \leq t|F(z)| \leq t^2|z|$, and $|F_n(z)| \leq t^n|z| \leq t^n s \to 0$ uniformly for $|z| \leq s$. Hence $f_n(w) = \phi^{-1}F_n(\phi(w)) \to \phi^{-1}(0) = w_0$ for every $w \in D$.

8.5 The Bieberbach Conjecture (optional!)

Because of applications, there is a lot of interest in functions f which are analytic and one-one in $B(0,1)$. Since $f'(0) \neq 0$, we can normalize so that $f(0) = 0, f'(0) = 1$ (else consider $(f(z) - f(0))/f'(0)$ instead of f), and look at the Taylor series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1.$$

Bieberbach proved in 1916 that $|a_2| \leq 2$, and asked whether we always have $|a_n| \leq n$. This became his conjecture, an object of an enormous amount of research, and was finally proved by Louis de Branges in 1984. Here is a proof of a special case.

8.6 Theorem (Dieudonné ca. 1930)

Suppose that f is analytic and one-one on $B(0,1)$, with $f(0)=0, f'(0)=1$, and maps $B(0,1)$ one-one onto a simply connected domain G which is symmetric about the real axis i.e. $\{\overline{z} : z \in G\} = G$. Then the Taylor coefficients $a_n = f^{(n)}(0)/n!$ satisfy $|a_n| \leq n$.

Proof: first write $f(x+iy) = u(x,y) + iv(x,y)$, with x,y,u,v real. Then $g(x+iy) = u(x,-y) - iv(x,-y)$ satisfies $g_x = u_x - iv_x$ and $g_y = -u_y + iv_y = v_x + iu_x = ig_x$. So g is analytic by Cauchy-Riemann.

Consequently, $g(z) = \overline{f(\overline{z})}$ also maps $\Delta = B(0,1)$ one-one analytically onto G with $g(0) = 0, g'(0) = 1$. By the uniqueness part of the mapping theorem, $g(z) = f(z)$, so that $f(z)$ is real if z is real. Thus all the a_n are real. Further, if $f(z)$ is real, then $f(z) = \overline{f(z)} = g(\overline{z}) = f(\overline{z})$, so z is real since f is one-one.

Thus for $0 < r < 1$, the function $v(re^{i\theta})$ has constant sign on $(0,\pi)$. Now, it's easy to check that for $k,n \in \mathbb{N}$, we have $\int_0^\pi \sin(k\theta)\sin(n\theta)d\theta = 0$ if $k \neq n$ and $= \pi/2$ if $k = n$. Also $|\sin n\theta| \leq n \sin\theta$ on $[0,\pi]$ (easy to prove by induction, using the formula for $\sin(A+B)$). Therefore, since

$$Im(f(re^{i\theta})) = v(re^{i\theta}) = \sum_{k=1}^{\infty} a_k r^k \sin k\theta,$$

we get

$$|(\pi/2)a_n r^n| = |\int_0^\pi v(re^{i\theta})\sin(n\theta)d\theta| \leq \int_0^\pi |v(re^{i\theta})| n \sin\theta d\theta =$$

$$= |\int_0^\pi v(re^{i\theta}) n \sin\theta d\theta| = (\pi/2)n|a_1|r.$$

Since $a_1 = 1$, we let $r \to 1$ and get $|a_n| \leq n$.

Note that $|a_n| = n$ is possible: the Koebe function $k(z) = z/(1-z)^2 = z + 2z^2 + 3z^3 + \ldots$ is analytic and one-one on $B(0,1)$.

9 Picard's theorem

We will prove that if f is an entire function which never takes the values 0 or 1 then f is constant. This will then generalise to meromorphic functions omitting 3 values from $\mathbb{C} \cup \{\infty\}$. This is another of the classic results of the theory (1879).

9.1 Lemma

Suppose $f : \Delta \to \mathbb{C}$ is analytic with $f(0) = 0, f'(0) = 1$, and suppose further that $|f(z)| \leq M < \infty$ for all $z \in \Delta$. Then $B(0, 1/6M) \subseteq f(\Delta)$, i.e. if $|w| < 1/6M$ then f takes the value w at some point in Δ.

Proof: we write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then $a_0 = 0, a_1 = 1$ and for $k \geq 1$ and $0 < r < 1$ we have

$$|a_k| = |f^{(k)}(0)/k!| = \left| \frac{1}{2\pi i} \int_{|z|=r} f(z) z^{-k-1} dz \right| \leq M r^{-k},$$

and so $|a_k| \leq M$. In particular, $M \geq 1$. Now, on $|z| = 1/4M$, we have

$$|f(z) - z| \leq \sum_{k=2}^{\infty} |a_k||z|^k \leq \sum_{k=2}^{\infty} M(4M)^{-k} =$$

$$= M(4M)^{-2} (1 - 1/4M)^{-1} = 1/4(4M-1) \leq 1/12M.$$

Thus $|f(z)| \geq |z| - |f(z) - z| \geq 1/4M - 1/12M = 1/6M$ for $|z| = 1/4M$. By Rouché's theorem, if $|w| < 1/6M$ then $f(z) - w$ has a zero in $|z| < 1/4M$.

Note: the larger M is, the less information we get. We can look at this as follows. Since $f'(0) = 1$, the smaller M is, the closer f has to be to the identity map.

9.2 Landau's theorem

Let $E = \{z : |z| \leq 1\}$. Suppose that f is analytic on E, and that $f'(0) = 1$. Then $f(\Delta)$ contains a disc of radius $1/24$.

Proof: let $M(r) = \max\{|f'(z)| : |z| \leq r\}$. Then $M(r)$ is non-decreasing. In fact $M(r) = \max\{|f'(z)| : |z| = r\}$ (maximum principle).

Now we put $K(r) = (1-r)M(r)$, and note that $K(1) = 0, K(0) = 1$. Let R be the sup of $r \leq 1$ such that $K(r) \geq 1$. Because $M(r) \leq M(1)$ for $r \leq 1$ we see that $K(r) \to 0$ as $r \to 1-$ and so $R < 1$.

We can take S very close to R such that $K(S) \geq 1$, and so there is some a with $|a| = S$ and

$$|f'(a)| = M(S) \geq (1-S)^{-1}.$$

Put $s = (1-S)/2$ so that $S + s > R$, since S is close to R, and so $K(S+s) < 1$.

Now consider f' on $|z - a| \leq s$. For $|z - a| \leq s$ we have

$$|f'(z)| \leq M(|a| + s) = M(S+s) < (1 - (S+s))^{-1} = 1/s.$$

So, for $|z-a| \leq s$ we get $|f(z)-f(a)| \leq s/s = 1$. Also $|sf'(a)| \geq 1/2$.

Set $g(u) = (f(a+su) - f(a))/sf'(a)$. Then g is analytic on $|u| \leq 1$, and satisfies $g'(0) = 1$ and $g(0) = 0$ and $|g(u)| \leq 2$ for all u there. Thus $g(\Delta)$ contains the disc $B(0, 1/12)$. But

$$f(z) = f(a) + sf'(a)g(u), \quad u = (z-a)/s, \quad |z-a| < s, \quad |u| < 1$$

and so $f(B(a,s))$ contains the disc $B(f(a), s|f'(a)|/12)$. This has radius $|sf'(a)|/12 \geq 1/24$.

9.3 Corollary

Let $R > 0$ and let f be analytic in $|z-a| \leq R$. Then $f(B(a,R))$ contains a disc of radius at least $R|f'(a)|/24$.

Proof: if $f'(a) \neq 0$, we put $h(z) = f(a+Rz)/Rf'(a)$ and apply Landau's theorem.

9.4 Schottky's theorem

Let α and β be positive real numbers with $\beta < 1$. Then there is a constant $C = C_{\alpha,\beta} > 0$ with the following property. If $f : \Delta = B(0,1) \to \mathbb{C}\backslash\{0,1\}$ is analytic, with $|f(0)| \leq \alpha$, then $|f(z)| \leq C$ for $|z| \leq \beta$.

Here C in general depends on α, β but not on the particular f.

Proof: we can assume that $\alpha \geq 2$, because if f satisfies the hypotheses with $\alpha < 2$ then it does so with α replaced by 2.

Assume first that $1/2 \leq |f(0)| \leq \alpha$. We use symbols D_j to denote positive constants which may depend on α and β, but do not depend on the particular function f, and we build up a system of functions, to the last of which we apply the corollary to Landau's theorem.

Since $f \neq 0$ in Δ, we can form an analytic function $F(z) = (1/2\pi i) \log f(z)$ in Δ. The real part of $\log f(0)$ is $\ln|f(0)| \in [\ln(1/2), \ln\alpha]$, which has modulus at most D_0 (this is the reason why we specified that $|f(0)| \geq 1/2$). We can also do this so that $|\mathrm{Im}(\log f(0))| = |\arg f(0)| \leq \pi$ (given one value of $\log f$ we can always add an integer multiple of $2\pi i$) and so we get $|F(0)| \leq 1 + (1/2\pi)|\log|f(0)|| \leq D_1$. Also, F does not take any integer values in Δ, because if $F(z) = n \in \mathbb{Z}$, we get $f(z) = \exp(2\pi i n) = 1$.

Now we define 'branches' of $(F(z)+1)^{1/2}$ and $F(z)^{1/2}$ in Δ, and we can do this by §5.6, because $F(z) \neq 0, -1$. We set $G(z) = (F(z)+1)^{1/2} + (F(z))^{1/2}$, and G is analytic on Δ. Also difference of two squares gives

$$((F+1)^{1/2} + F^{1/2})((F+1)^{1/2} - F^{1/2}) = 1$$

and so

$$1/G = (F+1)^{1/2} - F^{1/2}.$$

Now, $|G(0)| \leq (D_1+1)^{1/2} + D_1^{1/2} = D_2$. Further, $1/G(0) = (F(0)+1)^{1/2} - (F(0))^{1/2}$ and so we have $|1/G(0)| \leq D_2$ also.

We claim now that if $n \in \mathbb{N}$, then G does not take the value $(\sqrt{n+1} + \sqrt{n})^{\pm 1}$ in Δ. For if G took one of these values, we'd have

$$G(z)^2 + G(z)^{-2} = (\sqrt{n+1} + \sqrt{n})^2 + (\sqrt{n+1} + \sqrt{n})^{-2} =$$
$$= (\sqrt{n+1} + \sqrt{n})^2 + (\sqrt{n+1} - \sqrt{n})^2 = 4n + 2.$$

Since $G^2 + G^{-2} = ((F+1)^{1/2} + F^{1/2})^2 + ((F+1)^{1/2} - F^{1/2})^2 = 4F + 2$, we get $F(z) = n$, which we know is impossible, and this proves our claim.

We note next that $G(z) \neq 0$ in Δ, because otherwise we would have $F(z) + 1 = (-F(z)^{1/2})^2 = F(z)$. Hence we can define H on Δ by $H(z) = \log G(z)$. Again we can do this so that $|\mathrm{Im}(H(0))| = |\arg G(0)| \leq \pi$, and since $|G(0)| \leq D_2, |1/G(0)| \leq D_2$ we get

$$|H(0)| \leq \pi + |\log|G(0)|| \leq \pi + \log D_2 = D_3.$$

Now $H(z)$ does not take in Δ any of the values $\pm \ln\left(\sqrt{n+1}+\sqrt{n}\right)+k2\pi i \quad (n \in \mathbb{N}, k \in \mathbb{Z})$, because if it did we'd get

$$G(z) = \exp(H(z)) = \left(\sqrt{n+1}+\sqrt{n}\right)^{\pm 1}.$$

These values omitted by H form a 'grid' in the complex plane. Further, as $n \to +\infty$ we see that $\left(\sqrt{n+2}+\sqrt{n+1}\right) / \left(\sqrt{n+1}+\sqrt{n}\right) \to 1$. Thus the distance from a column to the next column along is bounded, and since the distance between one row and that above is 2π there is some positive constant T such that ANY disc of radius T must contain one of these values omitted by H.

We estimate H'. Suppose $|w| \leq \beta$. Then the closed disc $|z-w| \leq (1-\beta)/2$ is contained in Δ, and its image under H contains no disc of radius T. So by the corollary to Landau's theorem we have

$$|H'(w)|((1-\beta)/2)(1/24) \leq T,$$

and so $|H'(w)| \leq D_4$ on $|w| \leq \beta$. Therefore, in $|w| \leq \beta$ we get the following set of inequalities. First,

$$|H(w)| \leq |H(0)| + \beta D_4 \leq D_3 + \beta D_4 = D_5.$$

Since $G = e^H$, we have

$$|G| = \exp(\mathrm{Re}(H)) \leq \exp(|H|), \quad |G|^{-1} = \exp(-\mathrm{Re}(H)) \leq \exp(|H|),$$

and so we get $|G(w)| + |G(w)|^{-1} \leq D_6 = 2\exp(D_5)$. So

$$|G(w)| = |(F(w)+1)^{1/2} + F(w)^{1/2}| \leq D_6, \quad |G(w)|^{-1} = |(F(w)+1)^{1/2} - F(w)^{1/2}| \leq D_6.$$

Since $2F^{1/2} = G - 1/G$ these give $|F(z)|^{1/2} \leq D_7$ and so $|F(z)| \leq D_8$. Finally, $f(z) = \exp(2\pi i F(z))$ satisfies $|f(z)| \leq \exp(2\pi D_8) = D_9$.

This we proved under the assumptions that $1/2 \leq |f(0)| \leq \alpha$. If $|f(0)| < 1/2$, then we set $g(z) = 1 - f(z)$, which also omits 0 and 1, and satisfies $1/2 \leq |g(0)| < 2$. Thus we get $|g(w)| \leq D_9$ for $|w| \leq \beta$, this D_9 coming from the choice $\alpha = 2$. This gives $|f(w)| \leq 1 + D_9$. Thus we can always take $C = 1 + D_9$.

9.5 Picard's 'little' theorem

Let $f : \mathbb{C} \to \mathbb{C}\backslash\{0, 1\}$ *be analytic. Then* f *is constant.*

Proof: take $g(z) = f(Rz)$, with $R > 1$. Then in $|z| \leq 1/2$ we have $|f(Rz)| = |g(z)| \leq C_{|f(0)|,1/2}$. Since R is arbitrary, f is bounded, and so constant by Liouville's theorem.

Remark: it follows that if h is meromorphic in $\mathbb{C}$ and omits 3 distinct values $a, b, c \in \mathbb{C}^*$, then h is again constant. To see this, if $c = \infty$ look at $g(z) = (h(z) - a)/(b - a)$. If all of a, b, c are finite, look at $g(z) = \frac{(h(z)-a)(c-b)}{(h(z)-b)(c-a)}$. In either case g is analytic and never 0 or 1, and so constant. Since $h = T(g)$, for some Möbius transformation T, we get h constant.

Next we have a stronger version.

9.6 Picard's 'great' theorem

(a) Let $A = \{z \in \mathbb{C} : R < |z| < \infty\}$*, and let* a, b, c *be distinct elements of* $\mathbb{C}^*$*. Let* $f : A \to \mathbb{C}^*\backslash\{a, b, c\}$ *be meromorphic. Then* $\lim_{z\to\infty} f(z)$ *exists.*

(b) Suppose that f *is transcendental and meromorphic in* $\mathbb{C}$ *("transcendental" means: not a rational function). Then* f *takes every value in* $\mathbb{C}^*$*, with at most 2 exceptions, infinitely often in* $\mathbb{C}$*.*

The example e^z shows that 2 can't be replaced by a smaller number. Also $e^{1/z}$ is meromorphic, non-constant and bounded in $1 < |z| < \infty$, and so the conclusion of (a) cannot be replaced by 'f is constant'.

Proof. Assume that f is as in (a). By considering $T(f)$ instead of f we may assume that the omitted values are $0, 1, \infty$. Assuming that $\lim_{z\to\infty} f(z)$ does not exist, there exists a sequence $z_n \to \infty$ with $|f(z_n)| \leq 1$. For otherwise we'd have $|f(z)| \geq 1$ for $S < |z| < +\infty$, say, and $1/f$ would have a removable singularity at ∞, and $f(z)$ would have a limit as $z \to \infty$.

Let ϕ map the unit disc $B(0, 1)$ one-one analytically onto the simply connected domain $\{z : 1/2 < |z| < 2, |\arg z| < \pi\}$, with $\phi(0) = 1$. Such a ϕ exists by the Riemann mapping theorem. By looking at ϕ^{-1}, there is a real number $r \in (0, 1)$ such that the arc T given by $z = e^{it}, -\pi/2 \leq t \leq \pi/2$ has $T \subseteq \phi(\{z : |z| \leq r\})$. So the circle $|z| = 1$ is a subset of $\phi^2(\{z : |z| \leq r\})$.

For large n, define $h(z) = f(z_n\phi(z)^2)$, for $z \in B(0, 1)$. Since $(1/4)|z_n| \leq |z_n\phi^2(z)| \leq 4|z_n|$, we have h analytic and $\neq 0, 1$ on $B(0, 1)$. Also $|h(0)| = |f(z_n)| \leq 1$. So there is a constant $C > 0$, not depending on n, such that $|h(z)| \leq C$ for $|z| \leq r$. So $|f(w)| \leq C$ for $|w| = |z_n| = T_n$, because every such w is $z_n\phi^2(z)$ for some z with $|z| \leq r$.

So we've proved that f, which is analytic in $|z| > R$, has $|f(w)| \leq C$ on the circles $|w| = T_n \to +\infty$. By the maximum principle we have $|f(w)| \leq C$ between these circles. So f is bounded as $z \to \infty$ and so has a removable singularity at ∞, contradicting the assumption that $\lim_{z\to\infty} f(z)$ does not exist.

(b) follows at once, because if f takes three values finitely often we get $S > 0$ such that $f(z) \neq a, b, c$ for $|z| > S$. Hence $f(z)$ has a limit as $z \to \infty$ by (a), and since f is meromorphic in $\mathbb{C}$ we deduce that f is a rational function.

9.7 The connection between Picard's theorem and the Riemann mapping theorem (optional)

At first glance Riemann's mapping theorem and Picard's theorem appear totally unrelated and indeed opposed in character: the first asserts the existence of an analytic function mapping a simply connected domain $D \subset \mathbb{C}$ one-one onto the unit disc $B(0,1) = \Delta$, while the second asserts the nonexistence of a nonconstant analytic function on $\mathbb{C}$ omitting two finite values. In fact, however, the two are fundamentally related, insofar as they both follow from a more general, still more difficult, theorem.

To introduce this theorem we go back to the logarithm. If $a \in \mathbb{C}$ then the function $\phi(z) = a + e^z$ maps $\mathbb{C}$ onto $\mathbb{C} \setminus \{a\}$ (it's easy to check that e^z takes all non-zero finite values: just solve $e^z = w = \exp(\ln|w| + i \arg w)$). Moreover, $\phi(z)$ is locally injective i.e. each point z_0 lies in an open disc on which ϕ is injective (take radius π). We say that $\mathbb{C}$ is a covering surface of $\mathbb{C} \setminus \{a\}$ with covering map ϕ.

Suppose now that G is a simply connected domain in $\mathbb{C}$ and $f : G \to \mathbb{C} \setminus \{a\}$ is analytic. Then $f - a$ has no zeros in G and so we can define an analytic branch g of $\log(f - a)$ on G (see §5.6). Here $g : G \to \mathbb{C}$ is analytic, and we can write $f - a = e^g, f = \phi(g)$ on G.

Now suppose that D is any domain in $\mathbb{C}$ such that $\mathbb{C} \setminus D$ has at least two elements. The *uniformization theorem* asserts that there exists a locally injective analytic function ϕ mapping the unit disc $B(0,1)$ onto D. This time the covering surface is $B(0,1)$. Moreover, if G is a simply connected domain in $\mathbb{C}$ and $f : G \to D$ is analytic then there exists an analytic function g from G into the covering surface $B(0,1)$ such that $f = \phi(g)$ on G.

In the particular case where D as above is simply connected, the mapping ϕ is just the inverse of the function arising from Riemann's mapping theorem, and ϕ is a bijection. When D is not simply connected, ϕ is not a bijection, and proving the existence of ϕ is much harder.

To deduce Picard's theorem from the uniformization theorem, take $D = \mathbb{C} \setminus \{0,1\}$. If f is analytic on $\mathbb{C}$ and omits the values $0, 1$ we can write $f = \phi(g)$ where g maps $\mathbb{C}$ into the covering surface $B(0,1)$, and $\phi : B(0,1) \to D$ is the covering map. But then g is a bounded entire function and so constant, and thus $f = \phi(g)$ is also constant.

9.8 Some basic facts about iteration theory (optional)

The principal application of Picard's theorem and its analogues lies in the area of iteration theory (also called complex dynamics). Let f be an entire function. The iterates are defined by

$$f_0(z) = z, \quad f_1(z) = f(z), \quad f_{n+1}(z) = f(f_n(z)).$$

For each $z_0 \in \mathbb{C}$ we can then look at the *forward orbit*

$$O^+(z_0) = \{z_n = f_n(z_0) : n = 0, 1, 2, \ldots\}.$$

The main questions are then to ask:

(i) does $z_n = f_n(z_0)$ have a limit?

(ii) how does $O^+(z_0)$ vary if we vary z_0 slightly?

The simplest example is $f(z) = z^2$, for which $f_n(z) = z^{2^n}$. Since

$$\log |f_n(z)| = 2^n \log |z|$$

it's easy to see that $f_n(z) \to 0$ if $|z| < 1$, while $f_n(z) \to \infty$ if $|z| > 1$. For $|z| = 1$ we have $|f_n(z)| = 1$, and here $f_n(z)$ does not have to have a limit. For example $z_0 = e^{2\pi i/3}$ gives $z_1 = e^{4\pi i/3} \neq z_0$ and $z_2 = e^{8\pi i/3} = z_0$. This is an example of a *periodic cycle*.

In general we say that z_0 is *stable* for f if $O^+(z)$ stays close to $O^+(z_0)$ provided z is close enough to z_0. It is convenient to use the spherical metric, and the precise definition is that for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies $q(f_n(z), f_n(z_0)) < \varepsilon$ for $n = 0, 1, 2, \ldots$. This says in effect that the f_n are *equicontinuous* at z_0 i.e. given ε the same δ works for all the f_n.

The set of stable points for f is called the Fatou set F_f. Its complement is the Julia set $J_f = \mathbb{C}^* \setminus F_f$. For $f(z) = z^2$ we have $J_f = \{z : |z| = 1\}$, because these points (and no others) have arbitrarily close neighbouring points for which the forward orbit is very different.

It's fairly easy to check that z_0 is stable if and only if $f(z_0)$ is. Hence $z \in J_f$ iff $f(z) \in J_f$. We also state:

Theorem: *if f is entire but not a linear function (i.e. $az + b$), then $J_f \neq \emptyset$.*

Suppose now that z_0 is stable for some f, and let g_j be a sequence of f-iterates. We can take a subsequence h_j such that $h_j(z_0) \to \alpha \in \mathbb{C}^*$ as $j \to \infty$ (with respect to q of course). Since z_0 is stable we get that $h_j(z)$ is close to α for all large j and for all z in some disc $B(z_0, \delta)$.

Take a Möbius T such that $T(\alpha) = 0$. Then $k_j(z) = T(h_j(z))$ is close to 0, and so bounded, for all large j and for all $z \in B(z_0, \delta)$. Hence, by Arzela-Ascoli, the k_j have a convergent subsequence on $B(z_0, \delta)$, and so do the $h_j = T^{-1}(k_j)$. Similar reasoning works in the converse direction and we get:

Theorem: *$z_0 \in F_f$ if and only if the following is true. There is a neighbourhood $B(z_0, \delta)$ on which every sequence g_j of f-iterates has a convergent subsequence, converging locally uniformly on $B(z_0, \delta)$, with respect to the q metric, and with limit function possibly identically ∞.*

For $f(z) = z^2$ the limit functions are 0 and ∞.

The connection with Picard's theorem comes from the following local analogue, which is also proved via Schottky's theorem.

Montel's theorem: *let $D \subseteq \mathbb{C}$ be a domain, and let F_j be a sequence of meromorphic functions $F_j : D \to \mathbb{C}^* \setminus \{a, b, c\}$, where $a, b, c \in \mathbb{C}^*$ are distinct (and the same for every F_j). Then the sequence F_j has a subsequence converging locally uniformly on D, with respect to q, and with limit function possibly identically ∞.*

It follows from Montel's theorem that if the f_n all omit three given distinct values on a domain D then $D \subseteq F_f$.

Examples: let $f(z) = z^2 - 2$. Let $I = [-2, 2]$. Then $f(z) \in I$ iff $z^2 \in [0, 4]$ iff $z \in I$. Hence $\mathbb{C} \setminus I \subseteq F_f$. In fact, $J_f = I$ in this case.

Next, let $f(z) = \tan z$ (which is meromorphic but not entire). Here some modifications to the theory are needed, since f has a pole at $\pi/2$, and an essential singularity at ∞, so that $\lim_{z \to \pi/2} f_2(z)$ does not exist. However, we can note that $f(z)$ has positive imaginary part whenever z has, so that the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is contained in F_f. In fact $J_f = \mathbb{R} \cup \{\infty\}$ in this case.

For $f(z) = e^z$ Misiurewicz proved that $J_f = \mathbb{C}^*$ (all points are unstable).

We note finally that a point z is called periodic if $f_n(z) = z$ for some $n \in \mathbb{N}$ (and a fixpoint when $n = 1$). For $f(z) = z^2$ the point $z \in \mathbb{C}$ is periodic iff $z^{2^n} = z$ i.e. $z = 0$ or $z^{2^n - 1} = 1$, the latter holding iff $z = \exp(2\pi i k/(2^n - 1))$ for some integer k. It's then easy to see that if $|w| = 1$ we can find a periodic point z as close to w as we like. This is a general property of complex dynamics:

Theorem: *the periodic points are dense in the Julia set i.e. if $w \in J_f$ and $\varepsilon > 0$ there exists a periodic point z of f with $q(z, w) < \varepsilon$.*

This gives one way to produce pictures of Julia sets: just compute periodic points.

A link to a good Website for looking at pictures of Julia sets can be found on my Research WWW page (follow the links from my home page).