APPLIED ALGEBRA MTHS2002 (formerly HG2M02)
MODULE INFORMATION AND LECTURE NOTES
PROFESSOR J.K. LANGLEY

Target Students: mainly from the Business School or similar schools.
Available to JYA/Erasmus students, but NOT AVAILABLE to students in Science or Engineering.

If you wish to take, but have not already been accepted for, this module you should contact the
Director of Mathematics Service Teaching:
Dr Martin Kurth, Maths C14 (martin.kurth@nottingham.ac.uk).

Module lectured by:

Professor J.K. Langley, B46, Mathematical Sciences Building,
Tel. 0115 9514964,
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Office Hours: officially the hour preceding each module session, but you are welcome to consult me at other times (it may be best to contact me in advance by email).

Provisional timetable: Lectures Tuesday at 10.00 and Wednesday at 12.00 in Physics B21; Problem classes Thursday at 2.00 in ESLC B07 (not every week).
Lectures start in University Week 19 (begins 27/1/19).
Problems classes will be (provisionally) in University Weeks 20, 22, 23, 24, 26, 28, 33.

Module information, lecture notes, problems and solutions are on the **Moodle page.**
NOTE:the lectures will not be recorded this semester – this is because of very low attendance in semesters with recorded lectures. The notes do include all necessary calculations, so it should not matter if you miss the occasional session, but it is not recommended to take the module if you have a timetable clash. The notes may be slightly amended or updated periodically.
Books: it is **not** necessary to obtain a book for this module, as the lecture notes and Moodle resources should be sufficient, and if there is anything you do not understand you should ask the lecturer.

However, if you really want a book, the following may help.

S. Lipschutz, Theory and Problems of Linear Algebra, Schaum's Outline Series, McGraw-Hill 1968.
(probably the best, as it has lots of worked examples, and is likely to be available cheaply on the internet).
W. Cheney, D. Kincaid, Numerical Mathematics and Computing (7th Ed.). Brooks/Cole, 2013.
C Meyer, Matrix Analysis and Applied Linear Algebra. SIAM, 2000.
E. Kreyszig, Advanced Engineering Mathematics. Wiley.
D W Jordan, P Smith, Mathematical Techniques. OUP.

Module Aims and Objectives

This module introduces some of the fundamental concepts of matrix theory and linear algebra that arise naturally in mathematical models for several different fields, such as engineering, finance, natural sciences, etc. Problems that illustrate the applicability of the theory are covered where appropriate.

Pre-requisites for the module

Basic algebra, as provided by a pass in A-level Maths, or any quantitative methods module, such as Quantitative Methods 2A, taken within the University.

Module Assessment will consist of :
one **two hour written examination**, counting 90%;
one **assessed coursework**, counting 10%.

The examination has four questions, all compulsory. This is a change from previous years, but you can expect the questions themselves to have similar length and level to those from previous years, which may be found on Moodle.

Script viewing. After the exam there may be an opportunity to view your marked script. The purpose is to give the student feedback and insight into how they lost marks, in order to be better prepared for future examinations – it is **not** in order to seek remarking or additional marks.

School policy says that, for any student who is required to resit this module, the resit will be by examination only (in August or the following session), and that August (resit or first sit) examinations may only be taken at a University of Nottingham campus (UK, Malaysia or China).

Assessed coursework submission date (provisional):
Friday 27th March at 3.00 pm (Week 27).

Submitted assessed coursework should show all working, and be submitted strictly by the deadline. Instructions for handing in will be made available via Moodle.

Submitting the coursework requires you to have read and understood the guidelines on plagiarism, that the work submitted be your own and not plagiarised.

Students will normally be given two weeks to complete the assignment. Late submission of work will be penalised (5% per working day, 100% after one week) unless there are extenuating circumstances.

Any application (from an individual student) for an extension to a mathematics assessed coursework deadline should be made to the appropriate module lecturer, but will only be considered if it is accompanied by supporting evidence and a completed Extenuating Circumstances Form (ECF).

The module lecturer will decide whether or not to grant an extension and will indicate this (where appropriate) on the coursework script. Any student who is unhappy with a decision with respect to an extension request may appeal to the Director of Mathematics Service Teaching, Dr Martin Kurth (martin.kurth@nottingham.ac.uk).

Calculators

You are permitted a calculator during the exam, but the University allows *only* the following models, or ones which are "functionally equivalent":

Aurora HC133, Casio HS–5D, Deli – DL1654, Sharp EL–233, Aurora AX–582,

Casio FX83 family, Casio FX85 family, Casio FX570 family, Casio FX 991 family, Sharp EL–531 family.

There is no formula sheet in the exam

Module Syllabus:

Determinants

Evaluation of 2nd, 3rd and higher order determinants.
Expansion via rows and columns.
Effect of row and column operations. Cramer's rule.

Matrices

Matrix addition and multiplication.
The inverse matrix.
Solutions of systems of linear equations.
Eigenvalues and eigenvectors. Diagonalisation.

Applications

Difference equations and systems of differential equations.
Introduction to linear programming.
Optimisation problems. The simplex method.

These notes/slides may be updated/amended slightly as the module proceeds.

This module will feature extensively matrices and systems of equations. We start by defining matrices and then determinants.

An $m \times n$ matrix is an array with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Each entry a_{ij} is a real number and a_{ij} is the entry in the ith row and jth column (**r**ow fi**r**st, **c**olumn se**c**ond).

$$B = \begin{pmatrix} 1 & 3 & 7 & 2 \\ 8 & 9 & 0 & 6 \\ -1 & 5 & 2 & 8 \end{pmatrix}$$

is 3×4 and has $b_{21} = b_{34} = 8$ and $b_{14} = b_{33} = 2$.
A *square* matrix is one for which $m = n$.

Given a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

its determinant is defined to be

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Note that this is a *number*. For example,

$$\begin{vmatrix} 1 & 3 \\ 7 & 4 \end{vmatrix} = 1\times4-3\times7 = -17, \quad \begin{vmatrix} 5 & 6 \\ -4 & 8 \end{vmatrix} = 5(8)-6(-4) = 64.$$

Suppose we have

$$\begin{aligned} ax + by &= u, \\ cx + dy &= v, \end{aligned}$$

where the coefficients a, b, c, d, u, v are known, and we must find x, y.

Multiply the first equation by d, the second by b:

$$adx + bdy = ud, \quad bcx + bdy = vb.$$

Now subtract to get

$$(ad - bc)x = ud - vb.$$

Provided $ad - bc \neq 0$ this gives

$$x = \frac{ud - vb}{ad - bc} = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

Similarly, multiplying $cx + dy = v$ by a and $ax + by = u$ by c and subtracting gives $(ad - bc)y = va - uc$ and

$$y = \frac{va - uc}{ad - bc} = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

This is called *Cramer's rule*.

The method is as follows: suppose we are given

$$\begin{aligned} ax + by &= u, \\ cx + dy &= v, \end{aligned}$$

where the determinant $ad - bc$ is not 0.

Each variable is given as a fraction with denominator

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The numerator is the same determinant, but with the column corresponding to the variable replaced by the entries on the right-hand side of the equations.

Solve by Cramer's rule

$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 7. \end{aligned}$$

We get

$$x = \frac{\begin{vmatrix} 4 & 2 \\ 7 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{-18}{-7} = \frac{18}{7}.$$

(we replaced the first column in the numerator).

Similarly

$$y = \frac{\begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}} = \frac{-5}{-7} = \frac{5}{7}$$

(we replaced the second column). It is easy to check that these values solve the two equations.

Later we will see how to do this for more than two equations.

If $ad - bc = 0$ the equations may have no solution e.g.

$$x + 2y = 3, \quad 3x + 6y = 12;$$

here multiplying the first by 3 gives $9 = 3x + 6y = 12$, impossible.

Alternatively, when $ad - bc = 0$ there may be infinitely many solutions e.g.

$$x + 2y = 7, \quad 5x + 10y = 35.$$

Here the second equation is just the first times 5, and we can get a solution by taking any value for y and setting $x = 7 - 2y$.

There will be more on this theme later.

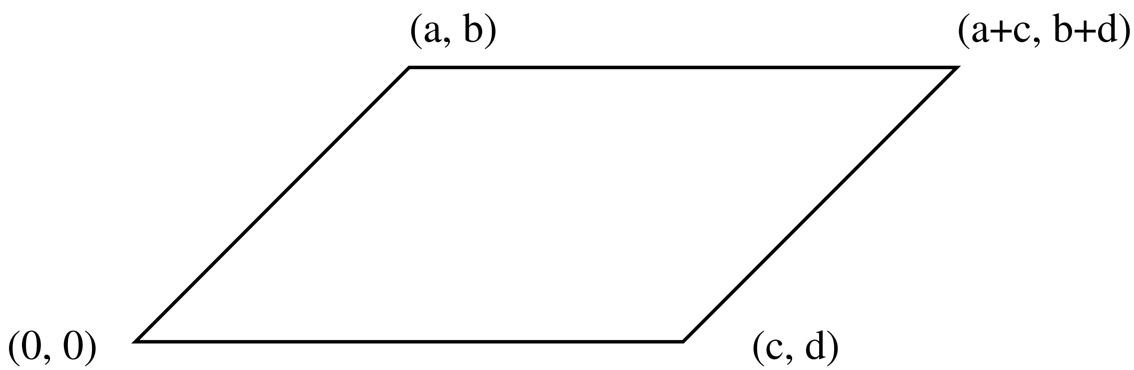

Figure 1: A parallelogram

1.5 2×2 DETERMINANTS AND AREA (OPTIONAL)

Form a parallelogram as in Figure 1.
The area of the parallelogram is the absolute value (modulus) of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

For example, if $(a, b) = (1, 4)$ and $(c, d) = (7, 2)$ then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} = 2 - 28 = -26$$

and the area is $|-26| = 26$.

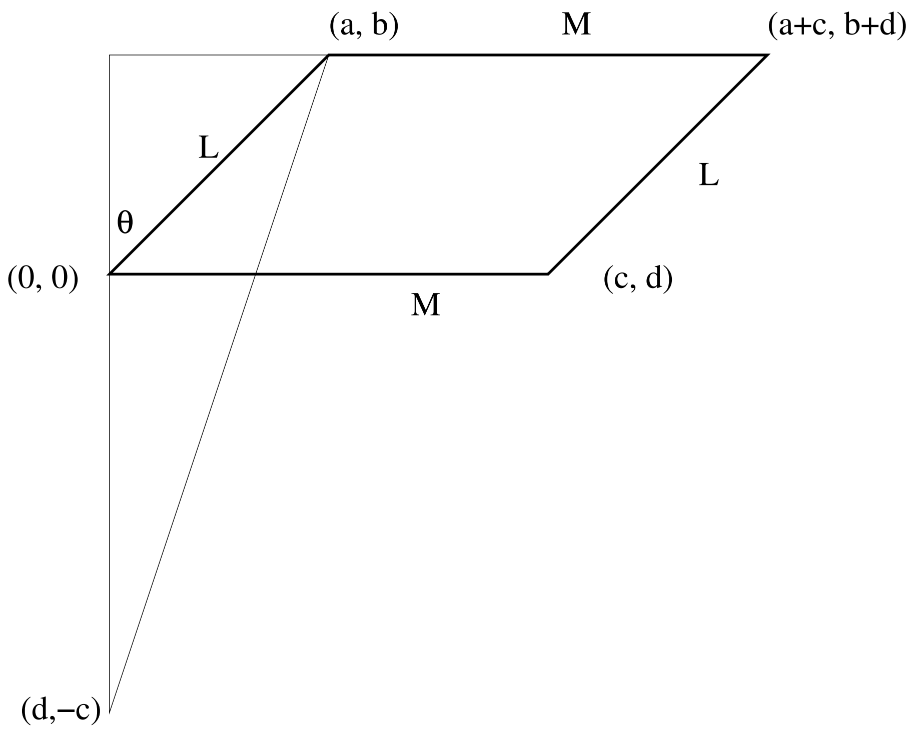

Figure 2: Why the area formula works

An (OPTIONAL) explanation of why this works is given using Fig. 2.
$L = \sqrt{a^2 + b^2}$ is the distance from $(0, 0)$ to (a, b).
$M = \sqrt{c^2 + d^2}$ is the distance from $(0, 0)$ to (c, d).

M is also the distance from $(0,0)$ to $(d,-c)$.
The area of the parallelogram is

$$A = \text{ base} \times \text{height} = ML\cos\theta = -ML\cos(180-\theta).$$

We apply the cosine rule to the triangle in Figure 2 with vertices at $(0,0)$, (a,b) and $(d,-c)$: this says that

$$\begin{aligned}
&M^2 + L^2 - 2ML\cos(180-\theta) \\
&= (\text{ distance from } (a,b) \text{ to } (d,-c))^2 \\
&= (a-d)^2 + (b+c)^2 \\
&= a^2 + d^2 - 2ad + b^2 + c^2 + 2bc \\
&= L^2 + M^2 - 2(ad-bc).
\end{aligned}$$

So $2A = -2ML\cos(180-\theta) = -2(ad-bc)$.

We next look at the 3×3 case

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

In any determinant, the MINOR of an entry is the determinant you get when you delete the row and column which contain the entry.

So for

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

the minor of a_1 is $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$;

the minor of a_3 is $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$;

the minor of b_2 is $\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$.

The formula to evaluate D is then

$$\begin{aligned}
D &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= a_1 \times \text{its minor} - a_2 \times \text{its minor} + a_3 \times \text{its minor} \\
&= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\
&= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).
\end{aligned}$$

The minus sign in front of a_2 is very important!

1.7 TWO EXAMPLES

$$
\begin{aligned}
D_1 &= \begin{vmatrix} 10 & 3 & 7 \\ -1 & 2 & 0 \\ 4 & 3 & 1 \end{vmatrix} \\
&= 10 \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} -1 & 0 \\ 4 & 1 \end{vmatrix} + 7 \begin{vmatrix} -1 & 2 \\ 4 & 3 \end{vmatrix} \\
&= 10 \times 2 - 3 \times (-1) + 7 \times (-11) = -54.
\end{aligned}
$$

$$
\begin{aligned}
D_2 &= \begin{vmatrix} 5 & 2 & -3 \\ 1 & 4 & 1 \\ 2 & 1 & 2 \end{vmatrix} \\
&= 5 \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \\
&= 5 \times 7 - 2 \times 0 - 3 \times (-7) = 56.
\end{aligned}
$$

We next look at the 4×4 case

$$D = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

As before, the MINOR of an entry is the determinant you get when you delete the row and column which contain the entry.

So for

$$D = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

the minor of b_2 is $\begin{vmatrix} a_1 & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$;

the minor of c_4 is $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$.

To evaluate D we go along the top row, multiplying each entry by its minor and alternating signs. The formula is

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}.$$

So to evaluate a 4×4 determinant involves evaluating 3×3 determinants 4 times!

Again the alternating $+/-$ signs are very important.

1.9 EXAMPLE

We evaluate

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & 2 & 0 & 3 \\ 2 & 0 & 7 & 0 \\ 1 & 4 & 0 & 2 \\ 3 & 2 & 0 & 0 \end{vmatrix} \\
&= 1 \times \begin{vmatrix} 0 & 7 & 0 \\ 4 & 0 & 2 \\ 2 & 0 & 0 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 7 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 0 \end{vmatrix} + 0 - 3 \times \begin{vmatrix} 2 & 0 & 7 \\ 1 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} \\
&= -7(-4) - 2(2 \times 0 - 7 \times (-6)) - 3(2 \times 0 + 7 \times (2 - 12)) \\
&= 28 - 84 + 210 = 154.
\end{aligned}
$$

This method is very slow and we next look at some rules which make determinants easier to compute.

1.10 RULES FOR SIMPLIFYING DETERMINANTS

We go back to

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

1.10.1 SWAPPING ROWS

If we swap the two rows we get

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - da = -(ad - bc).$$

Rule I: *swapping two rows of an $n \times n$ determinant multiplies the value of the determinant by -1.*

If we multiply the first row by a real number λ we get

$$\begin{vmatrix} a\lambda & b\lambda \\ c & d \end{vmatrix} = a\lambda d - b\lambda c = \lambda(ad - bc).$$

Rule II: *multiplying all entries in any one row of an $n \times n$ determinant by the same number λ multiplies the value of the determinant by λ.*

Note that if you multiply the whole matrix by λ,
then you are multiplying every row,
and so this multiplies the determinant by λ^n.

If we add λ times the first row to the second we get

$$\begin{aligned}\begin{vmatrix} a & b \\ a\lambda + c & b\lambda + d \end{vmatrix} &= a(b\lambda + d) - b(a\lambda + c) \\ &= ab\lambda - ba\lambda + ad - bc = ad - bc.\end{aligned}$$

Rule III: *adding/subtracting a multiple of one row of an $n \times n$ determinant to/from another row leaves the value of the determinant unchanged.*

To form the transpose of a matrix A you write its rows as columns and its columns as rows to form a new matrix $B = A^T$. Thus $b_{ij} = a_{ji}$. The transpose of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Now

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Rule IV: *taking the transpose of an $n \times n$ matrix leaves the value of the determinant unchanged*.

We can also combine Rule IV with Rules I, II and III to get

Rule I': *swapping two columns of an* $n \times n$ *determinant multiplies the value of the determinant by* -1.

Rule II': *multiplying all entries in any one column of an* $n \times n$ *determinant by the same number* λ *multiplies the value of the determinant by* λ.

Rule III': *adding/subtracting a multiple of one column of an* $n \times n$ *determinant to/from another column leaves the value of the determinant unchanged*.

These hold since the rows of A^T are columns of A.

Use the rules above to carry out these calculations.

$$D_1 = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 9 & 7 & 5 \\ 2 & 9 & 18 & 12 \\ 4 & 12 & 14 & 15 \end{vmatrix}$$

If we subtract twice row 1 from row 2 we can get rid of the 4 in row 2, column 1. We can write this as

$$R2 \to R2 - 2R1 \quad \text{or} \quad R2' = R2 - 2R1.$$

We also use $R3 \to R3 - R1, R4 \to R4 - 2R1$. None of these steps changes D_1, so we get the following.

Starting from

$$D_1 = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 9 & 7 & 5 \\ 2 & 9 & 18 & 12 \\ 4 & 12 & 14 & 15 \end{vmatrix},$$

$R2 \to R2 - 2R1, R3 \to R3 - R1, R4 \to R4 - 2R1$

gives

$$D_1 = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 0 & 3 & 7 & 3 \\ 0 & 6 & 18 & 11 \\ 0 & 6 & 14 & 13 \end{vmatrix}.$$

Now we use $R3 \to R3 - 2R2, R4 \to R4 - 2R2$ to get

$$D_1 = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 0 & 3 & 7 & 3 \\ 0 & 6 & 18 & 11 \\ 0 & 6 & 14 & 13 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 0 & 3 & 7 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 7 \end{vmatrix}.$$

If we now take the transpose, things become much simpler:

$$D_1 = \begin{vmatrix} 2 & 3 & 0 & 1 \\ 0 & 3 & 7 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 7 & 4 & 0 \\ 1 & 3 & 5 & 7 \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 3 & 0 & 0 \\ 7 & 4 & 0 \\ 3 & 5 & 7 \end{vmatrix} = 2 \times 3 \times \begin{vmatrix} 4 & 0 \\ 5 & 7 \end{vmatrix} = 2 \times 3 \times 4 \times 7 = 168.$$

Now consider

$$D_2 = \begin{vmatrix} 6 & 2 & 5 \\ 3 & 2 & 1 \\ 12 & 4 & 15 \end{vmatrix}.$$

This is only 3×3 but row operations make the calculation quicker.

Because 6 and 12 are both multiples of 3 we will first swap rows 1 and 2 (written $R1 \leftrightarrow R2$).

$$
\begin{aligned}
D_2 &= \begin{vmatrix} 6 & 2 & 5 \\ 3 & 2 & 1 \\ 12 & 4 & 15 \end{vmatrix} \\
(R1 \leftrightarrow R2) \quad &= -\begin{vmatrix} 3 & 2 & 1 \\ 6 & 2 & 5 \\ 12 & 4 & 15 \end{vmatrix} \\
(R2 \rightarrow R2 - 2R1, R3 \rightarrow R3 - 4R1) \quad &= -\begin{vmatrix} 3 & 2 & 1 \\ 0 & -2 & 3 \\ 0 & -4 & 11 \end{vmatrix} \\
(R3 \rightarrow R3 - 2R2) \quad &= -\begin{vmatrix} 3 & 2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix}.
\end{aligned}
$$

Taking the transpose again gives

$$
\begin{aligned}
D_2 &= -\begin{vmatrix} 3 & 2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix} = -\begin{vmatrix} 3 & 0 & 0 \\ 2 & -2 & 0 \\ 1 & 3 & 5 \end{vmatrix} \\
&= -3 \times \begin{vmatrix} -2 & 0 \\ 3 & 5 \end{vmatrix} = -3 \times (-2) \times 5 = 30.
\end{aligned}
$$

These steps make calculating determinants much quicker, but there is a further rule which simplifies things even more.

Consider

$$D = \begin{vmatrix} 2 & 3 & 5 & 1 \\ 4 & 7 & 0 & 8 \\ 0 & 2 & 0 & 0 \\ 4 & 3 & 0 & 5 \end{vmatrix}.$$

This looks complicated, but we can do the following. Swap rows 2 and 3: so

$$D = -\begin{vmatrix} 2 & 3 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ 4 & 7 & 0 & 8 \\ 4 & 3 & 0 & 5 \end{vmatrix}.$$

Now swap the new rows 1 and 2: we get

$$D = -\begin{vmatrix} 2 & 3 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ 4 & 7 & 0 & 8 \\ 4 & 3 & 0 & 5 \end{vmatrix} = +\begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 3 & 5 & 1 \\ 4 & 7 & 0 & 8 \\ 4 & 3 & 0 & 5 \end{vmatrix}.$$

Now expand out by row 1 in the usual way. Since 2 is the only non-zero entry in row 1 we take its minor (remembering the $+/-$ rule) to get

$$D = \begin{vmatrix} 0 & 2 & 0 & 0 \\ 2 & 3 & 5 & 1 \\ 4 & 7 & 0 & 8 \\ 4 & 3 & 0 & 5 \end{vmatrix} = -2\begin{vmatrix} 2 & 5 & 1 \\ 4 & 0 & 8 \\ 4 & 0 & 5 \end{vmatrix}.$$

What we have shown here is that D equals the 2 from the original row 3 multiplied by -1 times its minor $\begin{vmatrix} 2 & 5 & 1 \\ 4 & 0 & 8 \\ 4 & 0 & 5 \end{vmatrix}$.

So

$$
\begin{aligned}
D &= -2 \begin{vmatrix} 2 & 5 & 1 \\ 4 & 0 & 8 \\ 4 & 0 & 5 \end{vmatrix} \\
(R2 \to R2 - R3) \qquad &= -2 \begin{vmatrix} 2 & 5 & 1 \\ 0 & 0 & 3 \\ 4 & 0 & 5 \end{vmatrix} = 2 \begin{vmatrix} 0 & 0 & 3 \\ 2 & 5 & 1 \\ 4 & 0 & 5 \end{vmatrix} \\
&= (2)3(2 \times 0 - 5 \times 4) = -120.
\end{aligned}
$$

This leads us to the final rule: to evaluate a determinant we can do the following.

(i) choose any ONE row or column;
(ii) for each a_{ij} in that row or column, calculate a_{ij} times $(-1)^{i+j}$ times its minor;
(iii) add these up (over the one row or column you chose).

With this rule, evaluating an $n \times n$ determinant requires us to evaluate n minors, each of which is an $(n-1) \times (n-1)$ determinant. So if we use this rule enough times, we can evaluate an $n \times n$ determinant for any n.

1.13 EXAMPLE

Verify using the above rule(s) that

$$D = \begin{vmatrix} 1 & 0 & 4 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 8 & 4 \\ 0 & 0 & 5 & 0 \end{vmatrix} = -30.$$

Do this:

(a) expanding by row 2;

(b) expanding by column 2;

(c) expanding by row 4.

(a) Expanding by row 2 gives

$$D = \begin{vmatrix} 1 & 0 & 4 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 8 & 4 \\ 0 & 0 & 5 & 0 \end{vmatrix} = 3 \times (-1)^{2+2} \times \begin{vmatrix} 1 & 4 & 1 \\ 2 & 8 & 4 \\ 0 & 5 & 0 \end{vmatrix}$$

(using row 3)

$$= 3 \times 5 \times (-1)^{3+2} \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= -3 \times 5 \times (4 - 2) = -30.$$

(b) Expanding by column 2 gives exactly the same calculation.

(c) Expanding by row 4 gives

$$D = \begin{vmatrix} 1 & 0 & 4 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 8 & 4 \\ 0 & 0 & 5 & 0 \end{vmatrix} = 5 \times (-1)^{4+3} \times \begin{vmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

(use row 2 or column 2) $$= -5 \times 3 \times (-1)^{2+2} \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= -15 \times (4 - 2) = -30.$$

The $(-1)^{i+j}$ terms give alternating $+, -$, starting with $+$ in the top left corner $(i = j = 1)$ and finishing with $+$ in the bottom right corner $(i = j = n)$.
We get a "chessboard" pattern:

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \quad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}, \quad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}, \ldots$$

1.14 A USEFUL FACT

If a determinant has a row with all entries 0*, or a column with all entries* 0*, then its value is* 0*.*

Why? Just expand the determinant by that row or column.

1.15 THE $n \times n$ CRAMER'S RULE

Suppose we have n equations in n unknowns $x_1, \ldots, x_n$:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n.$$

The a_{ij} and b_i are known numbers. Cramer's rule for this case is as follows.

If the determinant of the coefficients

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

is non-zero, then there is a unique solution given by the following.

The value of each x_j is $1/D$ times the determinant which is obtained by replacing the entries in the jth column of D by $b_1, \dots, b_n$
(this is exactly what we did in the 2×2 case).
There is an OPTIONAL explanation of why this works on the Moodle page.

1.16 EXAMPLE

Use Cramer's rule to solve the equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 9 \\ 2x_1 + 3x_2 + 2x_3 &= 16 \\ x_1 \qquad\quad - 3x_3 &= -11 \end{aligned}$$

(solution $x_1 = 1, x_2 = 2, x_3 = 4$).

First we calculate the determinant D of the coefficients. This is

$$
\begin{aligned}
D &= \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & -3 \end{vmatrix} \\
(C3 \to C3 - C1) \quad &= \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 1 & 0 & -4 \end{vmatrix} \\
\text{(use } C3\text{)} \quad &= -4 \times \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -4 \times (-1) = 4.
\end{aligned}
$$

To calculate x_j, we need $1/4$ times the determinant with the jth column replaced by $\begin{pmatrix} 9 \\ 16 \\ -11 \end{pmatrix}$. So

$$
\begin{aligned}
x_1 &= \frac{1}{4}\begin{vmatrix} 9 & 2 & 1 \\ 16 & 3 & 2 \\ -11 & 0 & -3 \end{vmatrix} \\
\text{(use } R3\text{)} \qquad &= \frac{1}{4} \times \left(-11 \times \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 3 \times \begin{vmatrix} 9 & 2 \\ 16 & 3 \end{vmatrix}\right) \\
&= \frac{1}{4} \times (-11 \times 1 - 3 \times (27 - 32)) \\
&= \frac{1}{4} \times (-11 + 15) = 1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
x_2 &= \frac{1}{4}\begin{vmatrix} 1 & 9 & 1 \\ 2 & 16 & 2 \\ 1 & -11 & -3 \end{vmatrix} \\
(C3 \rightarrow C3 - C1) \qquad &= \frac{1}{4}\begin{vmatrix} 1 & 9 & 0 \\ 2 & 16 & 0 \\ 1 & -11 & -4 \end{vmatrix} \\
\text{(use } C3\text{)} \qquad &= \frac{1}{4} \times (-4) \times \begin{vmatrix} 1 & 9 \\ 2 & 16 \end{vmatrix} \\
&= -(16 - 18) = 2.
\end{aligned}
$$

Finally,

$$\begin{aligned}
x_3 &= \frac{1}{4}\begin{vmatrix} 1 & 2 & 9 \\ 2 & 3 & 16 \\ 1 & 0 & -11 \end{vmatrix} \\
(C3 \to C3 + 11C1) \qquad &= \frac{1}{4}\begin{vmatrix} 1 & 2 & 20 \\ 2 & 3 & 38 \\ 1 & 0 & 0 \end{vmatrix} \\
\text{(use } R3\text{)} \qquad &= \frac{1}{4}\begin{vmatrix} 2 & 20 \\ 3 & 38 \end{vmatrix} \\
&= \frac{1}{4} \times (76 - 60) = 4.
\end{aligned}$$

2 MATRICES AND BASIC OPERATIONS

2.1 SOME TYPES OF MATRIX

An $m \times n$ matrix is an array with m rows and n columns, say

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Each entry a_{ij} is a real number and a_{ij} means the entry in the ith row and jth column (always row first, column second).

We sometimes write $A = (a_{ij})_{m \times n}$ or just $A = (a_{ij})$ as shorthand.

An $m \times n$ matrix is called *square* if $m = n$.
Two special square matrices are

$$O_n = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{(the } n \times n \text{ zero matrix, all entries } 0\text{),}$$

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

I_n is the $n \times n$ identity matrix; all entries are 0 except that on the diagonal each entry is 1 (i.e. $a_{ii} = 1$).

A matrix with only one row (i.e. $m = 1$) is called a *row vector* e.g.

$$R = \begin{pmatrix} 1 & 3 & 6 & 7 & 5 \end{pmatrix}.$$

A matrix with only one column (i.e. $n = 1$) is called a *column vector* e.g.

$$C = \begin{pmatrix} 1 \\ 3 \\ 6 \\ 7 \\ 5 \end{pmatrix}.$$

The **main diagonal** (leading diagonal) of a square matrix is the diagonal running from top left to bottom right. It consists of the entries $a_{11}, a_{22}, \ldots$.

A square matrix is called **upper triangular** if all entries below the main diagonal are 0 e.g.

$$U = \begin{pmatrix} 1 & 4 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}.$$

We can express this as follows: a square matrix $A = (a_{ij})_{n\times n}$ is upper triangular if $a_{ij} = 0$ whenever $i > j$.

A square matrix is called **lower triangular** if all entries above the main diagonal are 0 e.g.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 6 & 2 & 19 \end{pmatrix}.$$

A square matrix $A = (a_{ij})_{n \times n}$ is lower triangular if $a_{ij} = 0$ whenever $i < j$.

A square matrix is called **diagonal** if all entries are 0 except on the main diagonal e.g.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 19 \end{pmatrix}.$$

$A = (a_{ij})_{n \times n}$ is diagonal if $a_{ij} = 0$ whenever $i \neq j$.

2.2 EXAMPLE

Compute $\det V$, where

$$V = \begin{pmatrix} 5 & 4 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}.$$

We expand by the first column:

$$\det V = \begin{vmatrix} 5 & 4 & 9 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{vmatrix} = 5 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = 5(2(6) - 0(3)) = 5 \times 2 \times 6.$$

The determinant is the product of the entries on the main diagonal. We generalise this as follows.

2.3 A USEFUL FACT

If a square matrix is upper triangular, lower triangular, or diagonal, the determinant is the product of the entries on the main diagonal.

2.4 ADDING AND SUBTRACTING MATRICES: MULTIPLYING BY A NUMBER

Suppose we have two matrices

$$A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{m\times n},$$

and λ is a real number. Then

$$A + B = C = (c_{ij})_{m\times n} \quad \text{where} \quad c_{ij} = a_{ij} + b_{ij}.$$

Thus to add two matrices we just add the entries in the same position.

Similarly

$$A - B = D = (d_{ij})_{m\times n} \quad \text{where} \quad d_{ij} = a_{ij} - b_{ij}.$$

Thus to subtract a matrix from another we just subtract the entries in the same position.

Finally

$$\lambda A = E = (e_{ij})_{m \times n} \quad \text{where} \quad e_{ij} = \lambda a_{ij},$$

i.e. we just multiply each entry by λ.

NOTE: to add or subtract matrices, they must have the same number of rows, and the same number of columns.

Let

$$A = \begin{pmatrix} -3 & 1 \\ 2 & 4 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 5 & -1 \end{pmatrix}.$$

We compute $A - B$ and $2A + 5B$.

$$A - B = \begin{pmatrix} -3-2 & 1-1 \\ 2--2 & 4-0 \\ 0-5 & 1--1 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ 4 & 4 \\ -5 & 2 \end{pmatrix},$$

$$2A+5B = \begin{pmatrix} 2(-3)+5(2) & 2(1)+5(1) \\ 2(2)+5(-2) & 2(4)+5(0) \\ 2(0)+5(5) & 2(1)+5(-1) \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -6 & 8 \\ 25 & -3 \end{pmatrix}.$$

Given a matrix
$$A = (a_{ij})_{m \times n},$$
the transpose A^T is the matrix we get by writing the rows of A as columns, and the columns of A as rows.
For example:
$$A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \\ 0 & 5 \end{pmatrix}, \quad A^T = \begin{pmatrix} -1 & 3 & 0 \\ 2 & 4 & 5 \end{pmatrix},$$
$$B = \begin{pmatrix} 6 & -2 & 3 \\ 0 & 1 & 5 \\ 7 & 2 & 0 \end{pmatrix}, \quad B^T = \begin{pmatrix} 6 & 0 & 7 \\ -2 & 1 & 2 \\ 3 & 5 & 0 \end{pmatrix}.$$

2.7 SOME FACTS CONCERNING THE TRANSPOSE

I. Given a matrix

$$A = (a_{ij})_{m\times n},$$

the transpose A^T is the matrix $C = (c_{ij})_{n\times m}$, where $c_{ij} = a_{ji}$.

II. The transpose of A^T is A itself i.e. $(A^T)^T = A$.

III. If A and B are $m \times n$, and α and β are real numbers, then

$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T.$$

IV. If A is a square matrix, then taking the transpose does not change the determinant (see the first chapter). Thus

$$\det(A^T) = \det A.$$

V. A **symmetric** matrix $S = (s_{ij})$ is a square matrix such that $S^T = S$ e.g.

$$S = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}.$$

This is the same as saying that $s_{ij} = s_{ji}$.

These have good properties in terms of their eigenvalues (see later).

2.8 EXAMPLE

If A and B are $n \times n$ matrices then the determinant of $A + B$ is generally NOT equal to $\det A + \det B$.

There are lots of examples to illustrate this: among the simplest are

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so that $0 = \det(A + B) \neq \det A + \det B = 1 + 1 = 2$.

Suppose we take two matrices A and B, and wish to form the product AB.

This is **ONLY** possible if the number of columns of A equals the number of rows of B i.e.

$$A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times q}.$$

The product will then be

$$AB = C = (c_{ij})_{m\times q}.$$

To calculate c_{ij} from

$$A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times q}, \quad AB = C = (c_{ij})_{m\times q},$$

we take the ith row of A and the jth column of B (remember: rows first, columns second).

We multiply each entry from the ith row of A by the corresponding entry from the jth column of B
(i.e. the first times the first, the second times the second, and so on) and we ADD, which gives us c_{ij}.

For this to make sense, the rows of A and the columns of B must have the same number of entries, which is why we need A to be $m \times n$ and B to be $n \times q$.

For example,

$$A = \begin{pmatrix} 2 & 4 \\ 0 & 5 \\ 1 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 8 \\ 4 & 2 & 0 \end{pmatrix}.$$

We will show (on the next slides) that

$$AB = \begin{pmatrix} 22 & 10 & 16 \\ 20 & 10 & 0 \\ 27 & 13 & 8 \end{pmatrix}, \quad BA = \begin{pmatrix} 14 & 65 \\ 8 & 26 \end{pmatrix}.$$

These illustrate a very important fact: in general AB and BA are not the same, even if both exist!

$$
\begin{aligned}
AB &= \begin{pmatrix} 2 & 4 \\ 0 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} 3 & 1 & 8 \\ 4 & 2 & 0 \end{pmatrix} \qquad ((3 \times 2) \times (2 \times 3)) \\
&= \begin{pmatrix} 2 \times 3 + 4 \times 4 & 2 \times 1 + 4 \times 2 & 2 \times 8 + 4 \times 0 \\ 0 \times 3 + 5 \times 4 & 0 \times 1 + 5 \times 2 & 0 \times 8 + 5 \times 0 \\ 1 \times 3 + 6 \times 4 & 1 \times 1 + 6 \times 2 & 1 \times 8 + 6 \times 0 \end{pmatrix} \\
&= \begin{pmatrix} 22 & 10 & 16 \\ 20 & 10 & 0 \\ 27 & 13 & 8 \end{pmatrix} \qquad (3 \times 3).
\end{aligned}
$$

$$
\begin{aligned}
BA &= \begin{pmatrix} 3 & 1 & 8 \\ 4 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 0 & 5 \\ 1 & 6 \end{pmatrix} \qquad ((2 \times 3) \times (3 \times 2)) \\
&= \begin{pmatrix} 3 \times 2 + 1 \times 0 + 8 \times 1 & 3 \times 4 + 1 \times 5 + 8 \times 6 \\ 4 \times 2 + 2 \times 0 + 0 \times 1 & 4 \times 4 + 2 \times 5 + 0 \times 6 \end{pmatrix} \\
&= \begin{pmatrix} 14 & 65 \\ 8 & 26 \end{pmatrix} \qquad (2 \times 2).
\end{aligned}
$$

I. If we have

$$A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times q}, \quad AB = C = (c_{ij})_{m\times q},$$

then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

However it is generally better to remember the method rather than to try to memorise this formula.

II. If A is $m \times n$ and B_1, B_2 are $n \times q$, then

$$A(B_1 + B_2) = AB_1 + AB_2.$$

III. If A_1 and A_2 are $m \times n$ and B is $n \times q$, then

$$(A_1 + A_2)B = A_1B + A_2B.$$

IV. If A is $m \times n$, B is $n \times q$ and C is $q \times r$, then

$$A(BC) = (AB)C$$

i.e. you can multiply B and C first, or A and B first. The answer you get will be $m \times r$.

V. If A is $m \times n$ and B is $n \times q$ then

$$(AB)^T = B^T A^T.$$

Note the change of order on the right.

This is because the ijth entry of $(AB)^T$ is the jith entry of AB,
which is formed by adding the products of terms from the jth row of A and the ith column of B,
which is the same as adding the products of terms from the ith row of B^T and the jth column of A^T.

VI. (This one is very important!)

If A and B are both $n \times n$ then

$$\det(AB) = \det A \cdot \det B$$

i.e. the determinant of the product is the product of the determinants.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We calculate I_2A, AI_2, AO_2, O_2A:

$$\begin{aligned}
I_2A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 \times a & 1 \times b \\ 1 \times c & 1 \times d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \\
AI_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a \times 1 & b \times 1 \\ c \times 1 & d \times 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \\
O_2A &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O_2 = AO_2.
\end{aligned}$$

These conclusions generalise: if A is $n \times n$ then

$$I_nA = AI_n = A, \quad AO_n = O_nA = O_n.$$

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

We calculate BA, CA, DA, AB, AC, AD; this will illustrate a useful connection between matrix multiplication and row/column operations.

First,

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix};$$

multiplying in front by B adds row 2 to row 1. Next,

$$CA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix};$$

multiplying in front by C swaps rows 1 and 2. Also,

$$DA = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ -c & -d \end{pmatrix};$$

multiplying in front by D multiplies $R1$ by 2 and $R2$ by -1.

In general, multiplying A in front by a suitable matrix has the same effect as a row operation.

Now look at

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix};$$

multiplying on the right by B adds $C1$ to $C2$. Next,

$$AC = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix};$$

multiplying on the right by C swaps $C1$ and $C2$. Also,

$$AD = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2a & -b \\ 2c & -d \end{pmatrix};$$

multiplying by D multiplies $C1$ by 2 and $C2$ by -1.

In general, multiplying A on the right by a suitable matrix has the same effect as a column operation.

4 ECHELON FORM AND ROW REDUCTION

This section describes a way of reducing a given matrix to a special form which is very useful for solving systems of equations and finding inverse matrices.

4.1 ECHELON MATRICES

Let $A = (a_{ij})_{m \times n}$ be a matrix.
A is an *echelon matrix*, or is in *echelon form*, if the following is true.
For each row we look at the number of 0s at the *start* of the row.
Each of rows 2 to m should either consist only of 0s or have at least one more 0 at the start than the row above it.

So

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are all in echelon form. The matrix

$$E = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 5 & 6 \\ 2 & 0 & 9 \end{pmatrix}$$

is not in echelon form (rows 2 and 3 break the condition). However we can perform row operations to turn E into a matrix which is in echelon form.

To reduce

$$E = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 5 & 6 \\ 2 & 0 & 9 \end{pmatrix}$$

to echelon form we can do this.

Swap rows 1 and 2 (written as $R1 \to R1' = R2, R2 \to R2' = R1$, or just $R1 \leftrightarrow R2$). This gives

$$\begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 2 & 0 & 9 \end{pmatrix}.$$

Now for
$$\begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 2 & 0 & 9 \end{pmatrix}$$
we subtract $2\times$ row 1 from row 3 (i.e. $R3 \to R3' = R3 - 2R1$):
$$\begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 0 & -10 & -3 \end{pmatrix}.$$
Multiply row 3 by 3 (i.e. $R3 \to R3' = 3R3$) to get
$$\begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 0 & -30 & -9 \end{pmatrix}.$$

Finally, for

$$\begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 0 & -30 & -9 \end{pmatrix}$$

we add 10 times row 2 to row 3 (i.e. $R3 \to R3' = R3 + 10R2$) to get

$$E' = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 2 \\ 0 & 0 & 11 \end{pmatrix}.$$

Here E' is in echelon form.

4.2 THEOREM

Every $m \times n$ *matrix* A *can be reduced to an* $m \times n$ *echelon form matrix* A' *using the three row operations:*
(i) swap two rows;
(ii) multiply one row by a non-zero real number;
(iii) add a multiple of one row to another row.

Each of these operations can be reversed:
to reverse (i) we just apply (i);
(ii) to reverse multiplying row i by $\lambda \neq 0$, multiply row i by $1/\lambda$;
(iii) to reverse adding λ times row i to row j, add $-\lambda$ times row i to row j.
Thus we can get back to A from A' via row operations.

We can even say *how* to find A' given A.

Start with an $m \times n$ matrix A, and find the first column with an entry $x \neq 0$.

Swap rows, so that this non-zero entry x is in row 1.

Subtract multiples of the new row 1 from the rows below, so that below the entry x we have just 0s.

Now repeat the process, just using rows 2 to m.

4.3 EXAMPLE

Let

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & 18 \\ 3 & 10 & 23 \end{pmatrix}.$$

We will reduce A to an echelon matrix A' using the row operations:
(i) swap two rows;
(ii) multiply one row by a non-zero real number;
(iii) add a multiple of one row to another row.

We will then show that using further row operations, A can be reduced to the identity matrix I_3.
This idea will be generalised in the subsequent section.

We write

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & 18 \\ 3 & 10 & 23 \end{pmatrix} \\
(R2 \to R2 - 2R1, R3 \to R3 - 3R1) \qquad &\Rightarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix} \\
(R2 \leftrightarrow R3) \qquad &\Rightarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}
$$

We do not write "$=$" in the second and third lines, because we are changing the matrix; also, "$R2 \leftrightarrow R3$" means "swap" $R2$ and $R3$. The last matrix is in echelon form.

To get to I_3 we write

$$\begin{aligned}
& & & \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \\
(R3 \to (1/4)R3) & & \Rightarrow & \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\
(R1 \to R1 - 7R3, R2 \to R2 - 2R3) & & \Rightarrow & \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(R1 \to R1 - 3R2) & & \Rightarrow & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.
\end{aligned}$$

Suppose we start with an $n \times n$ *square* matrix A, and reduce it to an echelon matrix A' using the basic row operations:
(i) swap two rows;
(ii) multiply one row by a non-zero real number;
(iii) add a multiple of one row to another row.

From the first chapter we know that each of these operations multiplies the determinant by a non-zero number (which can be 1).

So $\det A' = c \det A$, for some *non-zero* c.

Then there are two possible cases.

Case I: *the last row of A' has only 0s in it*

In this case $\det A' = 0$ and so we must have $\det A = 0$.

Case II: *the last row of A' has at least one non-zero entry*

In this case the non-zero entry in the last row can only be in the last place. So A' is upper triangular, and all the entries on the main diagonal of A' have to be non-zero; this implies that $\det A' \neq 0$ and $\det A \neq 0$.

We illustrate this case for $n = 4$ on the next slide.

We verify Case II for $n = 4$; since A' has echelon form,

$$A' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{pmatrix}.$$

Since the last row of A' has at least one non-zero entry, we have $b_{44} \neq 0$. But then $b_{33} \neq 0$, and so $b_{22} \neq 0$ and $b_{11} \neq 0$. Finally, $\det(A') = b_{11}b_{22}b_{33}b_{44} \neq 0$.

Moreover, we can now subtract multiples of $R4$ to get rid of b_{14}, b_{24}, b_{34},
then subtract multiples of $R3$ to get rid of b_{13}, b_{23},
then subtract a multiple of $R2$ to get rid of b_{12}.
We are left with a diagonal matrix.

In general, in Case II we can always go further.

Starting with the last row, subtract multiples of one row from all previous rows, to get a diagonal matrix with all diagonal entries non-zero.

Finally if necessary, we can multiply each row by a non-zero number to get I_n.

These observations lead us to a very important and useful theorem.

4.5 THEOREM

Let A be an $n \times n$ square matrix.

If $\det A = 0$ then A can be reduced via row operations to an echelon form matrix with a row of 0s.

If $\det A \neq 0$ then A can be reduced via row operations to the identity matrix I_n.

By row operations we mean:
(i) swap two rows;
(ii) multiply one row by a non-zero real number;
(iii) add a multiple of one row to another row.

Reduce these matrices either to echelon form with a row of 0s, or to the identity matrix, using row operations.

$$A = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -1 & 2 \\ 1 & 8 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -1 & 2 \\ 1 & 5 & 11 \end{pmatrix}.$$

This will be done in the next slides. Do the same for

$$C = \begin{pmatrix} 4 & 0 & 1 & 5 \\ 1 & 3 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 6 & 6 & 1 & 9 \end{pmatrix}.$$

N.B. in doing this we are only allowed to use *row operations*, **not** column operations (it is only when computing determinants that we are allowed to use both).

We start with

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 3 & 2 \\ -1 & -1 & 2 \\ 1 & 8 & 12 \end{pmatrix} \\
(R2 \to R2 + R1, R3 \to R3 - R1) \qquad &\Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 5 & 10 \end{pmatrix} \\
(R3 \to R3 - (5/2)R2) \qquad &\Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

We have echelon form and a row of 0s and so stop.

Now try

$$\begin{aligned}
B &= \begin{pmatrix} 1 & 3 & 2 \\ -1 & -1 & 2 \\ 1 & 5 & 11 \end{pmatrix} \\
(R2 \to R2 + R1, R3 \to R3 - R1) &\Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 9 \end{pmatrix} \\
(R3 \to R3 - R2) &\Rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}.
\end{aligned}$$

This has echelon form and the last row does not consist only of 0s.

We proceed to I_3 by first dividing $R3$ by 5:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix} \xRightarrow{(R3 \to (1/5)R3)} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we use $R3$ to clear the 2 and 4 in $C3$:

$$(R2 \to R2 - 4R3, R1 \to R1 - 2R3) \qquad \Rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The final steps are:

$$\xRightarrow{(R2 \to (1/2)R2)} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xRightarrow{(R1 \to R1 - 3R2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

For C, we start by swapping rows 1 and 2:

$$C = \begin{pmatrix} 4 & 0 & 1 & 5 \\ 1 & 3 & 0 & 2 \\ 1 & 2 & 0 & 1 \\ 6 & 6 & 1 & 9 \end{pmatrix} \xRightarrow{(R1 \leftrightarrow R2)} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 4 & 0 & 1 & 5 \\ 1 & 2 & 0 & 1 \\ 6 & 6 & 1 & 9 \end{pmatrix}.$$

Next, $R2 \to R2 - 4R1, R3 \to R3 - R1, R4 \to R4 - 6R1$ gives

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & -12 & 1 & -3 \\ 0 & -1 & 0 & -1 \\ 0 & -12 & 1 & -3 \end{pmatrix} \xRightarrow{(R4 \to R4 - R2)} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & -12 & 1 & -3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, $R3 \to 12R3 - R2$ gives echelon form with a row of 0s:

$$\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & -12 & 1 & -3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xRightarrow{(R3 \to 12R3 - R2)} \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & -12 & 1 & -3 \\ 0 & 0 & -1 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5 THE INVERSE MATRIX

5.1 DEFINITION OF THE INVERSE MATRIX

Let A be an $n \times n$ matrix. The inverse A^{-1} of A, if it exists, is an $n \times n$ matrix B such that $AB = BA = I_n$.

Not every matrix has an inverse. For example, to get

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we would need

$$a + 2c = 1, \quad \text{so} \quad 3a + 6c = 3,$$

but $3a + 6c = 0$, which is impossible. So $C = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ has no inverse matrix.

For a general square matrix A, how can we decide whether A^{-1} exists? And if A^{-1} exists, how do we find it?

5.2 THEOREM

Let A be an $n \times n$ matrix. Then the following two conditions are equivalent (each implies the other):
(i) the inverse A^{-1} of A exists;
(ii) $\det A \neq 0$.

For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0, \quad \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \quad \text{has no inverse.}$$

In one direction this is easy. If $B = A^{-1}$ then $AB = I_n$ so

$$1 = \det I_n = \det(AB) = \det A \det B.$$

Hence $\det A$ cannot be 0. This also tells us that if A^{-1} exists then its determinant is $1/\det A$, which is a useful fact in its own right.

A square matrix A is called **invertible**, or **non-singular**, if A^{-1} exists.

A is called **singular** if A^{-1} does not exist.

In general when we multiply matrices we do not have $AB = BA$. But the following is true and quite helpful.

5.3 A HELPFUL FACT

Let A and B be $n \times n$ matrices and let $I = I_n$.
If $AB = I$ then $BA = I$, and $B = A^{-1}$ and $A = B^{-1}$.

This means that if you think you have found $B = A^{-1}$ then you only need to check that $AB = I$ or $BA = I$: you don't have to check it both ways round.

This works because if $AB = I$ then

$$1 = \det A \det B$$

so $\det A \neq 0$ and so A^{-1} exists. Thus

$$A^{-1} = A^{-1}I = A^{-1}AB = IB = B.$$

We still need a method for finding A^{-1}.

Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

So A^{-1} exists precisely when $ad - bc \neq 0$. Also,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore if $ad - bc \neq 0$ then

$$A^{-1} = B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The rule is: swap the entries on the main diagonal, change the sign of the other two, divide by $\det A = ad - bc$.

Find the inverses of

$$E = \begin{pmatrix} 3 & 5 \\ -1 & 2 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Here $\det E = 6 - -5 = 11$, $\det F = 1 - -1 = 2$. So

$$E^{-1} = \frac{1}{11}\begin{pmatrix} 2 & -5 \\ 1 & 3 \end{pmatrix}, \quad F^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

For matrices which are 3×3 or larger, there is in general no simple formula for A^{-1}, but the **Gauss-Jordan method** in the next chapter enables us to find the inverse.

6 THE GAUSS-JORDAN METHOD

Suppose that A is an $n \times n$ matrix.

If we have already calculated $\det A$, we know whether A^{-1} exists (it exists precisely when $\det A \neq 0$).

The following method lets us find A^{-1}, if it exists.

It will even tell us *whether* A^{-1} exists, if we don't already know.

Suppose that A is an $n \times n$ matrix.

Make an $n \times 2n$ matrix, by writing A on the left and $I = I_n$ on the right:

$$B = \left(A \mid I\right).$$

So the first n columns of B are those of A;
the last n columns of B are those of I.

Apply row operations to $B = \begin{pmatrix} A \mid I \end{pmatrix}$ (swapping rows, multiplying a row by a non-zero real number, adding a multiple of one row to another) until you get

$$B^* = \begin{pmatrix} A^* \mid J \end{pmatrix},$$

where A^* is in echelon form, and J is some $n \times n$ matrix.

So you apply row operations to reduce A to echelon form, and the *same* ones to I.

These row operations lead to

$$B^* = \left(A^* \mid J\right),$$

where A^* is in echelon form.

If A^* has a row of zeros, then as we saw in the chapter on echelon form, $\det A = 0$ and so A^{-1} does not exist. In this case we proceed no further.

If A^* does not have a row of zeros, then we saw in the chapter on echelon form that we can apply more row operations to turn A^* into I_n.

So the method now is to apply more row operations to $B^* = \left(A^* \mid J\right)$ until you get a matrix

$$C = \left(I \mid D\right),$$

with $I = I_n$ on the left. The matrix D will be A^{-1}.

Warning I: you must apply the same row operations to both sides.
Warning II: when we were evaluating determinants, we were able to use row operations and column operations. For finding inverse matrices, we can ONLY use row operations.

Use the Gauss-Jordan method to decide if the following matrices have inverses, and if so find them.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 2 \\ 2 & 5 & 9 \end{pmatrix}.$$

Answers:

$$A^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 2 \end{pmatrix};$$

B^{-1} does not exist. Note that once you have found the inverse, you can always check your answer by multiplying it by the original matrix.

For

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix},$$

we start from

$$\begin{aligned}
\big(A \mid I_3\big) \;&=\; \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \\
(R3 \to R3 - 2R1) \qquad &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \\
(R2 \leftrightarrow R3) \qquad &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right).
\end{aligned}$$

Next we use

$$
\begin{array}{rcl}
 & & \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right) \\
(R2 \to (-1)R2) & \Rightarrow & \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right) \\
(R3 \to R3 - 2R2) & \Rightarrow & \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right).
\end{array}
$$

The left-hand block is in echelon form (and rows 2 and 3 are in the form we need).

To finish off all we need is to write

$$\begin{array}{cc} & \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right) \\ (R1 \to R1 - 3R2 - R3) \qquad \Rightarrow & \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & -4 & 1 & 2 \end{array}\right). \end{array}$$

Since we have I_3 on the left the block on the right is

$$A^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 2 \end{pmatrix}.$$

To handle $B = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 2 & 2 \\ 2 & 5 & 9 \end{pmatrix}$, write

$$
\begin{aligned}
(B \mid I_3) &= \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 2 & 5 & 9 & 0 & 0 & 1 \end{array}\right) \\
(R3 \to R3 - 2R1) \qquad &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right) \\
(R3 \to R3 + (1/2)R2) \qquad &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1/2 & 1 \end{array}\right)
\end{aligned}
$$

The left-hand block (is in echelon form and) has a row of 0s, so we stop: B^{-1} does not exist.

Use the Gauss-Jordan method to decide if the following matrices have inverses, and if so find them.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ -3 & 1 & 1 & -1 \\ 0 & 3 & 2 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 5 & 2 & 4 \\ 4 & 19 & 11 & 21 \\ 3 & 15 & 7 & 14 \\ 2 & 10 & 4 & 7 \end{pmatrix}.$$

For A we start from

$$
\begin{aligned}
\big(A \mid I\big) \;&=\; \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \\
(R3 \to R3 - 3R1) \;&\Rightarrow\; \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & -5 & -3 & -3 & 0 & 1 \end{array}\right) \\
(R3 \to 2R3 + 5R2) \;&\Rightarrow\; \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 14 & -6 & 5 & 2 \end{array}\right)
\end{aligned}
$$

The left-hand block is in echelon form and does not have a row of 0s. So we know that A^{-1} exists.

For A we continue with

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 14 & -6 & 5 & 2 \end{array}\right)$$

$$(R3 \to (1/14)R3) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{6}{14} & \frac{5}{14} & \frac{2}{14} \end{array}\right)$$

$$(R1 \to R1 - R3, R2 \to R2 - 4R3) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{20}{14} & -\frac{5}{14} & -\frac{2}{14} \\ 0 & 2 & 0 & \frac{24}{14} & -\frac{6}{14} & -\frac{8}{14} \\ 0 & 0 & 1 & -\frac{6}{14} & \frac{5}{14} & \frac{2}{14} \end{array}\right)$$

$$(R1 \to R1 - R2) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{14} & \frac{1}{14} & \frac{6}{14} \\ 0 & 2 & 0 & \frac{24}{14} & -\frac{6}{14} & -\frac{8}{14} \\ 0 & 0 & 1 & -\frac{6}{14} & \frac{5}{14} & \frac{2}{14} \end{array}\right)$$

For A we now just need

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{14} & \frac{1}{14} & \frac{6}{14} \\ 0 & 2 & 0 & \frac{24}{14} & -\frac{6}{14} & -\frac{8}{14} \\ 0 & 0 & 1 & -\frac{6}{14} & \frac{5}{14} & \frac{2}{14} \end{array}\right)$$

$$(R2 \to (1/2)R2) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{14} & \frac{1}{14} & \frac{6}{14} \\ 0 & 1 & 0 & \frac{12}{14} & -\frac{3}{14} & -\frac{4}{14} \\ 0 & 0 & 1 & -\frac{6}{14} & \frac{5}{14} & \frac{2}{14} \end{array}\right),$$

and so

$$A^{-1} = \frac{1}{14}\begin{pmatrix} -4 & 1 & 6 \\ 12 & -3 & -4 \\ -6 & 5 & 2 \end{pmatrix}.$$

For B we start from

$$\left(B \mid I\right) = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(R2 - 2R1, R3 + 3R1) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(R3 + 4R2, R4 + 3R2) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 5 & -6 & -5 & 4 & 1 & 0 \\ 0 & 0 & 5 & -6 & -6 & 3 & 0 & 1 \end{array}\right).$$

For B we can now write

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 5 & -6 & -5 & 4 & 1 & 0 \\ 0 & 0 & 5 & -6 & -6 & 3 & 0 & 1 \end{array}\right)$$

$$(R4 \to R4 - R3) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 5 & -6 & -5 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right).$$

The left-hand block is in echelon form and has a row of 0s. So B^{-1} does not exist.

For C we start from

$$(C \mid I) = \left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 4 & 1 & 0 & 0 & 0 \\ 4 & 19 & 11 & 21 & 0 & 1 & 0 & 0 \\ 3 & 15 & 7 & 14 & 0 & 0 & 1 & 0 \\ 2 & 10 & 4 & 7 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$(R2 - 4R1, R3 - 3R1, R4 - 2R1) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 5 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$(R4 \to -R4) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 5 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right)$$

Because we have echelon form on the left and no row of 0s we continue from

$$\left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 5 & -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right).$$

$(R1 \to R1 - 4R4, R2 \to R2 - 5R4, R3 \to R3 - 2R4)$ gives

$$\left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 0 & -7 & 0 & 0 & 4 \\ 0 & -1 & 3 & 0 & -14 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right).$$

For C we have nearly finished:

$$\left(\begin{array}{cccc|cccc} 1 & 5 & 2 & 0 & -7 & 0 & 0 & 4 \\ 0 & -1 & 3 & 0 & -14 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right)$$

$$(R1 - 2R3, R2 - 3R3) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 5 & 0 & 0 & 7 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 7 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right)$$

$$(R1 \to R1 + 5R2) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 42 & 5 & -17 & -5 \\ 0 & -1 & 0 & 0 & 7 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right)$$

All we need to do now is:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 42 & 5 & -17 & -5 \\ 0 & -1 & 0 & 0 & 7 & 1 & -3 & -1 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right)$$

$$(R2 \to -R2) \Rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 42 & 5 & -17 & -5 \\ 0 & 1 & 0 & 0 & -7 & -1 & 3 & 1 \\ 0 & 0 & 1 & 0 & -7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 \end{array}\right),$$

which tells us that

$$C^{-1} = \begin{pmatrix} 42 & 5 & -17 & -5 \\ -7 & -1 & 3 & 1 \\ -7 & 0 & 1 & 2 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Given an $n \times n$ matrix A, write down an $n \times 2n$ matrix

$$B = \left(A \mid I\right)$$

where $I = I_n$. Perform row operations on B to get

$$B^* = \left(A^* \mid J\right),$$

where A^* is in echelon form.

If A^* has a row of 0s, then $\det A = 0$ and A^{-1} does not exist.

If A^* does not have a row of 0s, you will be able to perform more row operations until you get

$$C = \left(I \mid D\right).$$

The matrix D on the right will be A^{-1} (and you can check it by calculating AD or DA).

The row operations you can use are:

swapping rows;

multiplying a row by a non-zero real number;

adding a multiple of one row to another.

You cannot use column operations.

7 SYSTEMS OF EQUATIONS AND MATRIX PRODUCTS

7.1 EXAMPLE

A company makes three products P, Q, R, using three raw materials A, B, C.
A ton of P uses 1 ton of A, 3 tons of B, 5 tons of C.
A ton of Q uses 2 tons of A, 5 tons of B, 12 tons of C.
A ton of R uses 3 tons of A, 9 tons of B, 16 tons of C.

Express the amounts of A, B, C which are required in terms of the amounts of P, Q, R which are produced.
Write this as a matrix equation.
Determine the production levels of P, Q, R if the company uses 65 tons of A, 175 tons of B and 370 tons of C.

Assume the company produces

x_1 tons of P, x_2 tons of Q, and x_3 tons of R,

and uses

b_1 tons of A, b_2 tons of B, and b_3 tons of C.

A ton of P uses 1 ton of A, 3 tons of B and 5 tons of C.
A ton of Q uses 2 tons of A, 5 tons of B, 12 tons of C.
A ton of R uses 3 tons of A, 9 tons of B, 16 tons of C.
Then

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1, \\ 3x_1 + 5x_2 + 9x_3 &= b_2, \\ 5x_1 + 12x_2 + 16x_3 &= b_3. \end{aligned}$$

We write

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &= b_1, \\
3x_1 + 5x_2 + 9x_3 &= b_2, \\
5x_1 + 12x_2 + 16x_3 &= b_3.
\end{aligned}
$$

as a matrix equation as follows:

$$
\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 9 \\ 5 & 12 & 16 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.
$$

We solve for x_1, x_2, x_3, given that $b_1 = 65, b_2 = 175, b_3 = 370$. We will look at three related methods, using

$$
A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 9 \\ 5 & 12 & 16 \end{pmatrix}.
$$

Method I. Using Cramer's rule

We first calculate

$$D = \det A = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 5 & 9 \\ 5 & 12 & 16 \end{vmatrix}.$$

Using $C3 \to C3 - 3C1$ and then expanding by $C3$ we get

$$D = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 5 & 0 \\ 5 & 12 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1 \neq 0.$$

So one way to solve is to use Cramer's rule to get

$$x_1 = \frac{1}{D} \begin{vmatrix} b_1 & 2 & 3 \\ b_2 & 5 & 9 \\ b_3 & 12 & 16 \end{vmatrix}, \quad x_2 = \frac{1}{D} \begin{vmatrix} 1 & b_1 & 3 \\ 3 & b_2 & 9 \\ 5 & b_3 & 16 \end{vmatrix}, \quad x_3 = \frac{1}{D} \begin{vmatrix} 1 & 2 & b_1 \\ 3 & 5 & b_2 \\ 5 & 12 & b_3 \end{vmatrix}.$$

We compute x_2: since $D = -1$ and $b_1 = 65, b_2 = 175, b_3 = 370$ we get, again using $(C3 \to C3 - 3C1)$,

$$x_2 = -\begin{vmatrix} 1 & 65 & 3 \\ 3 & 175 & 9 \\ 5 & 370 & 16 \end{vmatrix} = -\begin{vmatrix} 1 & 65 & 0 \\ 3 & 175 & 0 \\ 5 & 370 & 1 \end{vmatrix} = -(175 - 195) = 20.$$

OPTIONAL: check that we get $x_1 = 10$ and $x_3 = 5$.
This works, but it means that for every choice of b_1, b_2, b_3 we have to compute three determinants to find x_1, x_2, x_3.

Note that this method can *only* be used when the number of equations equals the number of variables and the determinant of the coefficients on the left-hand side is non-zero.

Method II. Using the inverse matrix

Our matrix equation is

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 9 \\ 5 & 12 & 16 \end{pmatrix}.$$

Because $\det A \neq 0$, we know that A has an inverse A^{-1}. So multiplying in front by A^{-1} on both sides gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Therefore, if we find A^{-1} we just need to multiply by a column vector to get the x_j.

This may be easier than calculating determinants.

Using the Gauss-Jordan method we find A^{-1} via

$$(A\,|\,I) \;=\; \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 5 & 9 & 0 & 1 & 0 \\ 5 & 12 & 16 & 0 & 0 & 1 \end{array}\right)$$

$$(R2 \to R2 - 3R1, R3 \to R3 - 5R1) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 2 & 1 & -5 & 0 & 1 \end{array}\right)$$

$$(R3 \to R3 + 2R2) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -11 & 2 & 1 \end{array}\right)$$

$$(R1 \to R1 - 3R3, R2 \to -R2) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 34 & -6 & -3 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & -11 & 2 & 1 \end{array}\right)$$

So we find A^{-1} using

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 34 & -6 & -3 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & -11 & 2 & 1 \end{array}\right)$$

$$(R1 \to R1 - 2R2) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 28 & -4 & -3 \\ 0 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & -11 & 2 & 1 \end{array}\right),$$

which tells us that

$$A^{-1} = \begin{pmatrix} 28 & -4 & -3 \\ 3 & -1 & 0 \\ -11 & 2 & 1 \end{pmatrix}.$$

In particular, if $b_1 = 65, b_2 = 175, b_3 = 370$ then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} 65 \\ 175 \\ 370 \end{pmatrix} = \begin{pmatrix} 28 & -4 & -3 \\ 3 & -1 & 0 \\ -11 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 65 \\ 175 \\ 370 \end{pmatrix} = \begin{pmatrix} 10 \\ 20 \\ 5 \end{pmatrix}.$$

Again this method can *only* be used when the number of equations equals the number of variables and the matrix formed from the coefficients on the left-hand side is invertible, which is equivalent to its determinant being non-zero.

Method III. Using row operations

Recall that the matrix equation we need to solve is

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 9 \\ 5 & 12 & 16 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 65 \\ 175 \\ 370 \end{pmatrix}.$$

We write this in the form of an *augmented matrix*:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 3 & 5 & 9 & 175 \\ 5 & 12 & 16 & 370 \end{array}\right).$$

The coefficients of the x_j go on the left, the b_j on the right, with a vertical line (or bar) separating them.

We use row operations to reduce the left-hand part of this matrix to echelon form.

Starting from

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 3 & 5 & 9 & 175 \\ 5 & 12 & 16 & 370 \end{array}\right)$$

we first use $R2 \to R2' = R2 - 3R1, R3 \to R3' = R3 - 5R1$ to get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 0 & -1 & 0 & -20 \\ 0 & 2 & 1 & 45 \end{array}\right).$$

This represents the equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 65, \\ -x_2 &= -20, \\ 2x_2 + x_3 &= 45. \end{aligned}$$

What we have done is to subtract 3 times the first equation from the second, and 5 times the first from the third. This process is reversible, and any solution of one system is a solution of the other.

To get echelon form on the left of the vertical bar we perform one more row operation on

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 0 & -1 & 0 & -20 \\ 0 & 2 & 1 & 45 \end{array}\right).$$

Here $R3 \to R3' = R3 + 2R2$ gives

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 0 & -1 & 0 & -20 \\ 0 & 0 & 1 & 5 \end{array}\right).$$

In the new augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 65 \\ 0 & -1 & 0 & -20 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

the block in the augmented matrix to the left of the vertical line is in echelon form. The matrix represents

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 65, \\ -x_2 &= -20, \\ x_3 &= 5, \end{aligned}$$

which is equivalent to the original system, and clearly gives $x_3 = 5$, $x_2 = 20$, $x_1 + 40 + 15 = 65$, so $x_1 = 10$.

This method is called *Gaussian elimination*, and it has a number of advantages:

unlike Methods I and II, it can be used when the number of equations is different to the number of variables x_j,

and when the number of equations equals the number of variables but the determinant of the coefficients is 0.

We generalise this idea in the next chapter.

8 THE METHOD OF GAUSSIAN ELIMINATION

Suppose we have m equations in n unknowns $x_1, \dots, x_n$ of the form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m.
\end{aligned}$$

The coefficients a_{ij} and b_j are given, and we need to find the x_j. A typical situation would be:
x_j is the amount produced of product j, for $j = 1, \dots, n$;
a_{ij} is the amount of raw material i needed per unit of product j;
b_i is the total amount of material i used, for $i = 1, \dots, m$.

Write the system as an augmented matrix

$$(A \mid B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

Here A is formed from the coefficients a_{ij}, and B from the b_j.

Apply row operations to the augmented matrix

$$(A \mid B)$$

(i.e. swap rows; multiply a row by $\lambda \neq 0$; add/subtract a multiple of one row to/from another).

At each stage the new matrix represents a system of equations which is equivalent to the original system. Remember to apply the operations to the whole augmented matrix, not just the left block A.

Do this till you get

$$(A^* \mid B^*)$$

where A^* is in echelon form.

Starting from the bottom, you will be able to read off whether you get no solution, one solution, or infinitely many solutions.

The best way to see this is through examples.

1. Solve the system of equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 4x_3 &= 2 \\ 3x_1 + 8x_2 - x_3 &= 18. \end{aligned}$$

We start from the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 2 & 5 & 4 & 2 \\ 3 & 8 & -1 & 18 \end{array}\right).$$

So we apply row operations until the block to the left of the vertical line is in echelon form:

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 2 & 5 & 4 & 2 \\ 3 & 8 & -1 & 18 \end{array}\right)$$

$$(R2 \to R2 - 2R1, R3 \to R3 - 3R1) \Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & -3 & -2 & -2 \\ 0 & -4 & -10 & 12 \end{array}\right)$$

$$(R3 \to 3R3 - 4R2) \Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & -22 & 44 \end{array}\right).$$

The new augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 2 \\ 0 & -3 & -2 & -2 \\ 0 & 0 & -22 & 44 \end{array}\right)$$

represents the equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 2 \\ -3x_2 - 2x_3 &= -2 \\ -22x_3 &= 44. \end{aligned}$$

These equations have the same solutions as the original system. Notice also that if we work upwards from the bottom, each equation contains (at least) one more variable than the one below it (this is why echelon form helps).

From

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 2 \\ -3x_2 - 2x_3 &= -2 \\ -22x_3 &= 44 \end{aligned}$$

we can now read off that $-22x_3 = 44$ so $x_3 = -2$. Then we get

$$-3x_2 = 2x_3 - 2 = -6, \quad x_2 = 2,$$

and finally

$$x_1 = 2 - 4x_2 - 3x_3 = 0.$$

It is easy to check that these solve the original equations. In this case there is a unique solution.

Notice that if you take the first two of the original equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 4x_3 &= 2 \end{aligned}$$

and substitute $x_1 = 2 - 4x_2 - 3x_3$ in the second you get

$$2 = 2(2 - 4x_2 - 3x_3) + 5x_2 + 4x_3 = 4 - 3x_2 - 2x_3$$

and so

$$-3x_2 - 2x_3 = -2,$$

which is the same as what we got by subtracting twice the first equation from the second.

Thus the row reduction method is in effect equivalent to substituting for variables, but is *much more efficient*, since it lets us work with matrices rather than equations.

2. Solve the system of equations

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 2x_4 &= 5 \\ 2x_1 + 4x_2 + x_3 &= 7 \\ x_1 \qquad\quad + 3x_3 + x_4 &= 8 \\ x_2 + 13x_3 + 7x_4 &= 9. \end{aligned}$$

The augmented matrix this time is

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 2 & 4 & 1 & 0 & 7 \\ 1 & 0 & 3 & 1 & 8 \\ 0 & 1 & 13 & 7 & 9 \end{array}\right).$$

Again we reduce the left side to echelon form:

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 2 & 4 & 1 & 0 & 7 \\ 1 & 0 & 3 & 1 & 8 \\ 0 & 1 & 13 & 7 & 9 \end{array}\right)$$

$$(R2 - 2R1, R3 - R1) \Rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & -2 & -7 & -4 & -3 \\ 0 & -3 & -1 & -1 & 3 \\ 0 & 1 & 13 & 7 & 9 \end{array}\right)$$

$$(R2 \leftrightarrow R4) \Rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & 1 & 13 & 7 & 9 \\ 0 & -3 & -1 & -1 & 3 \\ 0 & -2 & -7 & -4 & -3 \end{array}\right).$$

We continue via

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & 1 & 13 & 7 & 9 \\ 0 & -3 & -1 & -1 & 3 \\ 0 & -2 & -7 & -4 & -3 \end{array}\right)$$

$$(R3 + 3R2, R4 + 2R2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & 1 & 13 & 7 & 9 \\ 0 & 0 & 38 & 20 & 30 \\ 0 & 0 & 19 & 10 & 15 \end{array}\right)$$

$$(R4 \to R4 - (1/2)R3) \Rightarrow \left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & 1 & 13 & 7 & 9 \\ 0 & 0 & 38 & 20 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Finally we cancel a factor 2 in $R3$ to get the new augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 5 \\ 0 & 1 & 13 & 7 & 9 \\ 0 & 0 & 19 & 10 & 15 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The last equation tells us nothing: it just gives $0 = 0$.

The third equation says that

$$19x_3 + 10x_4 = 15, \quad x_3 = \frac{15}{19} - \frac{10x_4}{19}.$$

So we can give x_4 any value we like (it is a *free variable*).

Once we choose x_4, this gives x_3.

Now the second equation gives

$$x_2+13x_3+7x_4 = 9, \quad x_2 = 9-13x_3-7x_4 = -\frac{24}{19}-\frac{3x_4}{19}.$$

Finally, the first equation tells us that

$$x_1+3x_2+4x_3+2x_4 = 5, \quad x_1 = 5-3x_2-4x_3-2x_4 = \frac{107}{19}+\frac{11x_4}{19}.$$

There are infinitely many solutions, depending on the value of x_4.

3. Solve the system of equations

$$\begin{aligned} 4x_1 + x_2 + 3x_3 + 8x_4 &= 2 \\ 2x_1 + 7x_2 + 4x_3 + x_4 &= 6 \\ 3x_1 + 9x_2 + 2x_3 + 5x_4 &= 7 \\ 11x_1 + 24x_2 + 13x_3 + 15x_4 &= 20. \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{cccc|c} 4 & 1 & 3 & 8 & 2 \\ 2 & 7 & 4 & 1 & 6 \\ 3 & 9 & 2 & 5 & 7 \\ 11 & 24 & 13 & 15 & 20 \end{array}\right).$$

The calculations are complicated here.

Starting from

$$\left(\begin{array}{cccc|c} 4 & 1 & 3 & 8 & 2 \\ 2 & 7 & 4 & 1 & 6 \\ 3 & 9 & 2 & 5 & 7 \\ 11 & 24 & 13 & 15 & 20 \end{array}\right),$$

it is easiest to get echelon form using row 2. In order to avoid fractions we first double rows 3 and 4, and swap rows 1 and 2. This gives

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 4 & 1 & 3 & 8 & 2 \\ 6 & 18 & 4 & 10 & 14 \\ 22 & 48 & 26 & 30 & 40 \end{array}\right).$$

On the new matrix

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 4 & 1 & 3 & 8 & 2 \\ 6 & 18 & 4 & 10 & 14 \\ 22 & 48 & 26 & 30 & 40 \end{array}\right)$$

we use $R2' = R2 - 2R1, R3' = R3 - 3R1, R4' = R4 - 11R1$ to get

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -13 & -5 & 6 & -10 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & -29 & -18 & 19 & -26 \end{array}\right).$$

So far we have got

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -13 & -5 & 6 & -10 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & -29 & -18 & 19 & -26 \end{array}\right).$$

Now we bring the third row to the second place and multiply the second and fourth rows by 3 (to simplify the arithmetic):

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & -39 & -15 & 18 & -30 \\ 0 & -87 & -54 & 57 & -78 \end{array}\right).$$

We now have:

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & -39 & -15 & 18 & -30 \\ 0 & -87 & -54 & 57 & -78 \end{array}\right).$$

We use $R3' = R3 - 13R2$, $R4' = R4 - 29R2$ and get

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & 0 & 89 & -73 & 22 \\ 0 & 0 & 178 & -146 & 38 \end{array}\right).$$

This is nearly in echelon form.

Finally, for

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & 0 & 89 & -73 & 22 \\ 0 & 0 & 178 & -146 & 38 \end{array}\right)$$

we just need to use $R4' = R4 - 2R3$:

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & 0 & 89 & -73 & 22 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right).$$

The left-hand block is now in echelon form:

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 6 \\ 0 & -3 & -8 & 7 & -4 \\ 0 & 0 & 89 & -73 & 22 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right).$$

All these steps are reversible, so this new augmented matrix represents a system which is equivalent to the original system: the two systems have the same solutions.

But the last equation of the new system reads

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -6,$$

which is impossible. So there is no solution at all in this case: the system is *inconsistent*.

4. Solve the system of equations

$$\begin{aligned} x_1 + 4x_2 + 3x_3 + 6x_4 &= 2 \\ x_2 + 4x_3 + 2x_4 &= 7 \\ 2x_1 + 9x_2 + 10x_3 + 14x_4 &= 11 \\ x_1 + 9x_2 + 23x_3 + 16x_4 &= 37. \end{aligned}$$

. The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 2 \\ 0 & 1 & 4 & 2 & 7 \\ 2 & 9 & 10 & 14 & 11 \\ 1 & 9 & 23 & 16 & 37 \end{array}\right).$$

We reduce to echelon form via

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 2 \\ 0 & 1 & 4 & 2 & 7 \\ 2 & 9 & 10 & 14 & 11 \\ 1 & 9 & 23 & 16 & 37 \end{array}\right)$$

$$(R3 \to R3 - 2R1, R4 \to R4 - R1) \Rightarrow \left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 2 \\ 0 & 1 & 4 & 2 & 7 \\ 0 & 1 & 4 & 2 & 7 \\ 0 & 5 & 20 & 10 & 35 \end{array}\right)$$

$$(R3 \to R3 - R2, R4 \to R4 - 5R2) \Rightarrow \left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 2 \\ 0 & 1 & 4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The new augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 6 & 2 \\ 0 & 1 & 4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This time the last two equations just give $0 = 0$. The second is

$$x_2 + 4x_3 + 2x_4 = 7, \quad x_2 = 7 - 4x_3 - 2x_4,$$

and so x_3 and x_4 are free variables, which can each take any value. These values then determine x_2.

Finally, the first equation gives us x_1, via

$$2 = x_1+4x_2+3x_3+6x_4 = x_1+4(7-4x_3-2x_4)+3x_3+6x_4,$$

so that

$$x_1 = 13x_3 + 2x_4 - 26, \quad x_2 = 7 - 4x_3 - 2x_4.$$

Hence x_1 and x_2 are given in terms of the free variables x_3 and x_4: this is the *general solution*.

This system has infinitely many solutions, because x_3 and x_4 can be assigned any value.

5. The variables x, y, z solve the equations

$$\begin{aligned} x + y + az &= 42 \\ 5x + 11y + (1 + 6a)z &= 234 \\ 3x + 15y + (2 + 7a)z &= 180, \end{aligned}$$

where the *parameter* a is a real number (for example, if x, y, z represents outputs of three products and the numbers 42, 234, 180 on the right represent inputs of raw materials, then a affects how much of each raw material the third product uses).

For which values of a does the system have:
no solution; a unique solution; more than one solution?

The augmented matrix for

$$\begin{aligned} x + y + az &= 42 \\ 5x + 11y + (1+6a)z &= 234 \\ 3x + 15y + (2+7a)z &= 180 \end{aligned}$$

is

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 42 \\ 5 & 11 & 1+6a & 234 \\ 3 & 15 & 2+7a & 180 \end{array}\right).$$

We do not know a in advance, but we can still use Gaussian elimination.

Again we use row operations:

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 42 \\ 5 & 11 & 1+6a & 234 \\ 3 & 15 & 2+7a & 180 \end{array}\right)$$

$$(R2 \to R2 - 5R1, R3 \to R3 - 3R1) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & a & 42 \\ 0 & 6 & 1+a & 24 \\ 0 & 12 & 2+4a & 54 \end{array}\right)$$

$$(R3 \to R3 - 2R2) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & a & 42 \\ 0 & 6 & 1+a & 24 \\ 0 & 0 & 2a & 6 \end{array}\right).$$

Thus we obtain

$$\left(\begin{array}{ccc|c} 1 & 1 & a & 42 \\ 0 & 6 & 1+a & 24 \\ 0 & 0 & 2a & 6 \end{array}\right).$$

If $a = 0$ then the last equation reads $0 = 6$, which is impossible, so there is no solution (the system is inconsistent).

If $a \neq 0$ then the last equation gives $z = 6/2a = 3/a$. In this case the second equation gives

$$6y = 24 - (1+a)z = 24 - \frac{3+3a}{a} = 21 - \frac{3}{a}$$

and the first equation gives

$$x = 42 - az - y = 42 - 3 - \frac{1}{6}\left(21 - \frac{3}{a}\right) = 39 - \frac{21}{6} + \frac{1}{2a}.$$

In summary:

if $a = 0$ there is no solution;

for $a \neq 0$ there is a unique solution.

There are no values of a which give rise to more than one solution.

6. Solve (if possible) the equations

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 2x_4 &= 1 \\ 2x_1 + 7x_2 + 10x_3 + 5x_4 &= 2 \\ 3x_1 + 9x_2 + 12x_3 + 7x_4 &= 6 \\ x_1 + 2x_2 + 2x_3 + 2x_4 &= 4. \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 2 & 7 & 10 & 5 & 2 \\ 3 & 9 & 12 & 7 & 6 \\ 1 & 2 & 2 & 2 & 4 \end{array}\right).$$

We apply row operations to

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 2 & 7 & 10 & 5 & 2 \\ 3 & 9 & 12 & 7 & 6 \\ 1 & 2 & 2 & 2 & 4 \end{array}\right)$$

in order to reduce the block on the left to echelon form. First, $R2 \to R2' = R2 - 2R1, R3 \to R3' = R3 - 3R1, R4 \to R4' = R4 - R1$ gives

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & -1 & -2 & 0 & 3 \end{array}\right).$$

We now have

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & -1 & -2 & 0 & 3 \end{array}\right),$$

with 0s in the first place of rows 2, 3 and 4. Thus x_1 has been eliminated in the new equations 2, 3 and 4 (which explains the name *Gaussian elimination* and is a good idea, because having fewer variables usually makes the equations easier to solve).

For

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & -1 & -2 & 0 & 3 \end{array}\right),$$

$R4 \to R4' = R4 + R2$ gives

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Finally, in

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

we use $R4 \to R4' = R4 - R3$ to get echelon form:

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This new system has the same solutions as the original one, and each of its equations has at least one variable fewer than the one above it. So we work upwards from the last row.

The last equation of

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

just gives $0 = 0$, and we are left with 3 equations in 4 variables. The third equation tells us that $x_4 = 3$, so the second becomes $x_2 + 2x_3 + x_4 = x_2 + 2x_3 + 3 = 0$.

So one of x_2 and x_3 can take any value we like, which then determines the other variable.
We choose x_3 to be free, giving $x_2 = -2x_3 - 3$.

Finally, the first equation of

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

gives

$$\begin{aligned} 1 &= x_1 + 3x_2 + 4x_3 + 2x_4 \\ &= x_1 + 3x_2 + 4x_3 + 6 \\ &= x_1 + 3(-2x_3 - 3) + 4x_3 + 6 \\ &= x_1 - 2x_3 - 3, \end{aligned}$$

so $x_1 = 2x_3 + 4$. We have found the general solution in terms of the free variable x_3:

$$x_1 = 2x_3 + 4, \quad x_2 = -2x_3 - 3, \quad x_4 = 3.$$

The number of free variables is $1 = 4 - 3$ (there are 4 variables but only 3 equations).

Summary of the method

Given a system of m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2, \\ \vdots \quad & \quad \vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

write down the augmented matrix

$$C = \left(\, A \middle| B \, \right),$$

where the coefficients a_{ij} go in the left block A, and the terms b_i form the last column B.

Apply row operations to $C = \left(A \middle| B \right)$ until you get

$$C^* = \left(A^* \middle| B^* \right),$$

where A^* is in echelon form. The solutions of this system are the same as for the original system.

If you find a row of 0s in A^* with a non-zero entry c in the same row of B^*, then this equation reads

$$0x_1 + 0x_2 + \ldots + 0x_n = c \neq 0,$$

which is impossible, so in this case there is no solution: the equations are inconsistent. This is the only way that the system can have no solution.

If in $C^* = \left(A^* \middle| B^* \right)$ you do not find a row of 0s in A^* with a non-zero entry in the same row of B^*, then there is at least one solution, which you can find by working upwards from the last row.

In this case, free variables, which can take any value, may occur; this is because, as you work upwards, if two or more variables appear for the first time in the same equation then that equation determines only one of them.

Suppose we have m equations in n unknowns $x_1, \ldots, x_n$ of the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0. \end{aligned}$$

The coefficients a_{ij} are given, and all the b_i are 0.

Systems of this type are called *homogeneous*. In this case there is always at least one solution, because we can set all the x_j equal to 0. This is sometimes called the *trivial* solution.

However, there may be other solutions.

The augmented matrix for the system has the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right).$$

As before we apply row operations to make the left-hand block be in echelon form.

We will take as an example the system of equations

$$\begin{aligned}
4x_1 + x_2 + 3x_3 + 8x_4 &= 0 \\
2x_1 + 7x_2 + 4x_3 + x_4 &= 0 \\
3x_1 + 9x_2 + 2x_3 + 5x_4 &= 0 \\
11x_1 + 24x_2 + 13x_3 + 15x_4 &= 0.
\end{aligned}$$

This is the same as Example 3 from 8.1, which had no solution at all, but here we have changed all the b_j to 0. The augmented matrix is

$$\left(\begin{array}{cccc|c}
4 & 1 & 3 & 8 & 0 \\
2 & 7 & 4 & 1 & 0 \\
3 & 9 & 2 & 5 & 0 \\
11 & 24 & 13 & 15 & 0
\end{array}\right).$$

We do the reduction to echelon form following the same steps we used in Example 3 from 8.1, but all the numbers to the right of the vertical bar stay 0. We get

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 0 \\ 0 & -3 & -8 & 7 & 0 \\ 0 & 0 & 89 & -73 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We interpret this new augmented matrix

$$\left(\begin{array}{cccc|c} 2 & 7 & 4 & 1 & 0 \\ 0 & -3 & -8 & 7 & 0 \\ 0 & 0 & 89 & -73 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The last equation is vacuous: it just says that $0 = 0$.
The third equation tells us that $89x_3 - 73x_4 = 0$ i.e. $89x_3 = 73x_4$.
This means that we can assign any value we like to x_4, so x_4 is a *free variable*, and then $x_3 = 73x_4/89$.

The second equation now gives us

$$-3x_2 - 8x_3 + 7x_4 = 0, \quad x_2 = \frac{7x_4 - 8x_3}{3}.$$

Finally we get x_1 from the first equation:

$$2x_1 + 7x_2 + 4x_3 + x_4 = 0, \quad x_1 = -\frac{7x_2 + 4x_3 + x_4}{2}.$$

There are infinitely many solutions, depending on x_4:

$$x_3 = \frac{73x_4}{89}, \quad x_2 = \frac{13x_4}{89}, \quad x_1 = -\frac{236x_4}{89}.$$

Here we started with equations in 4 variables, then reduction to echelon form produced 3 non-zero rows, and the number of free variables was $1 = 4 - 3$, because each equation determines only one variable. A relation like this always holds for homogeneous systems (where all $b_j = 0$). Mathematicians call this the *rank-nullity theorem* (but we do not need this in HG2M02).

9 EIGENVALUES AND EIGENVECTORS

9.1 EXAMPLE

The prices of two commodities P and Q change rapidly. It is found that if p_n and q_n are the prices of P and Q respectively at the end of day n, then

$$p_{n+1} = 4p_n + 2q_n, \quad q_{n+1} = p_n + 3q_n.$$

We write this system as a matrix equation

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

If we start from $p_0 = 1, q_0 = 2$, then using

$$p_{n+1} = 4p_n + 2q_n, \quad q_{n+1} = p_n + 3q_n,$$

it is easy to check that the prices go

$$\begin{aligned} p_1 &= 4 \times 1 + 2 \times 2 = 8, \quad q_1 = 1 + 3 \times 2 = 7, \\ p_2 &= 4 \times 8 + 2 \times 7 = 46, \quad q_2 = 8 + 3 \times 7 = 29, \\ p_3 &= 4 \times 46 + 2 \times 29 = 242, \quad q_3 = 46 + 3 \times 29 = 133. \end{aligned}$$

Thus

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 46 \\ 29 \end{pmatrix}; \quad \begin{pmatrix} p_3 \\ q_3 \end{pmatrix} = \begin{pmatrix} 242 \\ 133 \end{pmatrix}.$$

This does not appear to give any obvious simple pattern.

If, however, we start from $p_0 = 2, q_0 = 1$ then we get

$$\begin{aligned} p_1 &= 4 \times 2 + 2 \times 1 = 10, \quad q_1 = 2 + 3 \times 1 = 5, \\ p_2 &= 4 \times 10 + 2 \times 5 = 50, \quad q_2 = 10 + 3 \times 5 = 25, \\ p_3 &= 4 \times 50 + 2 \times 25 = 250, \quad q_3 = 50 + 3 \times 25 = 125. \end{aligned}$$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 50 \\ 25 \end{pmatrix} = 25 \begin{pmatrix} p_0 \\ q_0 \end{pmatrix},$$

and so on.

In this case the prices multiply by a factor 5 after each day, so that it is easy to write down that $p_n = 5^n 2, q_n = 5^n$.

The column vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is called an *eigenvector* of the matrix

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

and 5 is its associated *eigenvalue*, because

$$A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Eigenvalues and eigenvectors play a very important role in the theory and applications of matrices.

In particular we will use them later on to solve problems of the form

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = A \cdot \begin{pmatrix} p_n \\ q_n \end{pmatrix},$$

where A is a matrix.

In some case eigenvalues and eigenvectors also give an easy way to compute powers A^n of A, and also to solve some systems of differential equations.

Let $A = (a_{ij})$ be a $n \times n$ matrix. A number λ is called an *eigenvalue* of A if we can find a column vector

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

such that

$$AX = \lambda X = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix} .$$

Here X is called an *eigenvector* for the eigenvalue λ. Multiplying X in front by A multiplies X by the factor λ.

To find these, the first step is to find the eigenvalues.

9.3 A USEFUL FACT

If B is an $n \times n$ matrix and

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a column vector such that

$$BX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

then either $\det B = 0$ or

$$X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Why? If $\det B \neq 0$ then B^{-1} exists and

$$BX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

gives

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = B^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = B^{-1}BX = I_n X = X.$$

Let $A = (a_{ij})$ be an $n \times n$ matrix and suppose λ is an *eigenvalue* of A. Then there exists a column vector

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $AX = \lambda X$.

This means that, with I_n the $n \times n$ identity matrix,

$$(A - \lambda I_n)X = AX - \lambda X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So if λ is an eigenvalue then $\det(A - \lambda I_n) = 0$ by §9.3.

So we need to solve

$$0 = \det(A - \lambda I_n) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}.$$

This is called the *characteristic equation* of A, and we need to solve it to find the possible eigenvalues λ.

Summary: to find the eigenvalues of an $n \times n$ matrix A, just solve $0 = \det(A - \lambda I_n)$ for λ.

9.5 EXAMPLE

Find all eigenvalues, and corresponding eigenvectors, for

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

We start by solving for eigenvalues:

$$\begin{aligned} 0 \;&=\; \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} \\ &=\; (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2). \end{aligned}$$

Thus the eigenvalues are $\lambda = 5, 2$.

To find an eigenvector for $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ and $\lambda = 5$, we need $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 5 \begin{pmatrix} x \\ y \end{pmatrix} = 5I_2 \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - 5I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $-x + 2y = 0, x - 2y = 0$, which both give $x = 2y$. So we can take $x = 2, y = 1$ to get eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

To find an eigenvector for $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ and $\lambda = 2$, we need $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} = 2I_2 \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - 2I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $2x + 2y = 0, x + y = 0$, which both give $x = -y$. So we can take $x = -1, y = 1$ to get eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Comment: the eigenvector for an eigenvalue is not unique. If $X \neq (0)$ and $AX = \lambda X$ then for any number $\mu \neq 0$ we have

$$A(\mu X) = \mu AX = \mu\lambda X = \lambda(\mu X)$$

so μX is also an eigenvector. Thus

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} \pi \\ -\pi \end{pmatrix}$$

would all be valid eigenvectors for $\lambda = 2$ in the last example (but $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ would not be valid).

9.6 EXAMPLE

Find all eigenvalues, and corresponding eigenvectors, for

$$A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}.$$

As before, we start by solving for eigenvalues:

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) + 2 \\ &= \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2). \end{aligned}$$

Thus the eigenvalues are $\lambda = 1, 2$.

To find an eigenvector for $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ and $\lambda = 1$, we need $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = I_2 \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $2x + y = 0, -2x - y = 0$, which both give $y = -2x$.
So we can take $x = 1, y = -2$ to get eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

To find an eigenvector for $A = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$ and $\lambda = 2$, we need $\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} = 2I_2 \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - 2I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So we need

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $x + y = 0, -2x - 2y = 0$, which both give $y = -x$.

So we can take $x = 1, y = -1$ to get eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

9.7 EXAMPLE

Find all eigenvalues, and corresponding eigenvectors, for

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}.$$

This matrix is 3×3 but the method is the same. First we find the eigenvalues via:

$$0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & -1 \\ 0 & 3-\lambda & 1 \\ 0 & -2 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{vmatrix}.$$

So we need

$$0 = -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda - 1)(\lambda - 2), \quad \lambda = 0, 1, 2.$$

To find an eigenvector for $A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$ and $\lambda = 0$

we need $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ such that

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and so $y - z = 0, \quad 3y + z = 0, \quad -2y = 0.$
Thus $y = 0$ and so $z = 0$, but we can take $x = 1$ and eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

To find an eigenvector for $A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$ and $\lambda = 1$

we need $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

From

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}-1 & 1 & -1\\ 0 & 2 & 1\\ 0 & -2 & -1\end{pmatrix} \begin{pmatrix}x\\y\\z\end{pmatrix}$$

we get

$$-x + y - z = 0, \quad 2y + z = 0, \quad -2y - z = 0.$$

The last two equations just give $z = -2y$, so that the first gives $0 = -x + y - z = -x + 3y$, i.e. $x = 3y$. So we can take $x = 3, y = 1, z = -2$ and eigenvector $\begin{pmatrix}3\\1\\-2\end{pmatrix}$.

To find an eigenvector for $A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$ and $\lambda = 2$

we need $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ such that

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - 2I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

From

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}-2 & 1 & -1\\ 0 & 1 & 1\\ 0 & -2 & -2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

we get

$$-2x + y - z = 0, \quad y + z = 0, \quad -2y - 2z = 0.$$

The last two equations just give $z = -y$, so that the first gives $0 = -2x + y - z = -2x + 2y$, i.e. $x = y$. So we can take $x = 1, y = 1, z = -1$ and eigenvector $\begin{pmatrix}1\\1\\-1\end{pmatrix}$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} f & g \\ h & j \end{pmatrix}, \quad C = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

We calculate

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f & g \\ h & j \end{pmatrix} = \begin{pmatrix} af + bh & ag + bj \\ cf + dh & cg + dj \end{pmatrix}.$$

So the columns of AB are the same as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} g \\ j \end{pmatrix}.$$

This says that we can find the jth column of AB by calculating A times the vector made from the jth column of B.

Also

$$BC = \begin{pmatrix} f & g \\ h & j \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} fp & gq \\ hp & jq \end{pmatrix}$$

(column 1 has been multiplied by p, column 2 by q).
Thus if we multiply B on the right by a diagonal matrix C, we multiply the columns of B by the corresponding entries of C.
These properties will be important in the next section.

Sometimes we can use eigenvectors to turn a square matrix into a diagonal matrix. Consider again

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

This had eigenvalues:
5, with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and 2, with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Suppose we make a matrix B with these eigenvectors as *columns*:

$$B = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

§9.8 says that AB is the matrix whose columns are

$$A\begin{pmatrix}2\\1\end{pmatrix} = 5\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}10\\5\end{pmatrix} \quad \text{and} \quad A\begin{pmatrix}-1\\1\end{pmatrix} = 2\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}$$

Thus (it is easy to check that)

$$AB = \begin{pmatrix}4 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\1 & 1\end{pmatrix} = \begin{pmatrix}10 & -2\\5 & 2\end{pmatrix}.$$

Multiplying B in front by A multiplies column 1 by the first eigenvalue 5, column 2 by the second eigenvalue 2.

But, again as in §9.8,

$$B\begin{pmatrix}5&0\\0&2\end{pmatrix}=\begin{pmatrix}2&-1\\1&1\end{pmatrix}\begin{pmatrix}5&0\\0&2\end{pmatrix}=\begin{pmatrix}10&-2\\5&2\end{pmatrix}=AB.$$

Multiplying B on the right by $D=\begin{pmatrix}5&0\\0&2\end{pmatrix}$ has the same effect as multiplying on the left by A: both multiply column 1 by the first eigenvalue 5, and column 2 by the second eigenvalue 2.

This means that

$$AB = BD, \quad \text{where} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

We can take this further: we have

$$\det B = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 3 \neq 0, \quad B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

(using the rule from Chapter 5), so we have

$$AB = BD, \quad B^{-1}AB = B^{-1}BD = D,$$

as well as

$$BDB^{-1} = ABB^{-1} = A.$$

In deriving D from A, we have *diagonalised* matrix A.

Suppose now we want to compute A^{60}: to do this directly would be very slow. But $A = BDB^{-1}$, so

$$\begin{aligned} A^2 &= BDB^{-1} \cdot BDB^{-1} = BDI_2DB^{-1} = BD^2B^{-1}, \\ A^3 &= A^2A = BD^2B^{-1} \cdot BDB^{-1} \\ &= BD^2I_2DB^{-1} = BD^3B^{-1}, \end{aligned}$$

and if we repeat this we get

$$A^{60} = BD^{60}B^{-1}.$$

This is a good idea because D^{60} is much easier to compute: we have

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 4 \end{pmatrix}$$

and

$$D^3 = \begin{pmatrix} 25 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 8 \end{pmatrix},$$

and repeating this gives

$$D^{60} = \begin{pmatrix} 5^{60} & 0 \\ 0 & 2^{60} \end{pmatrix}.$$

Thus

$$A^{60} = BD^{60}B^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^{60} & 0 \\ 0 & 2^{60} \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

and so we get

$$\begin{aligned} A^{60} &= \frac{1}{3} \begin{pmatrix} 2 \cdot 5^{60} & -2^{60} \\ 5^{60} & 2^{60} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 \cdot 5^{60} + 2^{60} & 2 \cdot 5^{60} - 2^{61} \\ 5^{60} - 2^{60} & 5^{60} + 2^{61} \end{pmatrix}. \end{aligned}$$

I have checked this by computer!

Since $5^{60} = 25^{30} > 10^{30}$, the entries of A^{60} are very large (more than 30 digits), so calculating A^{60} by repeatedly multiplying would be somewhat arduous!

Suppose that A is an $n \times n$ matrix and that we have found n distinct real eigenvalues $\lambda_1, \ldots, \lambda_n$ for A. This means that for each j there is a column vector

$$X_j = \begin{pmatrix} x_{1j} \\ \vdots \\ x_{nj} \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{with} \quad AX_j = \lambda_j X_j.$$

We form a matrix

$$B = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}.$$

What we have done is to make an $n \times n$ matrix B in which the jth column is X_j.

We also make an $n \times n$ diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

The entries on the diagonal are $\lambda_1, \dots, \lambda_n$ and it is very important to write these in the same order as $X_1, \dots, X_n$. Now

$$BD = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \dots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \dots & \lambda_n x_{2n} \\ \vdots & & & \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \dots & \lambda_n x_{nn} \end{pmatrix};$$

this is because multiplying B on the right by D multiplies the jth column of B by λ_j.

But it is also true that

$$AB = A\cdot\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \dots & \lambda_n x_{1n} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \dots & \lambda_n x_{2n} \\ \vdots & & & \\ \lambda_1 x_{n1} & \lambda_2 x_{n2} & \dots & \lambda_n x_{nn} \end{pmatrix};$$

this is because the jth column of AB is formed using the jth column of B, which is the eigenvector X_j.

So we have

$$AB = BD, \quad A = BDB^{-1}, \quad D = B^{-1}AB.$$

To summarise:
A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots \lambda_n$;
B is an $n \times n$ matrix in which the jth column is the eigenvector X_j corresponding to the eigenvalue λ_j;
D is an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, in the same order as $X_1, \ldots, X_n$.

Under these conditions the inverse matrix B^{-1} is guaranteed to exist (and you can find it by the Gauss-Jordan method).

This process is called *diagonalising* A.

9.11 EXAMPLE

Diagonalise the matrix A, and find A^5, when

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

We start by finding the eigenvalues via

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix}.$$

Because this is upper triangular, we need $0 = (1-\lambda)(2-\lambda)(-1-\lambda)$ and so $\lambda = 1, 2, -1$.

Useful fact: *if A is a square matrix which is upper triangular, lower triangular or diagonal, then its eigenvalues are the entries on the main diagonal.*

We find an eigenvector for $\lambda = 1$: we need

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives $z = 0$, $y + z = 0$, $-2z = 0$, so $y = -z = 0$,

and we can take $x = 1$ and eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

We find an eigenvector for $\lambda = 2$: we need

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - 2I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives $-x + z = 0$, $z = 0$, $-3z = 0$, so $x = z = 0$, and we can take $y = 1$ and eigenvector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

We find an eigenvector for $\lambda = -1$: we need

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = - \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -I_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A + I_3) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This gives $2x + z = 0$, $3y + z = 0$, $0 = 0$, and we can take $z = -6$ and $x = 3, y = 2$, to get eigenvector $\begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$.

So we write down

$$B = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Here the eigenvectors form the columns of B, and the corresponding eigenvalues (N.B. in the **same order**) are the diagonal entries of D.

We then have

$$AB = BD, \quad A = BDB^{-1}, \quad D = B^{-1}AB.$$

So we need to calculate B^{-1}:

$$
\begin{aligned}
(B|I_3) &= \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -6 & 0 & 0 & 1 \end{array}\right) \\
(R1 + (1/2)R3, R2 + (1/3)R3) &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & -6 & 0 & 0 & 1 \end{array}\right) \\
(R3 \to -(1/6)R3) &\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 1 & 0 & 0 & -1/6 \end{array}\right); \\
B^{-1} &= \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}.
\end{aligned}
$$

Thus we have

$$B^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

as well as

$$A^5 = (BDB^{-1})^5 = BD^5B^{-1} = B \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot B^{-1}.$$

Hence

$$A^5 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{6} \end{pmatrix}$$

so that

$$A^5 = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 32 & -2 \\ 0 & 0 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 32 & 11 \\ 0 & 0 & -1 \end{pmatrix}$$

(I have checked this!).

(a) Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

To find eigenvalues we write

$$0 = \det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

The only solution is $\lambda = 1$. To find an eigenvector we write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - I_2) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives $y = 0$, so an eigenvector is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have only found one eigenvector: so we cannot use the previous method to find powers of A. However, in this case it is easy to check that

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

OPTIONAL: keep multiplying by A and convince yourself that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

(b) Let

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To find eigenvalues we write

$$0 = \det(B - \lambda I_2) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

This equation has no real solutions, since $\lambda^2 + 1 \geq 1$ for real λ.

The eigenvalue equation $\lambda^2 + 1 = 0$ has complex roots $\lambda = \pm i$, where $i^2 = -1$. In fact

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} = i \begin{pmatrix} 1 \\ i \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

If you have not met complex numbers before, it does not matter.

In general, $n \times n$ matrices don't always have n distinct real eigenvalues, but in the HG2M02 exam, **they always will**: these exceptional cases will not appear in the exam.

Take a *symmetric* 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

a, b, c, d real. The eigenvalues are found by setting

$$\begin{aligned} 0 &= \begin{vmatrix} a-\lambda & b \\ b & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - b^2 \\ &= \lambda^2 - (a+d)\lambda + ad - b^2. \end{aligned}$$

For $\lambda^2 - (a+d)\lambda + ad - b^2 = 0$ the quadratic formula gives

$$\begin{aligned}
\lambda &= \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-b^2)}}{2} \\
&= \frac{a+d \pm \sqrt{a^2+d^2+2ad-4ad+4b^2}}{2} \\
&= \frac{a+d \pm \sqrt{a^2+d^2-2ad+4b^2}}{2} \\
&= \frac{a+d \pm \sqrt{(a-d)^2+4b^2}}{2}.
\end{aligned}$$

Because $(a-d)^2 \geq 0$ and $4b^2 \geq 0$ both eigenvalues are real. Also the eigenvalues are distinct (and so A can be diagonalised), except when $a-d=b=0$, in which case A is already a diagonal matrix.

In fact we have the following important result – it will not be needed in HG2M02 but you may see it used in other modules.

Theorem: *Let A be an $n \times n$ matrix with real entries which is symmetric (i.e. $A = A^T$). Then all eigenvalues of A are real and A can be diagonalised i.e. there exist an invertible matrix B and a diagonal matrix D such that $AB = BD$, $A = BDB^{-1}$, $D = B^{-1}AB$.*

10 DISCRETE MODELS AND DIFFERENCE EQUATIONS

We go back to the example which began the last chapter.

10.1 EXAMPLE

The prices of two commodities P and Q change rapidly. If p_n and q_n are the prices of P and Q respectively at the end of day n, then

$$p_{n+1} = 4p_n + 2q_n, \quad q_{n+1} = p_n + 3q_n.$$

Determine p_n and q_n, if $p_0 = 1, q_0 = 2$.

Systems like this are called *difference equations*, because the differences $p_{n+1} - p_n$ and $q_{n+1} - q_n$ can be expressed in terms of p_n, q_n.

We write the system as a matrix equation

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 4p_n + 2q_n \\ p_n + 3q_n \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

We saw in Chapter 9 that the prices go

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} 46 \\ 29 \end{pmatrix}; \quad \begin{pmatrix} p_3 \\ q_3 \end{pmatrix} = \begin{pmatrix} 242 \\ 133 \end{pmatrix},$$

which does not exhibit any obvious pattern.

However, with the eigenvalue/eigenvector techniques from the last chapter, we can now solve this problem completely.

In Example 9.5 we found that

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

has the following eigenvalues:

5, with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and 2, with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

This means that

$$A \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, \quad A \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

The next step is to write the starting values $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ in terms of these two eigenvectors. So we need to find real numbers a, b with

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = a \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This requires

$$1 = 2a - b, \quad 2 = a + b.$$

Adding gives $3 = 3a$, so $a = 1$, and $b = 1$ also. So

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Now

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = A \cdot \begin{pmatrix} p_n \\ q_n \end{pmatrix}.$$

So

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = A \cdot \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}, \quad \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = A \cdot \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = A^2 \cdot \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}.$$

Repeating this gives

$$\begin{aligned} \begin{pmatrix} p_n \\ q_n \end{pmatrix} &= A^n \cdot \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \\ &= A^n \cdot \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] \\ &= A^n \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + A^n \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

But, because these are eigenvectors,

$$A \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad A \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Multiplying $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in front by A is the same as multiplying it by 5. So

$$\begin{aligned} A^2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} &= A\left(A\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = A\left(5\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) \\ &= 5A\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 5^2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \end{aligned}$$

Repeatedly using this idea gives

$$A^n \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5^n \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad A^n \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2^n \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Combining these formulas we get

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = A^n \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + A^n \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 5^n \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^n \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We can read off: $p_n = 2 \cdot 5^n - 2^n$; $q_n = 5^n + 2^n$. Hence

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 5^1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix},$$

and

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = 5^2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 46 \\ 29 \end{pmatrix},$$

and

$$\begin{pmatrix} p_3 \\ q_3 \end{pmatrix} = 5^3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2^3 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 242 \\ 133 \end{pmatrix}.$$

These are the values we got earlier, but we now have a formula valid for *every* n.

An alternative: using diagonalisation and a change of variables

In §9.9 we took

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

and we wrote down a matrix B with the eigenvectors of A as *columns*:

$$B = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

We then got

$$AB = BD, \quad \text{where} \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix},$$

and

$$AB = BD, \quad B^{-1}AB = D, \quad A = BDB^{-1}.$$

Suppose we define new variables r_n, s_n by

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = B \cdot \begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 2r_n - s_n \\ r_n + s_n \end{pmatrix}.$$

Then

$$\begin{pmatrix} r_n \\ s_n \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

and

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = B^{-1} \cdot \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = B^{-1}A \cdot \begin{pmatrix} p_n \\ q_n \end{pmatrix} = B^{-1}AB \cdot \begin{pmatrix} r_n \\ s_n \end{pmatrix}.$$

But this gives

$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = D \cdot \begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 5r_n \\ 2s_n \end{pmatrix}.$$

This new system

$$r_{n+1} = 5r_n, \quad s_{n+1} = 2s_n$$

is much simpler and easier to solve, with solution

$$r_n = 5^n r_0, \quad s_n = 2^n s_0.$$

To find r_0 and s_0 given that $p_0 = 1, q_0 = 2$ we just write

$$1 = p_0 = 2r_0 - s_0, \quad 2 = q_0 = r_0 + s_0,$$

which solves to give $r_0 = s_0 = 1$. Thus $r_n = 5^n$, $s_n = 2^n$ and

$$p_n = 2r_n - s_n = 2 \cdot 5^n - 2^n, \quad q_n = r_n + s_n = 5^n + 2^n,$$

which is the same as what we got by the first method.
The two methods are closely related and it does not matter which you use along as your working is clear.
However, I tend to prefer the first method.

10.2 EXAMPLE

Suppose that $x_0 = 5$ and $y_0 = -7$, and that

$$x_{n+1} = x_n - 2y_n, \quad y_{n+1} = x_n + 4y_n.$$

Hence

$$x_1 = 19, \quad y_1 = -23, \quad x_2 = 65, \quad y_2 = -73, \ldots$$

To find x_n, y_n for *every* n we write $\mathbf{v}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, and

$$\mathbf{v}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n - 2y_n \\ x_n + 4y_n \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

So we have

$$\mathbf{v}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}.$$

We need to find eigenvalues for A, and their eigenvectors.
To find the eigenvalues we need

$$0 = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix},$$

$$0 = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Hence the eigenvalues are $\lambda = 2, 3$.

To find an eigenvector for $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ and $\lambda = 2$ we write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - 2I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives $-x - 2y = 0, x + 2y = 0$, $x = -2y$ and we can take $x = 2$, $y = -1$ and eigenvector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

To find an eigenvector for $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ and $\lambda = 3$ we write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - 3I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This gives $-2x - 2y = 0, x + y = 0$, $x = -y$ and we can take $x = 1$, $y = -1$ and eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus we have

$$A \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad A \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Multiplying n times in front by A gives

$$A^n \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2^n \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad A^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The next step is to express $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ in terms of these eigenvectors:

$$\begin{pmatrix} 5 \\ -7 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = c \begin{pmatrix} 2 \\ -1 \end{pmatrix} + d \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$5 = 2c + d, \quad -7 = -c - d.$$

Adding these two equations gives $-2 = c$, $c = -2$, and so $-7 = 2 - d$, $d = 9$. So

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 9 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now

$$\begin{aligned}
\mathbf{v}_n &= \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\
&= A^n \cdot \left[-2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 9 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\
&= -2A^n \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 9A^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= -2 \cdot 2^n \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + 9 \cdot 3^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\end{aligned}$$

so

$$x_n = -2^{n+2} + 3^{n+2}, \quad y_n = 2^{n+1} - 3^{n+2}.$$

Check: using

$$x_n = -2^{n+2} + 3^{n+2}, \quad y_n = 2^{n+1} - 3^{n+2},$$

gives

$$x_0 = -4 + 9 = 5, \quad y_0 = 2 - 9 = -7,$$

$$x_1 = -8 + 27 = 19, \quad y_1 = 4 - 27 = -23,$$

$$x_2 = -16 + 81 = 65, \quad y_2 = 8 - 81 = -73,$$

which agrees with what we got earlier.

We can consider similar problems with more than two variables. The method is essentially the same, and will use eigenvectors.

Example. Three variables p_n, q_n, r_n satisfy the formulas

$$\begin{aligned} p_{n+1} &= 7p_n - 5q_n - 8r_n \\ q_{n+1} &= 2q_n \\ r_{n+1} &= 4p_n - 4q_n - 5r_n. \end{aligned}$$

Find p_n, q_n, r_n if we are given that $p_0 = 7$, $q_0 = 2$ and $r_0 = 4$.

As before we can write this as a matrix system

$$\begin{pmatrix} p_{n+1} \\ q_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} 7p_n - 5q_n - 8r_n \\ 2q_n \\ 4p_n - 4q_n - 5r_n \end{pmatrix}$$

$$= A \cdot \begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix},$$

$$A = \begin{pmatrix} 7 & -5 & -8 \\ 0 & 2 & 0 \\ 4 & -4 & -5 \end{pmatrix}.$$

We first find eigenvalues and eigenvectors for A.

To obtain eigenvalues for A we write as before

$$\begin{aligned}
0 &= \begin{vmatrix} 7-\lambda & -5 & -8 \\ 0 & 2-\lambda & 0 \\ 4 & -4 & -5-\lambda \end{vmatrix} \\
&= (2-\lambda)(-1)^{2+2} \begin{vmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{vmatrix} \\
&= (2-\lambda)((7-\lambda)(-5-\lambda)+32) \\
&= (2-\lambda)(-35-2\lambda+\lambda^2+32) \\
&= (2-\lambda)(\lambda^2-2\lambda-3) \\
&= (2-\lambda)(\lambda-3)(\lambda+1).
\end{aligned}$$

Here we expanded by row 2. This gives eigenvalues $2, 3, -1$.

To find an eigenvector for $\lambda = 2$ we write

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - \lambda I_3) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7-\lambda & -5 & -8 \\ 0 & 2-\lambda & 0 \\ 4 & -4 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 5 & -5 & -8 \\ 0 & 0 & 0 \\ 4 & -4 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which gives two equations

$$5x - 5y - 8z = 0, \quad 4x - 4y - 7z = 0.$$

These give

$$20x - 20y - 32z = 0, \quad 20x - 20y - 35z = 0, \quad 3z = 0, \quad 20x = 20y.$$

So we can take $x = y = 1$ and $z = 0$.

So our eigenvector for $\lambda = 2$ is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Note here that

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad A^n \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2^n \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Next we find an eigenvector for $\lambda = 3$ by writing

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7-\lambda & -5 & -8 \\ 0 & 2-\lambda & 0 \\ 4 & -4 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -5 & -8 \\ 0 & -1 & 0 \\ 4 & -4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which gives three equations

$$4x - 5y - 8z = 0, \quad -y = 0, \quad 4x - 4y - 8z = 0,$$

although the third is really just the first minus the second.

The second equation gives $y = 0$ and the first gives $4x - 8z = 0$, $x = 2z$. So we can take $x = 2$, $z = 1$.

Hence our eigenvector for $\lambda = 3$ is $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Note this time that

$$A \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad A^n \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 3^n \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Finally we find an eigenvector for $\lambda = -1$ by writing

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 7-\lambda & -5 & -8 \\ 0 & 2-\lambda & 0 \\ 4 & -4 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -5 & -8 \\ 0 & 3 & 0 \\ 4 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which gives three equations

$$8x - 5y - 8z = 0, \quad 3y = 0, \quad 4x - 4y - 4z = 0.$$

Again the second equation gives $y = 0$ and the first gives $8x - 8z = 0$, $x = z$. So we can take $x = z = 1$.

Thus our eigenvector for $\lambda = -1$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Note this time that

$$A \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad A^n \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (-1)^n \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

To find p_n, q_n, r_n if we are given that $p_0 = 7$, $q_0 = 2$ and $r_0 = 4$ we have to express $\begin{pmatrix} p_0 \\ q_0 \\ r_0 \end{pmatrix}$ in terms of the eigenvectors. So we write

$$\begin{pmatrix} p_0 \\ q_0 \\ r_0 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and solve for a, b, c. So we get

$$7 = a + 2b + c, \quad 2 = a, \quad 4 = b + c.$$

Hence $5 = 2b + c, b = 5 - 4 = 1, 3 = c$ and so

$$\begin{pmatrix} p_0 \\ q_0 \\ r_0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Now we have

$$\begin{aligned}
\begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix} &= A^n \cdot \begin{pmatrix} 7 \\ 2 \\ 4 \end{pmatrix} = A^n \cdot \left[2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \\
&= 2 \cdot A^n \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + A^n \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot A^n \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
&= 2 \cdot 2^n \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3^n \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + 3 \cdot (-1)^n \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

We can read off

$$\begin{aligned}
p_n &= 2^{n+1} + 2 \cdot 3^n + 3 \cdot (-1)^n, \\
q_n &= 2^{n+1}, \\
r_n &= 3^n + 3 \cdot (-1)^n.
\end{aligned}$$

10.4 EXAMPLE

The prices x_n, y_n of two stocks on day n satisfy $x_0 = 11, y_0 = 32$ and

$$x_{n+1} = 5x_n - y_n, \quad y_{n+1} = 6x_n.$$

An investor wishes to buy, on day 0, the stock which will perform better *in the long run*. Which should they buy?

As before we write

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A = \begin{pmatrix} 5 & -1 \\ 6 & 0 \end{pmatrix}.$$

We find the eigenvalues of A by writing

$$0 = \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 6 & -\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

The eigenvalues are $2, 3$. For $\lambda = 2$ we write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - 2I) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which gives $3x - y = 0 = 6x - 2y$.

We can take $x = 1, y = 3$ and eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

For $\lambda = 3$ we write

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - 3I) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which gives $2x - y = 0 = 6x - 3y$.

We can take $x = 1, y = 2$ and eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

We now express the initial values in terms of the eigenvectors:

$$\begin{pmatrix} 11 \\ 32 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

so

$$11 = a + b, \quad 32 = 3a + 2b, \quad 22 = 2a + 2b,$$

which gives

$$a = 32 - 22 = 10, \quad b = 1.$$

This now gives

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A^n \left(10 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$
$$= 10\, A^n \begin{pmatrix} 1 \\ 3 \end{pmatrix} + A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 10 \cdot 2^n \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3^n \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Thus

$$x_n = 10 \cdot 2^n + 3^n, \quad y_n = 30 \cdot 2^n + 2 \cdot 3^n.$$

As n gets very large 3^n becomes much bigger than 2^n and

$$\frac{y_n}{x_n} = \frac{30 \cdot 2^n + 2 \cdot 3^n}{10 \cdot 2^n + 3^n} = \frac{30 \cdot (2/3)^n + 2}{10 \cdot (2/3)^n + 1} \to \frac{0+2}{0+1} = 2.$$

But $y_0/x_0 = 32/11 \approx 2.9 > 2$, so the stock with price x_n is the better investment.

11 EIGENVALUES AND DIFFERENTIAL EQUATIONS

This section will look at some systems of differential equations for which diagonalisation is very useful.

11.1 EXAMPLE

Solve the differential equations

$$\frac{dx}{dt} = 4x + 2y, \quad \frac{dy}{dt} = x + 3y.$$

Here x and y depend on (time) t, and the system of equations expresses their rates of change as combinations of x and y.

One way to solve this is to differentiate the first equation, and then use the second equation, to get

$$\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}(4x+2y) = 4\frac{dx}{dt} + 2\frac{dy}{dt} \\ &= 4(4x+2y) + 2(x+3y) = 18x + 14y \\ &= 18x + 7\left(\frac{dx}{dt} - 4x\right).\end{aligned}$$

This equation can be solved for x, which then gives us y.

However, there is a much better method using matrices and eigenvalues.

We write x', y' for dx/dt and dy/dt and express the equations

$$x' = \frac{dx}{dt} = 4x + 2y, \quad y' = \frac{dy}{dt} = x + 3y,$$

in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4x + 2y \\ x + 3y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix},$$

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

In Example 9.5 we saw that

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

has the following eigenvalues:

5, with eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and 2, with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Also, we wrote down a matrix B with the eigenvectors of A as *columns*, and a diagonal matrix D:

$$B = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

We then got

$$AB = BD, \quad B^{-1}AB = D, \quad A = BDB^{-1}.$$

Now define new variables u, v by

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u - v \\ u + v \end{pmatrix}.$$

Then we get

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= B^{-1} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \\ \begin{pmatrix} u' \\ v' \end{pmatrix} &= B^{-1} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = B^{-1} \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= B^{-1} \cdot A \cdot B \cdot \begin{pmatrix} u \\ v \end{pmatrix} = D \cdot \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 5u \\ 2v \end{pmatrix}. \end{aligned}$$

This has given us a much simpler system to solve:

$$u' = 5u, \quad v' = 2v.$$

Now $u' = 5u$ gives us

$$5 = \frac{u'}{u} = \frac{d}{dt}(\ln u), \quad \ln u = 5t + c,$$

where c is a constant, and so $u = e^{\ln u} = e^{5t+c} = ae^{5t}$, where $a = e^c$ is a constant. Similarly, $v' = 2v$ solves to give $v = be^{2t}$ where b is a constant.

So the solution of the system is (with a, b constants)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ae^{5t} \\ be^{2t} \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = B \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} ae^{5t} \\ be^{2t} \end{pmatrix}$$

and so $x = 2ae^{5t} - be^{2t}$, $y = ae^{5t} + be^{2t}$.

We can check this: $x = 2ae^{5t} - be^{2t}$, $y = ae^{5t} + be^{2t}$ gives

$$x' = 10ae^{5t} - 2be^{2t} = 4(2ae^{5t} - be^{2t}) + 2(ae^{5t} + be^{2t}) = 4x + 2y,$$

$$y' = 5ae^{5t} + 2be^{2t} = 2ae^{5t} - be^{2t} + 3(ae^{5t} + be^{2t}) = x + 3y.$$

Here the diagonalisation trick works very well, because it gives equations which are much easier to solve than the ones we start from.

Note that in this method you do not need to find B^{-1}.

We can also write our solution as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} ae^{5t} \\ be^{2t} \end{pmatrix} = \begin{pmatrix} 2ae^{5t} - be^{2t} \\ ae^{5t} + be^{2t} \end{pmatrix} \quad (1)$$

$$= a \cdot e^{5t} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + b \cdot e^{2t} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Here the terms

$$e^{5t} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad e^{2t} \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

each have the form

$$e^{t \times \text{eigenvalue}} \times \text{eigenvector}.$$

However, the strong advantage of the diagonalisation method is that it shows that *all* solutions have the form (1).

11.2 EXAMPLE

Solve the system of differential equations

$$x' = \frac{dx}{dt} = y - z, \quad y' = \frac{dy}{dt} = 3y + z, \quad z' = \frac{dz}{dt} = -2y.$$

We write this in matrix form as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} y - z \\ 3y + z \\ -2y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}.$$

In §9.7 we found that

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 0 \end{pmatrix}$$

has eigenvalues and corresponding eigenvectors:

$$\lambda = 0, \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \qquad \lambda = 1, \qquad \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix};$$

$$\lambda = 2, \qquad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

As before, we write down a matrix B with the eigenvectors as columns and a diagonal matrix D with the eigenvalues on the diagonal (in the same order):

$$B = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This gives

$$AB = BD, \quad B^{-1}AB = D.$$

Define new variables u, v, w by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = B \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

Then we get (note that we do not need to calculate B^{-1})

$$\begin{aligned}
\begin{pmatrix} u \\ v \\ w \end{pmatrix} &= B^{-1} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \\
\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} &= B^{-1} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = B^{-1} \cdot A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= B^{-1} \cdot A \cdot B \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = D \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0u \\ 1v \\ 2w \end{pmatrix}.
\end{aligned}$$

So we need to solve

$$u' = 0u, \quad v' = v, \quad w' = 2w$$

and the solutions are

$$u = ae^{0t} = a, \quad v = be^{t}, \quad w = ce^{2t},$$

where a, b, c are constants.

This gives

$$
\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= B \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ be^t \\ ce^{2t} \end{pmatrix} \\
&= \begin{pmatrix} a + 3be^t + ce^{2t} \\ be^t + ce^{2t} \\ -2be^t - ce^{2t} \end{pmatrix} \\
&= a \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \cdot e^t \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + c \cdot e^{2t} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} .
\end{aligned}
$$

Again we can check our answer:

$$x = a+3be^t+ce^{2t}, \quad y = be^t+ce^{2t}, \quad z = -2be^t-ce^{2t},$$

gives

$$\begin{aligned} x' &= 3be^t + 2ce^{2t}, \quad y - z = be^t + ce^{2t} - (-2be^t - ce^{2t}) = x' \\ y' &= be^t + 2ce^{2t}, \quad 3y + z = 3(be^t + ce^{2t}) + (-2be^t - ce^{2t}) = y' \\ z' &= -2be^t - 2ce^{2t}, \quad -2y = -2(be^t + ce^{2t}) = z' \end{aligned}$$

as required.

12 BASIC LINEAR PROGRAMMING

In this section we will solve problems involving maximising quantities, of a type which can arise for example in a production process.

12.1 LINEAR INEQUALITIES

Suppose that a, b, c are real numbers, with a, b non-zero. The equation

$$ax + by = c, \quad y = -\frac{ax}{b} + \frac{c}{b},$$

defines a line in the x, y plane with slope $-a/b = \frac{dy}{dx}$.

Now suppose that a, b are positive, and we are given the *inequality*

$$ax + by \leq c.$$

If x or y increases, then so does $ax + by$, since a, b are positive. So $ax + by \leq c$ gives all points on or below the line.

Example: consider the set of points given by

$$x + y \leq 1.$$

Now $x + y = 1$ gives a straight line through $(1, 0)$ and $(0, 1)$.
Reducing x or y reduces $x + y$, and so we need the set of points which lie on or below the line: see Figure 3.

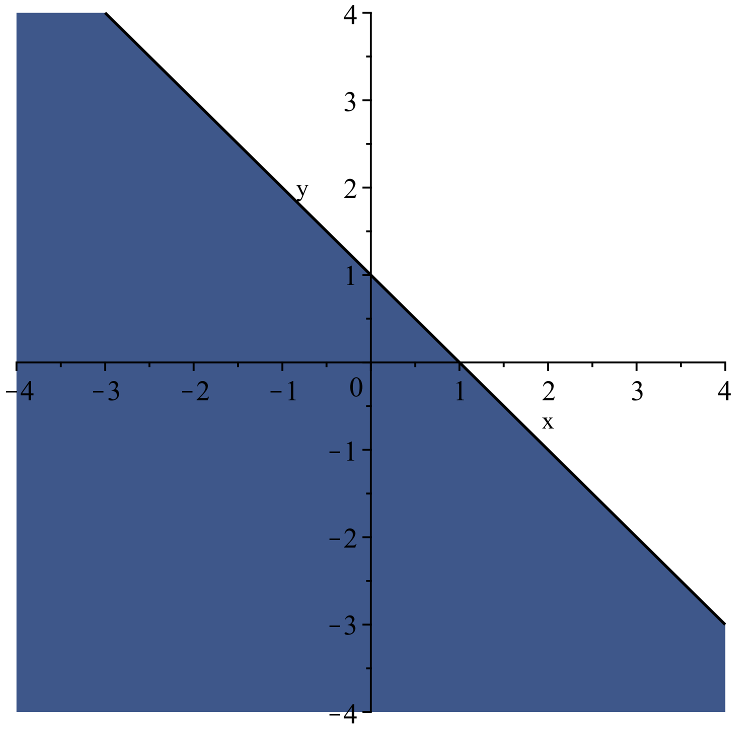

Figure 3: The points satisfying the constraint $x + y \leq 1$

12.2 EXAMPLE

A company makes products A, B using materials P, Q.

A ton of A requires 1 ton of P and 4 tons of Q;
a ton of B needs 3 tons of P and 1 of Q.

It earns: £500 for a ton of A; £100 for a ton of B.
It has in stock 8 tons of P and 10 tons of Q.
How much of each should it make to maximise revenue?

Suppose it makes x_1 tons of A, and x_2 tons of B.
The amount of P used is $x_1 + 3x_2$, so $x_1 + 3x_2 \leq 8$.
The amount of Q used is $4x_1 + x_2$, so $4x_1 + x_2 \leq 10$.

The "obvious" solution is to use all its stock of P, Q, so

$$x_1 + 3x_2 = 8, \quad 4x_1 + x_2 = 10, \quad 12x_1 + 3x_2 = 30.$$

This solves via $11x_1 = 30 - 8$ to give $x_1 = x_2 = 2$.
If it does this, the revenue is $\pounds(500\times 2+100\times 2) = \pounds 1200$.

But suppose instead that the company decides to produce *only* A, and none of B. Then $x_1 \leq 8$ and $4x_1 \leq 10$, so the maximum production of A would be $\frac{10}{4} = 2.5$ tons.
The revenue from this would be $\pounds 500 \times 2.5 = \pounds 1250$, which is *more* than $\pounds 1200$.

So the "obvious" solution, to use up all of its stock of P and Q, does not give the maximum revenue.
Is it possible to earn more than $\pounds 1250$?

This is a *linear programming* problem.

The company has to maximise its revenue

$$z = 500x_1 + 100x_2,$$

subject to the *constraints*

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 3x_2 \leq 8, \quad 4x_1 + x_2 \leq 10.$$

The problem is called linear because x_1 and x_2 only occur to the power 1.

If we plot the problem graphically, with x_1 on the horizontal axis, and x_2 on the vertical, what do the constraints look like?

The equation $x_1 + 3x_2 = 8$ is a line in the plane given by $x_2 = \frac{8}{3} - \frac{1}{3}x_1$, with slope $-\frac{1}{3}$.

When $x_2 = 0$ we have $x_1 = 8$, and when $x_1 = 0$ we get $x_2 = 8/3$.
Since $x_1 \geq 0$ and $x_2 \geq 0$ and $x_1 + 3x_2 \leq 8$, we are restricted to the shaded area in Figure 4.

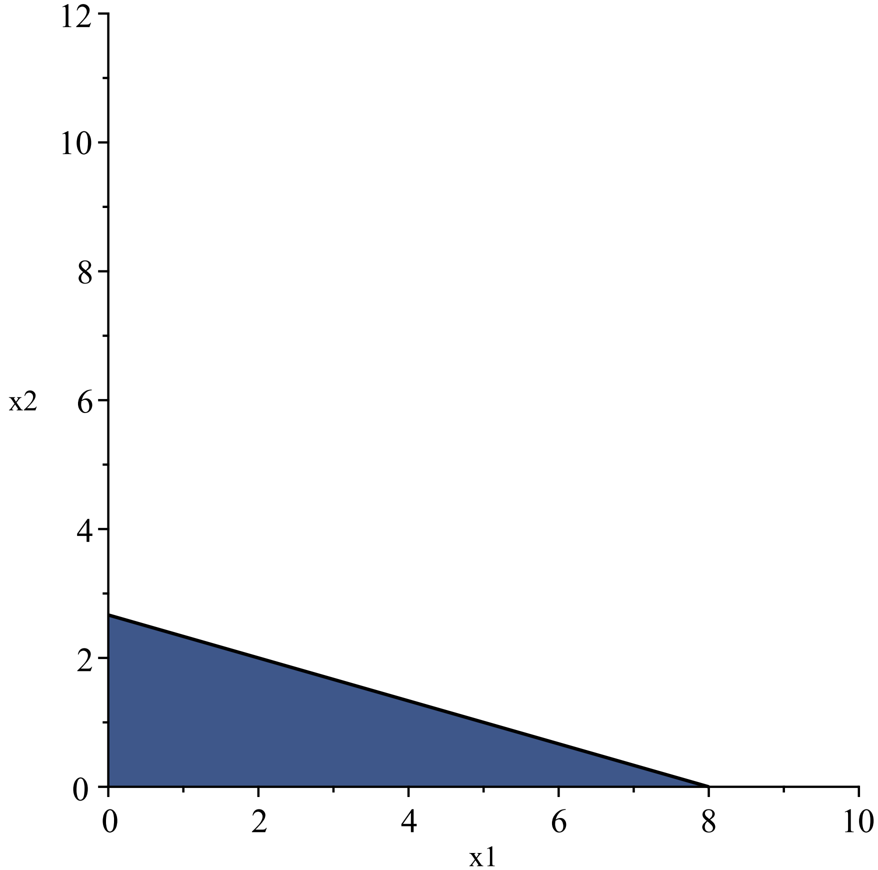

Figure 4: The constraint $x_1 + 3x_2 \leq 8$ with $x_j \geq 0$

But we also need $4x_1 + x_2 \le 10$.
Again the equation $4x_1 + x_2 = 10$ is a straight line $x_2 = 10 - 4x_1$, with slope -4 (so steeper than the previous line).

This time $x_2 = 0$ gives $x_1 = 2.5$, and $x_1 = 0$ gives $x_2 = 10$.

The two lines intersect where $x_1 = x_2 = 2$ (worked out earlier).

Combining this with our previous picture we get the *feasible region* in Figure 6: this is the set of possible x_1, x_2 values.

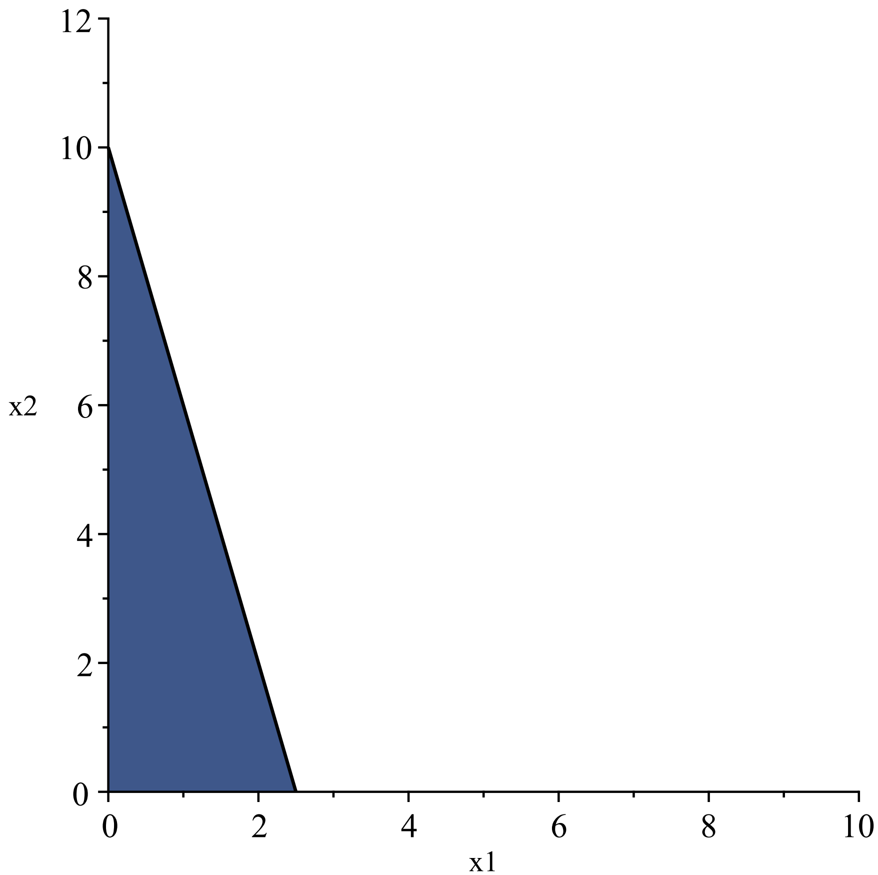

Figure 5: The constraint $4x_1 + x_2 \leq 10$ with $x_j \geq 0$

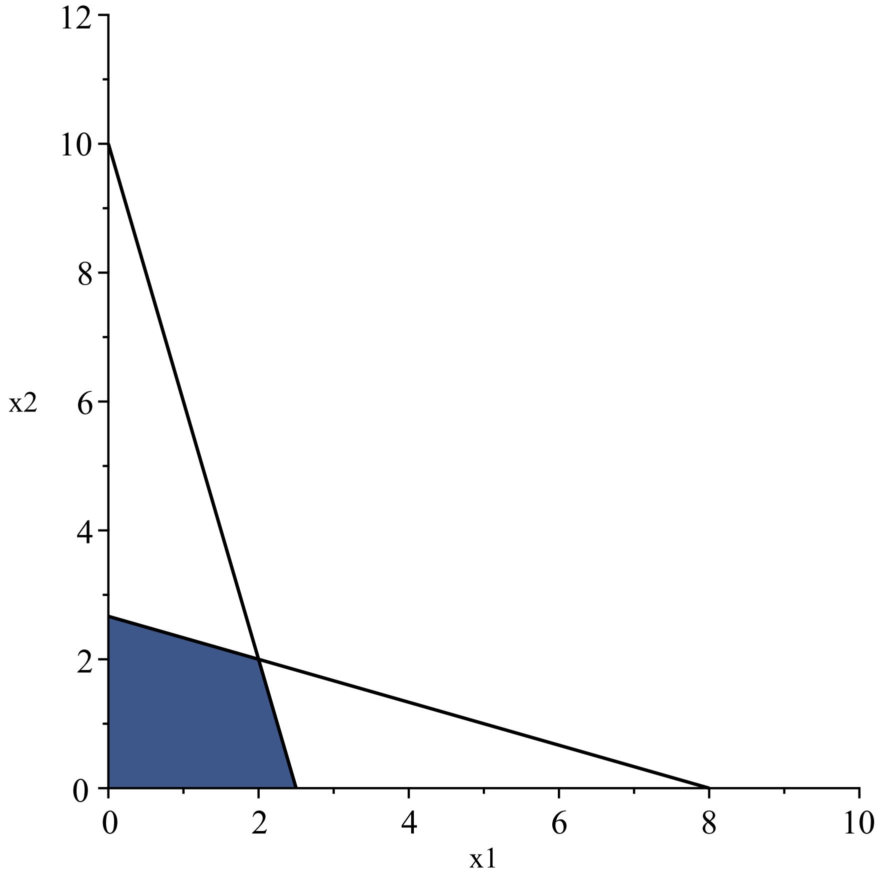

Figure 6: Combining the constraints: the feasible region for Example 12.2

So we need to find the maximum of $z = 500x_1 + 100x_2$ on this region.

The maximum cannot be at an interior point of the region: if we are *inside* then $z = 500x_1 + 100x_2$ can be increased by increasing x_1 and/or x_2.

So the maximum is taken somewhere on one of the edges.

But I claim that as we move along any edge, z always increases or decreases.

When $x_1 = 0$ we have $z = 100x_2$, and when $x_2 = 0$ we have $z = 500x_1$.

When $x_1 + 3x_2 = 8$ we have

$z = 500x_1 + 100x_2 = 500(8 - 3x_2) + 100x_2 = 4000 - 1400x_2$,

so we can increase z by decreasing x_2.

When $4x_1 + x_2 = 10$ we have

$z = 500x_1 + 100x_2 = 500x_1 + 100(10 - 4x_1) = 1000 + 100x_1$,

so we can increase z by increasing x_1.

CONCLUSION: the maximum is where two edges meet, and so at a vertex.

So we just need to compute z at each vertex:

$$z(0,0) = 0, \quad z(0,8/3) = 500 \times 0 + 100 \times \frac{8}{3} \approx 267,$$
$$z(2,2) = 500 \times 2 + 100 \times 2 = 1200,$$
$$z(2.5,0) = 500 \times 2.5 = 1250.$$

The maximum revenue is £1250, achieved when $x_1 = 2.5$ and $x_2 = 0$.

Suppose we need to find the maximum of $z = ax_1 + bx_2$, subject to the conditions $x_1 \geq 0$, $x_2 \geq 0$ and

$$c_1x_1+d_1x_2 \leq e_1, \quad c_2x_1+d_2x_2 \leq e_2, \ldots, c_nx_1+d_nx_2 \leq e_n.$$

Here the c_j, d_j and e_j are positive numbers which we are given, and a, b are real numbers.

Use the equations of the lines $c_jx_1 + d_jx_2 = e_j$ to sketch the set of points (x_1, x_2) which satisfy all the constraints. This is called the *feasible region* and its boundary consists of finitely many straight edges: the vertices/corners are where the lines intersect.
The maximum of $z = ax_1 + bx_2$ will be taken at one of the vertices of the feasible region.

12.4 EXAMPLE

Find the maximum of $z = 5x_1 + 7x_2$ subject to

$$x_1 \geq 0, x_2 \geq 0, \quad x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 6, \quad 2x_1 + x_2 \leq 10.$$

Find also the maximum of $u = 3x_1 - x_2$ subject to the same constraints.

This time we have one more constraint than in the previous example, but the method is essentially the same.

We plot the lines

$$x_1 + 2x_2 = 10, \quad x_1 + x_2 = 6, \quad 2x_1 + x_2 = 10,$$

with x_1 along the horizontal axis and x_2 along the vertical axis, and we need to find the points of intersection. Now

$$x_1 + 2x_2 = 10, \quad x_1 + x_2 = 6,$$

intersect where $x_2 = 10 - 6 = 4$ and so $x_1 = 6 - 4 = 2$. Similarly,

$$x_1 + x_2 = 6, \quad 2x_1 + x_2 = 10,$$

intersect at $x_1 = 4, x_2 = 2$.

Finally

$$x_1 + 2x_2 = 10, \quad 2x_1 + x_2 = 10,$$

intersect where

$$2x_1 + 4x_2 = 20, \quad 2x_1 + x_2 = 10,$$

which gives

$$3x_2 = 20 - 10 = 10, \quad x_2 = \frac{10}{3}, \quad x_1 = 10 - 2x_2 = \frac{10}{3},$$

and so this point is $\left(\frac{10}{3}, \frac{10}{3}\right)$.

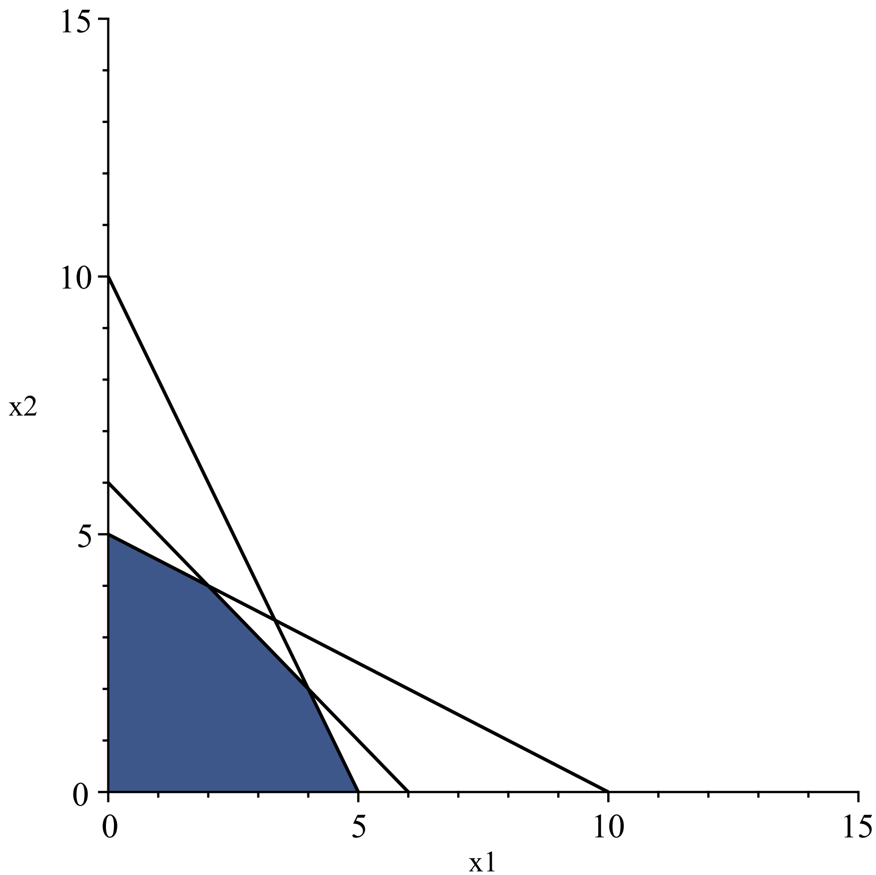

Figure 7: The feasible region for $x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 6, \quad 2x_1 + x_2 \leq 10$; the lines intersect at $(2, 4)$, $(4, 2)$ and $(10/3, 10/3)$

In the sketch in Figure 7 the vertices are

$$(0,0), \quad (5,0), \quad (4,2), \quad (2,4), \quad (0,5).$$

Note that two of the lines intersect at $(10/3, 10/3)$, but this is not a vertex of the feasible region, because at this point $x_1 + x_2 = 20/3 > 6$.

To find the maximum of $z = 5x_1 + 7x_2$ we calculate

$$z(0,0) = 0, \quad z(5,0) = 25, \quad z(4,2) = 34,$$
$$z(2,4) = 38, \quad z(0,5) = 35.$$

So the maximum possible value of z is 38.
Also $u = 3x_1 - x_2$ gives $u(0,0) = 0$ and

$$u(5,0) = 15, \quad u(4,2) = 10, \quad u(2,4) = 2, \quad u(0,5) = -5.$$

So the maximum possible value of u is 15.

It is not always easy to sketch the lines and the feasible region accurately. The next three diagrams will display our original conditions

$$x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 6, \quad 2x_1 + x_2 \leq 10$$

and two variants of them, in order to show how varying just one inequality can change fundamentally the shape of the feasible region (shaded).

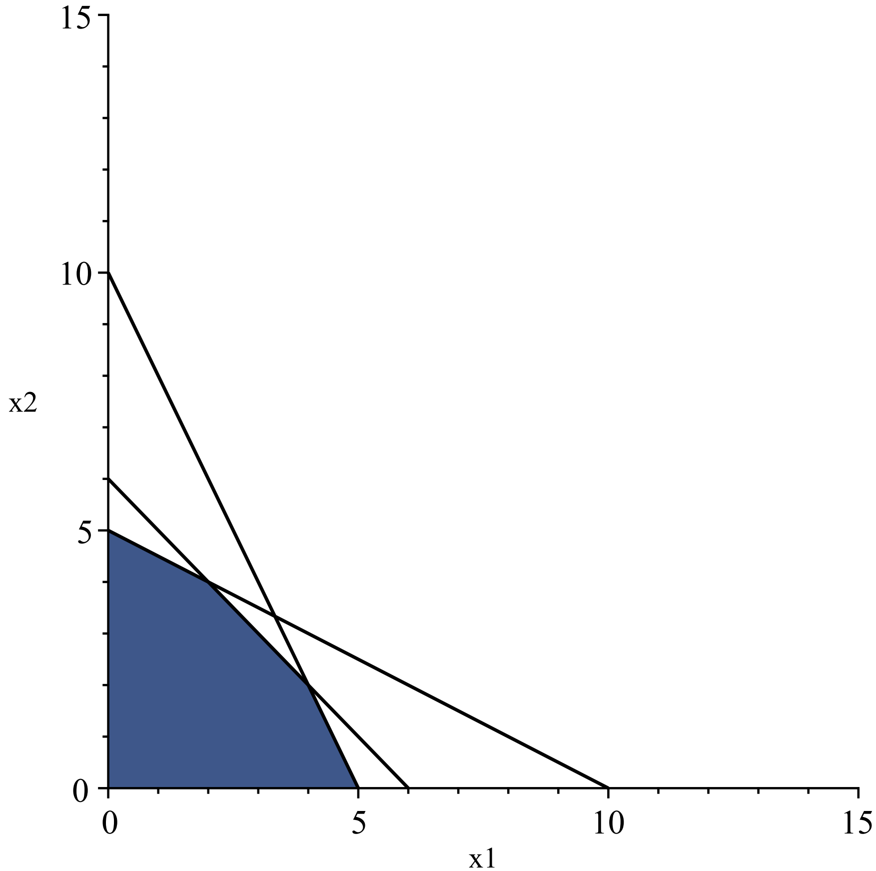

Figure 8: $x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 6, \quad 2x_1 + x_2 \leq 10$

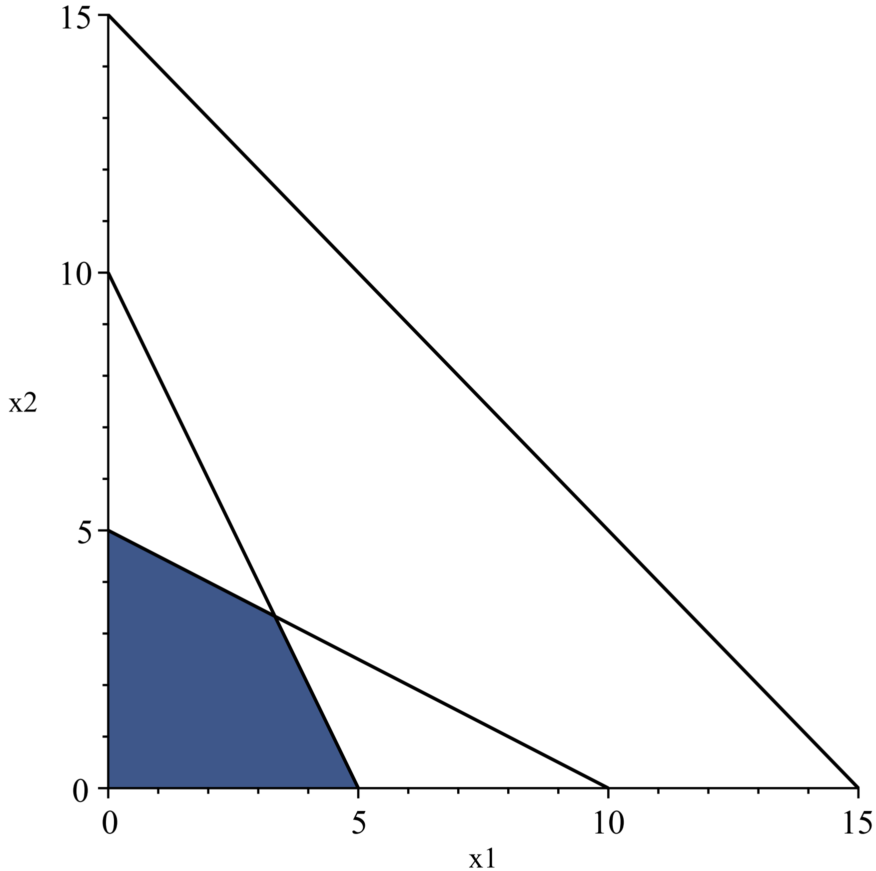

Figure 9: $x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 15, \quad 2x_1 + x_2 \leq 10$

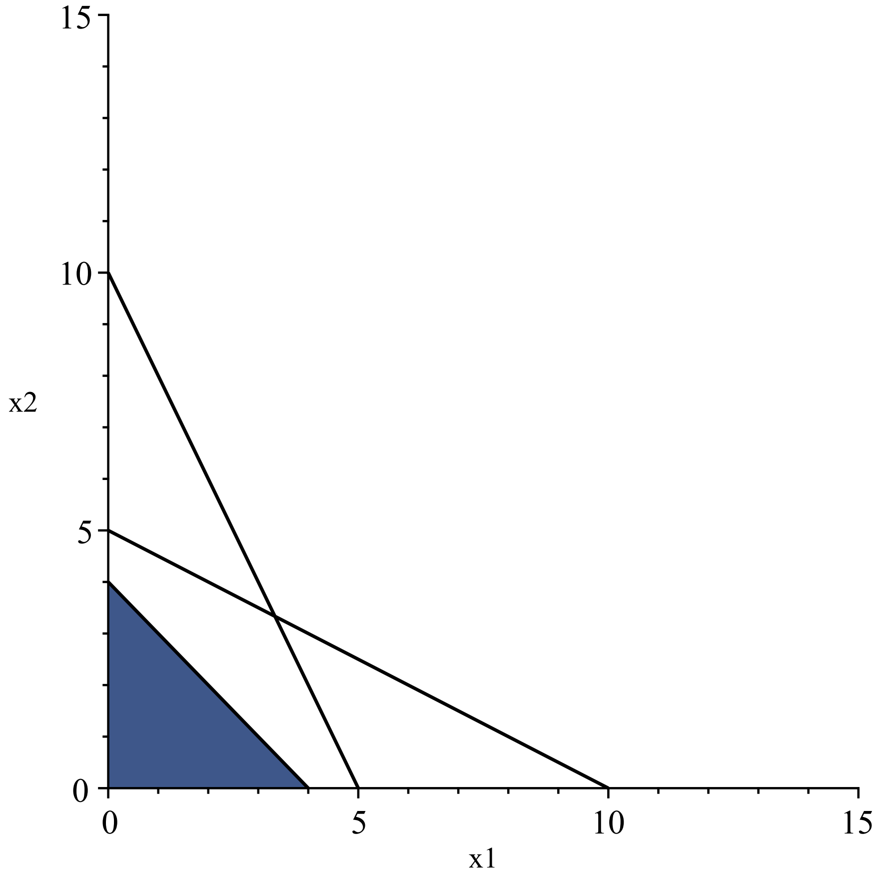

Figure 10: $x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 4, \quad 2x_1 + x_2 \leq 10$

To determine the feasible region, you may find the following *modified geometric method* useful.

Take the equations given by the constraints:

$$x_1 + 2x_2 = 10, \quad x_1 + x_2 = 6, \quad 2x_1 + x_2 = 10.$$

Determine the points where they intersect each other, as well as the x_1 and x_2 axes.

Discard any which do not satisfy *all of the constraints*.

To find the maximum of z on the feasible region, check all the points which are left.

In our example:
$x_1 + 2x_2 = 10$ meets the axes at:
$(10, 0)$ (discard because $x_1 + x_2 > 6$, or since $2x_1 + x_2 > 10$);
$(0, 5)$ (retain because $x_1 + x_2 = 5 < 6$ and $2x_1 + x_2 = 5 < 10$).

Next, $x_1 + x_2 = 6$ meets the axes at
$(6, 0)$ (discard because $2x_1 + x_2 > 10$)
and $(0, 6)$ (discard because $x_1 + 2x_2 > 10$)
and meets $x_1 + 2x_2 = 10$ at $(2, 4)$ (retain since $2x_1 + x_2 = 8 < 10$).

Finally, $2x_1 + x_2 = 10$ meets the axes at $(5, 0)$ (retain because $x_1 + x_2 = 5 < 6$ and $x_1 + 2x_2 = 5 < 10$)
and $(0, 10)$ (discard because $x_1 + x_2 > 6$, or because $x_1 + 2x_2 > 10$);
this line meets $x_1 + 2x_2 = 10$ at $(10/3, 10/3)$ (discard because $x_1 + x_2 = 20/3 > 6$)
and meets $x_1 + x_2 = 6$ at $(4, 2)$ (retain because $x_1 + 2x_2 = 8 < 10$).

Thus we are left with the following points to evaluate $z = 5x_1 + 7x_2$ at:

$$(0, 0), \quad (5, 0), \quad (4, 2), \quad (2, 4), \quad (0, 5).$$

If you use the geometric method in the exam, you should:

draw a sketch of the feasible region with its vertices clearly labelled;

make it clear which intersection points you have included and why.

We go back to the first problem: maximise

$$z = 500x_1 + 100x_2,$$

subject to the constraints

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 3x_2 \leq 8, \quad 4x_1 + x_2 \leq 10.$$

The idea of dummy variables is to turn the inequalities

$$x_1 + 3x_2 \leq 8, \quad 4x_1 + x_2 \leq 10,$$

into *equations*.

Now $x_1 + 3x_2 \leq 8$ is the same as

$$x_3 = 8 - x_1 - 3x_2 \geq 0,$$

and $x_3 = 8 - x_1 - 3x_2$ is the same as $x_1 + 3x_2 + x_3 = 8$.

Similarly $4x_1 + x_2 \leq 10$ is the same as

$$x_4 = 10 - 4x_1 - x_2 \geq 0,$$

and $x_4 = 10 - 4x_1 - x_2$ is the same as $4x_1 + x_2 + x_4 = 10$.

So we can write our conditions

$$z = 500x_1 + 100x_2,$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 3x_2 \leq 8, \quad 4x_1 + x_2 \leq 10,$$

in the form of *equations*

$$\begin{aligned} z - 500x_1 - 100x_2 &= 0 \\ x_1 + 3x_2 + x_3 &= 8 \\ 4x_1 + x_2 + x_4 &= 10, \end{aligned}$$

where x_1, x_2, x_3, x_4 all have to be ≥ 0.
Here x_3 and x_4 are called *dummy variables*.

The advantage of this idea is that we can use row operations on the equations!

There are 3 equations in 5 variables, which are z and the x_j. So the augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & -500 & -100 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 8 \\ 0 & 4 & 1 & 0 & 1 & 10 \end{array}\right).$$

We will apply row operations, but instead of reducing to echelon form we will aim to get rid of all negative entries in the top row. The benefit of this will be apparent when we have done it.

We start by getting rid of the -500 in row 1 of

$$\left(\begin{array}{ccccc|c} 1 & -500 & -100 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 8 \\ 0 & 4 & 1 & 0 & 1 & 10 \end{array}\right).$$

To do this, we use row 3 (we will explain why later). Using $R1' = 2R1 + 250R3$, $R2' = 4R2 - R3$ we get

$$\left(\begin{array}{ccccc|c} 2 & 0 & 50 & 0 & 250 & 2500 \\ 0 & 0 & 11 & 4 & -1 & 22 \\ 0 & 4 & 1 & 0 & 1 & 10 \end{array}\right).$$

The new augmented matrix represents the equations

$$\begin{aligned} 2z + 50x_2 + 250x_4 &= 2500 \\ 11x_2 + 4x_3 - x_4 &= 22 \\ 4x_1 + x_2 + x_4 &= 10, \end{aligned}$$

and we still need all $x_j \geq 0$.
Now all coefficients in the first equation are ≥ 0, and the larger we make x_2 and x_4, the smaller z will be. So

$$2z \leq 2500, \quad z \leq 1250.$$

Can we achieve $z = 1250$? To do this needs $x_2 = x_4 = 0$.
In this case the third equation gives $4x_1 = 10, x_1 = 2.5$, and the second gives $4x_3 = 22, x_3 = 5.5$.
All these values are ≥ 0, so $z = 1250$ is achievable, and is the maximum.

This dummy variables method is known as the *simplex algorithm* and has given the same result as we got by the geometric method.

This method has worked nicely, but why did we use row 3 and not row 2?

Suppose we use row 2 to remove the -500 in row 1 of

$$\left(\begin{array}{ccccc|c} 1 & -500 & -100 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 8 \\ 0 & 4 & 1 & 0 & 1 & 10 \end{array}\right),$$

Using $R1' = R1 + 500R2$, $R3' = R3 - 4R2$ we get

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1400 & 500 & 0 & 4000 \\ 0 & 1 & 3 & 1 & 0 & 8 \\ 0 & 0 & -11 & -4 & 1 & -22 \end{array}\right).$$

This augmented matrix represents the equations

$$\begin{aligned} z + 1400x_2 + 500x_3 &= 4000 \\ x_1 + 3x_2 + x_3 &= 8 \\ -11x_2 - 4x_3 + x_4 &= -22. \end{aligned}$$

If we use these equations then the first one tells us that $z \le 4000$, because $x_2, x_3 \ge 0$.
But to achieve $z = 4000$ we would need $x_2 = x_3 = 0$, in which case the third equation gives $x_4 = -22$.
This is not allowed – we need all $x_j \ge 0$.

So in this case the method has gone wrong. It has told us that $z \le 4000$, but we cannot achieve $z = 4000$ without violating the condition that all x_j must be ≥ 0.
To avoid this we need a way to choose the right row.

12.6 EXAMPLE

Use the simplex method to maximise $z = 5x_1 + 7x_2$ subject to the conditions

$$x_1 \geq 0, x_2 \geq 0, \quad x_1{+}2x_2 \leq 10, \quad x_1{+}x_2 \leq 6, \quad 2x_1{+}x_2 \leq 10.$$

The geometric method told us that the maximum possible value of z is 38, achieved when $x_1 = 2$, $x_2 = 4$. We write our problem as

$$\begin{aligned} z - 5x_1 - 7x_2 &= 0 \\ x_1 + 2x_2 + x_3 &= 10 \\ x_1 + x_2 + x_4 &= 6 \\ 2x_1 + x_2 + x_5 &= 10. \end{aligned}$$

These must be satisfied, with $x_1, x_2, x_3, x_4, x_5 \geq 0$.

The augmented matrix is

$$\left(\begin{array}{cccccc|c} 1 & -5 & -7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

We write this as

$$\begin{array}{c} \begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left(\begin{array}{cccccc|c} 1 & -5 & -7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \end{array}.$$

We apply three steps.

Step I: select a column.

To do this we look along row 1 of

$$\begin{array}{c} \begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left(\begin{array}{cccccc|c} 1 & -5 & -7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \end{array}$$

until we see the first negative entry: -5 in column 2.
So we have selected column 2 (highlighted):

$$\begin{array}{c} \begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left(\begin{array}{cccccc|c} 1 & -\mathbf{5} & -7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \end{array}.$$

Step II: select a row.

To do this we look at the *positive* entries below the -5 in column 2, and calculate what we get when we divide each one into the corresponding entry in column 7:

$$\left(\begin{array}{cccccc|c|c}
z & x_1 & x_2 & x_3 & x_4 & x_5 & b & C7/C2 \\
\hline
1 & -\mathbf{5} & -7 & 0 & 0 & 0 & 0 & \\
0 & 1 & 2 & 1 & 0 & 0 & 10 & \frac{10}{1} \\
0 & 1 & 1 & 0 & 1 & 0 & 6 & \frac{6}{1} \\
0 & 2 & 1 & 0 & 0 & 1 & 10 & \frac{10}{2}
\end{array}\right).$$

The correct row to select is the one which gives the *least* ratio in column 8: here we select row 4 because

$$\frac{10}{2} < \frac{6}{1} < \frac{10}{1}.$$

This slide shows column 2 and row 4 highlighted:

$$\left(\begin{array}{cccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b & C7/C2 \\ \hline 1 & \mathbf{-5} & -7 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 & \frac{10}{1} \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 & \frac{6}{1} \\ 0 & \mathbf{2} & 1 & 0 & 0 & 1 & 10 & \frac{10}{2} \end{array}\right).$$

Step III: use the selected row to clear all the other non-zero entries in the selected column.

Use $R1' = 2R1 + 5R4$, $R2' = 2R2 - R4$, $R3' = 2R3 - R4$ on

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & -5 & -7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

to get

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & -9 & 0 & 0 & 5 & 50 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

In the new matrix

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & -9 & 0 & 0 & 5 & 50 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

the entries in the b-column are all ≥ 0 (this should be the case if you chose the right row at Step II).

We still have a negative entry in the first row. So we repeat the three steps.

Step I: select a column.

Look along row 1 of

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & -\mathbf{9} & 0 & 0 & 5 & 50 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

until we see the first negative entry. This is -9 in column 3 (highlighted). So this time we have selected column 3.

Step II: select a row.

To do this we look at the *positive* entries below the -9 in column 3, and calculate what we get when we divide each one into the corresponding entry in column 7:

$$\left(\begin{array}{cccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b & C7/C3 \\ \hline 2 & 0 & -\mathbf{9} & 0 & 0 & 5 & 50 & \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 & \frac{10}{3} \\ 0 & 0 & \mathbf{1} & 0 & 2 & -1 & 2 & \frac{2}{1} \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 & \frac{10}{1} \end{array}\right).$$

Here $\frac{2}{1} < 10/3 < \frac{10}{1}$ so we select row 3 (highlighted).

Step III: use the selected row 3 to clear all the other non-zero entries in the selected column.

Use $R1' = R1 + 9R3, R2' = R2 - 3R3, R4' = R4 - R3$ on

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & -9 & 0 & 0 & 5 & 50 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

to get

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 0 & 18 & -4 & 68 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 2 & 8 \end{array}\right).$$

In the new matrix

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 0 & 18 & -4 & 68 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 2 & 8 \end{array}\right)$$

the entries in the b-column are still ≥ 0 (this is worth checking to ensure you haven't made a mistake).

We still have a negative entry in the first row. So we repeat the three steps again.

Step I: select a column.

Look along row 1 until we see the first negative entry. This is -4 in column 6 (highlighted).

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 0 & 18 & -\mathbf{4} & 68 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 2 & 8 \end{array}\right).$$

Step II: select a row. To do this we look at the *positive* entries below the -4 in column 6, and calculate what we get when we divide each one into the corresponding entry in column 7:

$$\left(\begin{array}{cccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b & C7/C6 \\ \hline 2 & 0 & 0 & 0 & 18 & -\mathbf{4} & 68 & \\ 0 & 0 & 0 & 2 & -6 & \mathbf{2} & 4 & \frac{4}{2} \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 & \\ 0 & 2 & 0 & 0 & -2 & 2 & 8 & \frac{8}{2} \end{array}\right).$$

Here $\frac{4}{2} < \frac{8}{2}$ so we select row 2 (highlighted).

Step III: use the selected row 2 to clear all the other non-zero entries in the selected column.

Use $R1' = R1 + 2R2, R3' = 2R3 + R2, R4' = R4 - R2$ on

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 0 & 18 & -4 & 68 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 0 & 0 & -2 & 2 & 8 \end{array}\right)$$

to get

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 4 & 6 & 0 & 76 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 2 & 2 & -2 & 0 & 8 \\ 0 & 2 & 0 & -2 & 4 & 0 & 4 \end{array}\right).$$

Again it is worth checking that in the new matrix

$$\left(\begin{array}{cccccc|c}
z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\
\hline
2 & 0 & 0 & 4 & 6 & 0 & 76 \\
0 & 0 & 0 & 2 & -6 & 2 & 4 \\
0 & 0 & 2 & 2 & -2 & 0 & 8 \\
0 & 2 & 0 & -2 & 4 & 0 & 4
\end{array}\right).$$

the entries in the b-column are all ≥ 0.

There are no negative entries in the first row. So we have finished the algorithm!

Now we can read off from

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 0 & 4 & 6 & 0 & 76 \\ 0 & 0 & 0 & 2 & -6 & 2 & 4 \\ 0 & 0 & 2 & 2 & -2 & 0 & 8 \\ 0 & 2 & 0 & -2 & 4 & 0 & 4 \end{array}\right).$$

The first row says that

$$2z + 4x_3 + 6x_4 = 76.$$

In particular we get $2z \leq 76$, since $x_3, x_4 \geq 0$, so $z \leq 38$. To achieve $z = 38$ we need $x_3 = x_4 = 0$.

To check that our answer makes sense, we verify that $x_3 = x_4 = 0$ does not make any x_j negative. Row 2, with $x_3 = x_4 = 0$, gives $2x_5 = 4$, $x_5 = 2$. Row 3 gives $2x_2 = 8$, $x_2 = 4$. Row 4 gives $2x_1 = 4$, $x_1 = 2$.

So we can achieve $z = 38$, which is therefore the maximum.

This matches what we got geometrically:
the maximum of z is 38, achieved when $x_1 = 2$, $x_2 = 4$.

The simplex algorithm can sometimes be much quicker than this, as the next example shows.

Use the simplex method to maximise $u = 3x_1 - x_2$ subject to the conditions

$$x_1 \geq 0, x_2 \geq 0, \quad x_1 + 2x_2 \leq 10, \quad x_1 + x_2 \leq 6, \quad 2x_1 + x_2 \leq 10.$$

The geometric method told us that the maximum possible value of u is 15, achieved when $x_1 = 5$, $x_2 = 0$. We write our problem as

$$\begin{aligned} u - 3x_1 + x_2 &= 0 \\ x_1 + 2x_2 + x_3 &= 10 \\ x_1 + x_2 + x_4 &= 6 \\ 2x_1 + x_2 + x_5 &= 10. \end{aligned}$$

These must be satisfied, with $x_1, x_2, x_3, x_4, x_5 \geq 0$.

The augmented matrix is

$$\left(\begin{array}{cccccc|c} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

We write this as

$$\left(\begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

We apply the three steps.

Step I: select a column.

To do this we look along row 1 of

$$\begin{array}{c} \begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left(\begin{array}{cccccc|c} 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \end{array}$$

until we see the first negative entry: -3 in column 2.

So we have selected column 2 (highlighted):

$$\begin{array}{c} \begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \end{array} \\ \left(\begin{array}{cccccc|c} 1 & \mathbf{-3} & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \end{array}.$$

Step II: select a row.

To do this we look at the *positive* entries below the -3 in column 2, and calculate what we get when we divide each one into the corresponding entry in column 7 (this is the same as what we got for $z = 5x_1 + 7x_2$).

$$\left(\begin{array}{cccccc|c|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b & C7/C2 \\ \hline 1 & -\mathbf{3} & 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 & \frac{10}{1} \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 & \frac{6}{1} \\ 0 & \mathbf{2} & 1 & 0 & 0 & 1 & 10 & \frac{10}{2} \end{array}\right).$$

The correct row to select is the one which gives the *least* ratio in column 8: here we select row 4 because

$$\frac{10}{2} < \frac{6}{1} < \frac{10}{1}.$$

Step III: use the selected row to clear all the other non-zero entries in the selected column.

Use $R1' = 2R1 + 3R4$, $R2' = 2R2 - R4$, $R3' = 2R3 - R4$ on

$$\left(\begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 6 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

to get

$$\left(\begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 5 & 0 & 0 & 3 & 30 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right).$$

In the new matrix

$$\left(\begin{array}{cccccc|c}
u & x_1 & x_2 & x_3 & x_4 & x_5 & b \\
\hline
2 & 0 & 5 & 0 & 0 & 3 & 30 \\
0 & 0 & 3 & 2 & 0 & -1 & 10 \\
0 & 0 & 1 & 0 & 2 & -1 & 2 \\
0 & 2 & 1 & 0 & 0 & 1 & 10
\end{array}\right).$$

the entries in the b-column are all ≥ 0 (this should be the case if you chose the right row at Step II)

AND we have no negative entries in row 1. So we can stop.

$$\left(\begin{array}{cccccc|c} u & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 2 & 0 & 5 & 0 & 0 & 3 & 30 \\ 0 & 0 & 3 & 2 & 0 & -1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -1 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

The first row says that

$$2u + 5x_2 + 3x_5 = 30.$$

In particular we get $2u \leq 30$, since $x_2, x_5 \geq 0$, so $u \leq 15$.
To achieve $u = 15$ we need $x_2 = x_5 = 0$.
We check that $x_2 = x_5 = 0$ does not make any x_j negative. Row 2, with $x_2 = x_5 = 0$, gives $2x_3 = 10$, $x_3 = 5$. Row 3 gives $2x_4 = 2$, $x_4 = 1$. Row 4 gives $2x_1 = 10$, $x_1 = 5$.

Just to check these again: we have

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5 + 0 + 5 = 10, \\ x_1 + x_2 + x_4 &= 5 + 0 + 1 = 6, \\ 2x_1 + x_2 + x_5 &= 10 + 0 + 0 = 10. \end{aligned}$$

We can achieve $u = 15$, which is therefore the maximum.

This matches what we got geometrically:
the maximum of u is 15, achieved when $x_1 = 5$, $x_2 = 0$.

Our rule for choosing rows explains why, in 12.5, row 3 was the correct choice for getting rid of the -500 in

$$\left(\begin{array}{ccccc|c} 1 & -\mathbf{500} & -100 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 8 \\ 0 & 4 & 1 & 0 & 1 & 10 \end{array}\right).$$

The positive entries below the -500 are 1 and 4, and when we divide these into the entries in the last column we get

$$\frac{8}{1} = 8 > \frac{10}{4} = 2.5.$$

This shows that row 3 was the right choice.

12.7 EXAMPLE

Maximise $z = 40x_1 + 88x_2$ subject to

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 4x_2 \leq 30, \quad 5x_1 + 2x_2 \leq 60.$$

We will use the simplex algorithm here. Write

$$x_1 + 4x_2 + x_3 = 30, \quad 5x_1 + 2x_2 + x_4 = 60.$$

So we have introduced one dummy variable for each constraint, and we need $x_j \geq 0$ for every j. The labelled augmented matrix (also called a simplex table) is

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right).$$

Step I: select a column.
We look along row 1 of

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right)$$

until we see the first negative entry: -40 in column 2. So we select column 2.

We now have

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & \mathbf{-40} & -88 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right).$$

Step II: select a row. We look at the *positive* entries below the -40 in column 2, and divide each one into the entry in column 6:

$$\left(\begin{array}{ccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & b & C6/C2 \\ \hline 1 & \mathbf{-40} & -88 & 0 & 0 & 0 & \\ 0 & 1 & 4 & 1 & 0 & 30 & \frac{30}{1} \\ 0 & 5 & 2 & 0 & 1 & 60 & \frac{60}{5} \end{array}\right).$$

Here $\frac{60}{5} < \frac{30}{1}$ so we select row 3.

Step III: use the selected row 3 to clear all the other non-zero entries in the selected column.

So we use row 3 on

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 30 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right).$$

We use $R1' = R1 + 8R3, R2' = 5R2 - R3$ to get

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & -72 & 0 & 8 & 480 \\ 0 & 0 & 18 & 5 & -1 & 90 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right).$$

Again the entries in the b-column are all ≥ 0.

However

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & -72 & 0 & 8 & 480 \\ 0 & 0 & 18 & 5 & -1 & 90 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right)$$

has a negative entry in row 1, so we repeat Steps I to III. The first negative entry in row 1 is -72 so we select column 3.

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & \mathbf{-72} & 0 & 8 & 480 \\ 0 & 0 & 18 & 5 & -1 & 90 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{array}\right).$$

Step II: select a row. We look at the *positive* entries below the -72 in column 3, and divide each one into the entry in column 6:

$$\left(\begin{array}{ccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & b & C6/C3 \\ \hline 1 & 0 & \mathbf{-72} & 0 & 8 & 480 & \\ 0 & 0 & 18 & 5 & -1 & 90 & \frac{90}{18} \\ 0 & 5 & 2 & 0 & 1 & 60 & \frac{60}{2} \end{array}\right).$$

Here $\frac{90}{18} < \frac{60}{2}$ so we select row 2.

We now have

$$\begin{array}{c}
\begin{array}{ccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & b & C6/C3 \end{array} \\
\left(\begin{array}{ccccc|c|c}
1 & 0 & -\mathbf{72} & 0 & 8 & 480 & \\
0 & 0 & \mathbf{18} & 5 & -1 & 90 & \frac{90}{18} \\
0 & 5 & 2 & 0 & 1 & 60 & \frac{60}{2}
\end{array}\right)
\end{array}.$$

Step III: use the selected row 2 to clear all the other non-zero entries in the selected column.

We apply $R1' = R1 + 4R2, R3' = 9R3 - R2$ to get

$$\begin{array}{c}
\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 0 & 0 & 20 & 4 & 840 \\
0 & 0 & 18 & 5 & -1 & 90 \\
0 & 45 & 0 & -5 & 10 & 450
\end{array}\right)
\end{array}.$$

Again the entries in the b-column are all ≥ 0.

All entries in row 1 are ≥ 0 so we stop.

We now read off from

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & 0 & 0 & 20 & 4 & 840 \\ 0 & 0 & 18 & 5 & -1 & 90 \\ 0 & 45 & 0 & -5 & 10 & 450 \end{array}\right).$$

Row 1 says that $z + 20x_3 + 4x_4 = 840$, so $z \leq 840$ since $x_3, x_4 \geq 0$.

To achieve $z = 840$ we need $x_3 = x_4 = 0$.
In this case row 2 gives $18x_2 = 90$ so $x_2 = 5$,
and row 3 gives $45x_1 = 450$ so $x_1 = 10$.

So $z = 840$ is achievable, with $x_1 = 10, x_2 = 5$, and is the maximum of z.

We check our answer using the geometric method. We need to maximise $z = 40x_1 + 88x_2$ subject to

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_1 + 4x_2 \leq 30, \quad 5x_1 + 2x_2 \leq 60.$$

Here $x_1 + 4x_2 = 30$ meets the axes at $(30, 0)$ and $(0, 7.5)$; also $5x_1 + 2x_2 = 60$ meets the axes at $(12, 0)$ and $(0, 30)$.

The two lines intersect where

$$x_1 + 4x_2 = 30, \quad 5x_1 + 2x_2 = 60, \quad 10x_1 + 4x_2 = 120,$$

which gives $9x_1 = 120 - 30 = 90$, $x_1 = 10, x_2 = 5$.
The vertices are

$$(0, 0), \quad (12, 0), \quad (10, 5), \quad (0, 7.5).$$

Calculating $z = 40x_1 + 88x_2$ for each vertex gives

$$z(0,0) = 0, \quad z(12,0) = 480, \quad z(10,5) = 840, \quad z(0,7.5) = 660.$$

So the maximum is $z = 840$, achieved at $x_1 = 10, x_2 = 5$.

The next slide shows the feasible region.

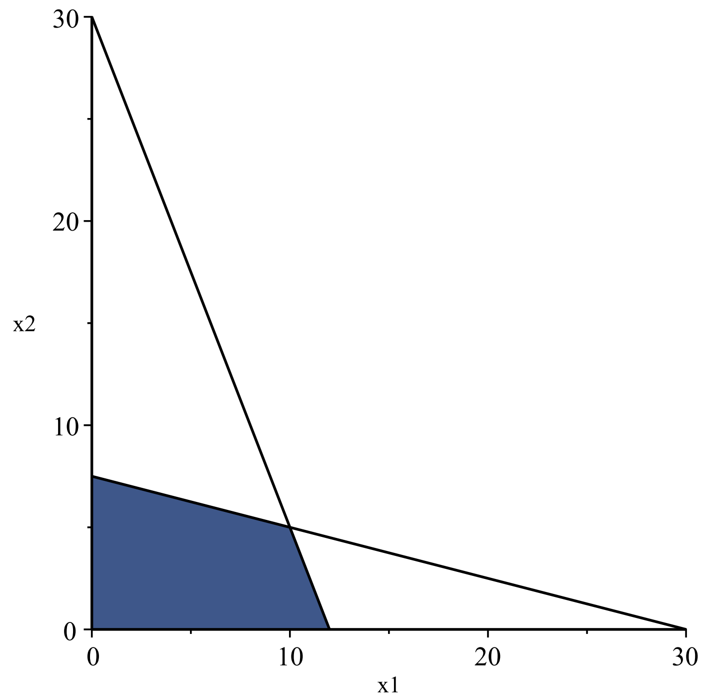

Figure 11: $x_1 + 4x_2 \leq 30, \quad 5x_1 + 2x_2 \leq 60$

This shows the feasible region.

The simplex method may seem more complicated than the geometric approach, but it has several advantages:
(i) it is an algorithm, so can be programmed on a computer;
(ii) it can be used when z initially depends on more than two variables (we have used only two so far);
(iii) it can be used with any number of constraints, which would make the feasible region difficult to identify;
(iv) it avoids all the problems associated with deciding which points actually form vertices of the feasible region.

However, if a maximisation problem of this type occurs on the HG2M02 exam, it will be possible to solve it by either method.

A firm makes x_1 tons of product A, plus x_2 tons of product B, and x_3 tons of product C.
A ton of A uses 1 ton of steel, and no aluminium.
A ton of B uses 3 tons of aluminium and no steel.
A ton of C uses 4 tons of steel, and 1 ton of aluminium.

A ton of A sells for £1000, a ton of B for £2000, and a ton of C for £1000. So the revenue (in thousands of pounds) to be maximised is

$$z = x_1 + 2x_2 + x_3.$$

The firm has 7 tons of steel and 8 tons of aluminium, so

$$x_1 + 4x_3 \leq 7, \quad 3x_2 + x_3 \leq 8.$$

We write the problem in the form

$$\begin{aligned} z - x_1 - 2x_2 - x_3 &= 0 \\ x_1 + 4x_3 + x_4 &= 7 \\ 3x_2 + x_3 + x_5 &= 8. \end{aligned}$$

Here x_4 and x_5 are dummy variables, which must be ≥ 0, as must x_1, x_2, x_3.

We write this as a simplex table

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & -1 & -2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right).$$

As before we aim to remove all negative entries in row 1.

We start by getting rid of the first -1 in row 1:

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & -\mathbf{1} & -2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right).$$

We can do this using $R1' = R1 + R2$, which gives

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & 0 & -2 & 3 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right).$$

In the new matrix

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 1 & 0 & -2 & 3 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right),$$

we now clear the -2 in row 1 using $R1' = 3R1 + 2R3$:

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 3 & 0 & 0 & 11 & 3 & 2 & 37 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right).$$

All entries in row 1 are now ≥ 0.

We have

$$\left(\begin{array}{cccccc|c} z & x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 3 & 0 & 0 & 11 & 3 & 2 & 37 \\ 0 & 1 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 3 & 1 & 0 & 1 & 8 \end{array}\right).$$

The first row says that

$$3z + 11x_3 + 3x_4 + 2x_5 = 37,$$

so $3z \leq 37$ and $z \leq 37/3$ because all x_j are ≥ 0.

To attain $z = 37/3$ we need $x_3 = x_4 = x_5 = 0$.
In this case row 2 gives $x_1 = 7 \geq 0$ and row 3 gives $3x_2 = 8 \geq 0$, and this is attainable.
Thus the best strategy is to discard product C ($x_3 = 0$) and make 7 tons of A and $8/3$ tons of B.

Maximise $z = x_1 + 2x_2$, subject to

$$x_1 \geq 0, x_2 \geq 0, \quad -2x_1 + 3x_2 \leq 5, \quad -3x_1 + 5x_2 \leq 8.$$

Notice that here the constraints include *negative* coefficients. We write the formulas as

$$\begin{aligned} z - x_1 - 2x_2 &= 0 \\ -2x_1 + 3x_2 + x_3 &= 5 \\ -3x_1 + 5x_2 + x_4 &= 8. \end{aligned}$$

The simplex table is

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & -2 & 3 & 1 & 0 & 5 \\ 0 & -3 & 5 & 0 & 1 & 8 \end{array}\right).$$

We try to apply the algorithm to

$$\left(\begin{array}{ccccc|c} z & x_1 & x_2 & x_3 & x_4 & b \\ \hline 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & -2 & 3 & 1 & 0 & 5 \\ 0 & -3 & 5 & 0 & 1 & 8 \end{array}\right).$$

We would like to get rid of the first negative entry in row 1, which is -1, but this has only negative entries below it.

Suppose we try the second negative entry in row 1, which is -2.

Calculating the ratios between columns 6 and 3 we get

$$\left(\begin{array}{ccccc|c|c} z & x_1 & x_2 & x_3 & x_4 & b & C6/C3 \\ \hline 1 & -1 & -2 & 0 & 0 & 0 & \\ 0 & -2 & 3 & 1 & 0 & 5 & \frac{5}{3} \\ 0 & -3 & 5 & 0 & 1 & 8 & \frac{8}{5} \end{array}\right).$$

Here $\frac{8}{5} = 1.6 < \frac{5}{3}$ so we use row 3.

So in

$$\begin{pmatrix} z & x_1 & x_2 & x_3 & x_4 & \vert & b \\ \hline 1 & -1 & -2 & 0 & 0 & \vert & 0 \\ 0 & -2 & 3 & 1 & 0 & \vert & 5 \\ 0 & -3 & 5 & 0 & 1 & \vert & 8 \end{pmatrix}$$

we use $R1' = 5R1 + 2R3$ and $R2' = 5R2 - 3R3$ to get

$$\begin{pmatrix} z & x_1 & x_2 & x_3 & x_4 & \vert & b \\ \hline 5 & -11 & 0 & 0 & 2 & \vert & 16 \\ 0 & -1 & 0 & 5 & -3 & \vert & 1 \\ 0 & -3 & 5 & 0 & 1 & \vert & 8 \end{pmatrix}.$$

We still have a negative entry in row 1, namely -11, but we cannot go further, because there are no positive entries below it. The algorithm appears to have broken down.

However, we can go back to the original problem: maximise

$$z = x_1 + 2x_2,$$

subject to

$$x_1 \geq 0, x_2 \geq 0, \quad -2x_1 + 3x_2 \leq 5, \quad -3x_1 + 5x_2 \leq 8.$$

If we set $x_2 = 0$, then the constraints will be satisfied for any $x_1 \geq 0$. So we can make $z = x_1 + 2x_2$ as large as we like, by choosing $x_2 = 0$ and x_1 arbitrarily large.

The algorithm failed to find a maximum because there isn't one!

This will not happen if all the coefficients in the constraints are ≥ 0 (and definitely will not happen in HG2M02!).

If you try to solve this problem geometrically you can write the constraints as

$$x_2 \leq \frac{5}{3} + \frac{2}{3}x_1, \quad x_2 \leq \frac{8}{5} + \frac{3}{5}x_1$$

and for $x_1 \geq 0$ the second implies the first because

$$\frac{8}{5} < \frac{5}{3}, \quad \frac{3}{5} < \frac{2}{3}.$$

The feasible region consists of all points with $x_1 \geq 0$ and below the line $x_2 = \frac{8}{5} + \frac{3}{5}x_1$, and is not bounded ($x_1$ can be as large as we like).

This ends the examinable material for HG2M02.

The rest of these notes consists of OPTIONAL material. Some of it may be discussed in the lectures, but it will NOT be on the exam.

Take n equations, in n unknowns $x_1, \ldots, x_n$,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

where n is very large. Then any method (Cramer's rule, the inverse matrix, or Gaussian elimination) will be very costly in terms of time and/or computing power.

If, however, the matrix of coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is a *diagonal* matrix ($a_{ij} = 0$ for $i \neq j$) with non-zero diagonal entries $a_{jj} \neq 0$, then the solution is easy: the jth equation is just $a_{jj}x_j = b_j$, and the solution is $x_j = b_j/a_{jj}$.

Jacobi iteration concerns the case where A is *close to* a diagonal matrix, in which case it gives a way to compute *approximate* solutions numerically, with less computing power needed.

A square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is called *diagonally dominant* if each diagonal entry has modulus greater than the sum of the moduli of all the other entries in its row i.e.

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|.$$

For example

$$B = \begin{pmatrix} 5 & -2 & 2 \\ 1 & -7 & 4 \\ 2 & -3 & 6 \end{pmatrix}$$

is diagonally dominant, because

$$|5| > |-2| + |2|, \quad |-7| > |1| + |4|, \quad |6| > |2| + |-3|,$$

but

$$C = \begin{pmatrix} 5 & -2 & 2 \\ 1 & -7 & 4 \\ 2 & -4 & 6 \end{pmatrix}$$

is not diagonally dominant (look at row 3).

Suppose now that the square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

is diagonally dominant. Write

$$D = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} & 0 & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & 0 \end{pmatrix}.$$

So to make D we take only the diagonal entries from A, and set all other entries to be 0, and for C we take only the non-diagonal entries. Note that $C = A - D$.

Note also that all the diagonal entries a_{ii} in A are non-zero, because A is diagonally dominant. So

$$D^{-1} = \begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn}^{-1} \end{pmatrix}.$$

Now consider the matrix

$$E = -D^{-1}C = -\begin{pmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & a_{nn}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ a_{21} & 0 & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & \dots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & \dots & -\frac{a_{2n}}{a_{22}} \\ \vdots & & & \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \dots & 0 \end{pmatrix}.$$

This is a "small" matrix in the sense that if you add up the moduli of the terms in each row then the sum is always less than 1.

Because E has "small" entries, it is possible to show that if we take any column vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then $E^m X$ tends to the zero vector as $m \to \infty$.

Jacobi iteration then does the following: we want to solve

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

which we write in the form

$$AX = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

We let

$$X_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \vdots \\ \frac{b_n}{a_{nn}} \end{pmatrix} = D^{-1}\mathbf{b}.$$

So X_1 is the solution to the equation $DX = \mathbf{b}$. Now we write

$$\begin{aligned} X_2 &= X_1 + EX_1, \\ X_3 &= X_1 + EX_2 = X_1 + E(X_1 + EX_1) = X_1 + EX_1 + E^2X_1 \\ X_4 &= X_1 + EX_3 = X_1 + EX_1 + E^2X_1 + E^3X_1, \\ &\vdots \end{aligned}$$

and so on.

The general formula for X_m for $m \geq 2$ is

$$X_m = X_1 + EX_1 + E^2X_1 + \ldots + E^{m-1}X_1.$$

Because E^mX_1 tends to the zero vector as $m \to \infty$ we find that these vectors X_m converge to a vector

$$\begin{aligned} X^* &= X_1 + EX_1 + E^2X_1 + \ldots + E^{m-1}X_1 + \ldots = \sum_{p=0}^{\infty} E^pX_1 \\ &= X_1 + E(X_1 + EX_1 + E^2X_1 + \ldots) \\ &= X_1 + EX^* = X_1 - D^{-1}CX^*. \end{aligned}$$

Therefore

$$DX^* = DX_1 - DD^{-1}CX^* = \mathbf{b} - CX^*, \quad AX^* = (D+C)X^* = \mathbf{b}.$$

Thus the approximations $X_1, X_2, X_3, \ldots$ converge to the required solution X^* of the equation $AX = \mathbf{b}$.

13.2 EXAMPLE

Solve the equations

$$
\begin{aligned}
10x + 3y + 4z &= 17 \\
2x + 10y - 5z &= 7 \\
3x + y + 5z &= 9
\end{aligned}
$$

approximately using Jacobi iteration.

In practice we would never use Jacobi iteration on such a small system, but the idea here is just to illustrate the method.

It is easy to check that the the true solution is given by $x = y = z = 1$: it is unique because the determinant of the coefficients is

$$\begin{vmatrix} 10 & 3 & 4 \\ 2 & 10 & -5 \\ 3 & 1 & 5 \end{vmatrix} = 363 \neq 0.$$

To set up Jacobi iteration we use

$$A = \begin{pmatrix} 10 & 3 & 4 \\ 2 & 10 & -5 \\ 3 & 1 & 5 \end{pmatrix}, \quad A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{b} = \begin{pmatrix} 17 \\ 7 \\ 9 \end{pmatrix}.$$

The matrix A is diagonally dominant since

$$10 > 3 + 4, \quad 10 > 2 + |-5|, \quad 5 > 3 + 1.$$

Now we split A into the diagonal part and the rest:

$$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 3 & 4 \\ 2 & 0 & -5 \\ 3 & 1 & 0 \end{pmatrix}, \quad A = D + C.$$

We also need

$$D^{-1} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$

$$E = -D^{-1}C = - \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 3 & 4 \\ 2 & 0 & -5 \\ 3 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -0.3 & -0.4 \\ -0.2 & 0 & 0.5 \\ -0.6 & -0.2 & 0 \end{pmatrix}.$$

Now we can make our approximate solutions:

$$X_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad X_1 = D^{-1}\mathbf{b} = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix},$$

$$\begin{aligned} X_2 &= X_1 + EX_1 \\ &= \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix} + \begin{pmatrix} 0 & -0.3 & -0.4 \\ -0.2 & 0 & 0.5 \\ -0.6 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix} = \begin{pmatrix} 0.77 \\ 1.26 \\ 0.64 \end{pmatrix}. \end{aligned}$$

Then we write

$$\begin{aligned} X_3 &= X_1 + EX_2 \\ &= \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix} + \begin{pmatrix} 0 & -0.3 & -0.4 \\ -0.2 & 0 & 0.5 \\ -0.6 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 0.77 \\ 1.26 \\ 0.64 \end{pmatrix} = \begin{pmatrix} 1.066 \\ 0.866 \\ 1.086 \end{pmatrix}. \end{aligned}$$

This is closer to $x = y = z = 1$. Iterate twice more:

$$\begin{aligned} X_4 &= X_1 + EX_3 \\ &= \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix} + \begin{pmatrix} 0 & -0.3 & -0.4 \\ -0.2 & 0 & 0.5 \\ -0.6 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 1.066 \\ 0.866 \\ 1.086 \end{pmatrix} = \begin{pmatrix} 1.0058 \\ 1.0298 \\ 0.9872 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} X_5 &= X_1 + EX_4 \\ &= \begin{pmatrix} 1.7 \\ 0.7 \\ 1.8 \end{pmatrix} + \begin{pmatrix} 0 & -0.3 & -0.4 \\ -0.2 & 0 & 0.5 \\ -0.6 & -0.2 & 0 \end{pmatrix} \begin{pmatrix} 1.0058 \\ 1.0298 \\ 0.9872 \end{pmatrix} = \begin{pmatrix} 0.99618 \\ 0.99244 \\ 0.99056 \end{pmatrix}, \end{aligned}$$

which hopefully convinces you that X_m is tending to the right solution!

14 APPENDIX (OPTIONAL!): SOME PROOFS

All these proofs are completely optional: you do not need to read them, and they will not be on the exam.

Several of the most important topics in this module, for example the existence of inverse matrices, and the rule for the determinant of a product, depend on the connection between row operations and multiplying by a matrix. Let $A = (a_{ij})$ be any $n \times n$ matrix.

Consider the determinant

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}. \tag{2}$$

By definition this is

$$D = a_{11}M_{11} - a_{12}M_{12} + \ldots = \sum_{p=1}^{n} (-1)^{p+1} a_{1p} M_{1p}, \tag{3}$$

where M_{1p} is the minor of a_{1p}, which is the determinant you get by deleting row 1 and column p.

We define b_{pq} for $p \neq q$ as follows: b_{pq} is the determinant obtained by deleting from D rows 1 and 2, plus columns p and q. Obviously $b_{pq} = b_{qp}$.

The idea is to express each M_{1p} in terms of these b_{pq}. We have

$$M_{1p} = \begin{vmatrix} a_{21} & \ldots & a_{2,p-1} & a_{2,p+1} & \ldots & a_{2n} \\ \vdots & & & & & \\ a_{n1} & \ldots & a_{n,p-1} & a_{n,p+1} & \ldots & a_{nn} \end{vmatrix}.$$

Expanding M_{1p} by its first row we get

$$\begin{aligned} M_{1p} &= a_{21}b_{1p} - a_{22}b_{2p} + \ldots + (-1)^p a_{2,p-1}b_{p-1,p} + \\ &\quad +(-1)^{p+1}a_{2,p+1}b_{p+1,p} + \ldots + (-1)^n a_{2,n}b_{np} \\ &= \sum_{1\le q\le n, q\ne p} a_{2q}b_{qp}s_{pq}, \\ s_{pq} &= (-1)^{q+1} \quad \text{if } q < p, \quad s_{pq} = (-1)^q \quad \text{if } q > p. \end{aligned}$$

Substituting this into (3) we get

$$D = \sum_{1\le p,q\le n, q\ne p} a_{1p}a_{2q}b_{qp}r_{pq},$$

where

$$r_{pq} = (-1)^{p+q} \quad \text{if } q < p, \quad r_{pq} = (-1)^{p+q+1} \quad \text{if } q > p.$$

Notice that $r_{pq} = -r_{qp}$ here.

14.2 THEOREM

Swapping rows 1 and 2 of a determinant multiplies the value of the determinant by -1.

If we swap rows 1 and 2 the new determinant is

$$\begin{aligned} D_1 &= \sum_{1\le p,q\le n, q\neq p} a_{2p}a_{1q}b_{qp}r_{pq} \\ &= \sum_{1\le p,q\le n, p\neq q} a_{1q}a_{2p}b_{pq}(-r_{qp}) = -D. \end{aligned}$$

14.3 THEOREM

Swapping any two rows of a determinant multiplies the value of the determinant by -1.

This is easy to check for 2×2 determinants. Suppose we know it is true for $n-1 \times n-1$ determinants. Swap rows i and j of D. If i and j are both at least 2 then in the formula (3) all the minors M_{1p} have two rows swapped, so are multiplied by -1, and therefore so is D.

Now suppose we swap row 1 with row i, where $i \geq 2$. We can do this by swapping row 2 with row i, then row 2 with row 1, then row 2 with row i again. The effect of this is to multiply D by -1 three times, which is the same as multiplying D by -1.

14.4 THEOREM

If two rows of a determinant are the same, then the determinant is 0*.*

This is because swapping the identical rows multiplies D by -1, but also leaves D unchanged.

14.5 THEOREM

If we add one row of a determinant to another row, the value of the determinant is unchanged.

Suppose we add row i (where $i \geq 2$) to row 1. Then by (3) the new determinant is

$$\begin{aligned} D_1 &= \sum_{p=1}^{n} (-1)^{p+1} (a_{1p} + a_{ip}) M_{1p} \\ &= \sum_{p=1}^{n} (-1)^{p+1} a_{1p} M_{1p} + \sum_{p=1}^{n} (-1)^{p+1} a_{ip} M_{1p}. \end{aligned}$$

This is the sum of two determinants: the first is D; the second is 0, because it has two rows the same.

If we want to add row i to row $j \neq 1, i$, we can just do the following: swap rows 1 and j, add row i to row 1, then swap rows 1 and j again. The net effect is no change.

14.6 THEOREM

If we multiply one row of a determinant by λ, the value of the determinant is multiplied by λ.

This is obvious from (3) if we multiply row 1 by λ. The general case follows by first swapping rows so that the multiplied row is row 1, and then swapping back.

The combination of these theorems proves all our basic facts about determinants, except for the one involving the transpose, which says that $\det(A^T) = \det A$. We will see that one later.

For example, consider the case where we expand the determinant D of an $n \times n$ matrix $A = (a_{ij})$ by the mth row, not the first. The rule from the notes is

$$D = a_{m1} \times (-1)^{m+1} \times M_{m1} + a_{m2} \times (-1)^{m+2} \times M_{m2} + \dots,$$

with a_{ij} the entry in row i, column j, and M_{ij} its minor.

Let R_j denote row j of A. We swap row m with the row above it, then with the one above that, and carry on until the old row m becomes the first row, but all the other rows are in the same order as before.

Doing this gives us matrices with rows as shown:

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_{m-2} \\ R_{m-1} \\ R_m \\ R_{m+1} \\ \vdots \\ R_n \end{pmatrix}, \quad \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_{m-2} \\ R_m \\ R_{m-1} \\ R_{m+1} \\ \vdots \\ R_n \end{pmatrix}, \quad \cdots \begin{pmatrix} R_1 \\ R_m \\ R_2 \\ R_3 \\ \vdots \\ R_{m-2} \\ R_{m-1} \\ R_{m+1} \\ \vdots \\ R_n \end{pmatrix}, \quad \begin{pmatrix} R_m \\ R_1 \\ R_2 \\ R_3 \\ \vdots \\ R_{m-2} \\ R_{m-1} \\ R_{m+1} \\ \vdots \\ R_n \end{pmatrix}.$$

To achieve this has required $m - 1$ swaps in total.
Denote by C the last of these matrices.

Then the entry c_{1j} from row 1 of C is a_{mj}, and its minor is the same as M_{mj}. So, given that we have performed $m-1$ row swaps,

$$\begin{aligned} D &= (-1)^{m-1}\det(C) \\ &= (-1)^{m+1}\det(C) \\ &= (-1)^{m+1}(a_{m1}M_{m1} - a_{m2}M_{m2} + \ldots) \\ &= a_{m1}(-1)^{m+1}M_{m1} + a_{m2}(-1)^{m+2}M_{m2} + \ldots \end{aligned}$$

as required.

In this section we show how the three row operations we used in the Gauss-Jordan and other methods are linked to multiplying in front by special matrices.

Consider first swapping two rows.
Doing this multiplies the determinant by -1. Also, it is easy to check that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}, \quad \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This generalises as follows.
Given any $n \times n$ *matrix* A*, swapping rows* i *and* $j \neq i$ *is equivalent to multiplying* A *in front by a matrix whose determinant is* -1 *and which is its own inverse.*

In fact it is easy to check that the matrix you need is just I_n with rows i and j swapped.

Now consider multiplying a row by a real number $\lambda \neq 0$.

Doing this multiplies the determinant by λ. Also, it is easy to check that

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\lambda & b\lambda \\ c & d \end{pmatrix}, \quad \begin{vmatrix} \lambda & 0 \\ 0 & 1 \end{vmatrix} = \lambda,$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Given any $n \times n$ matrix A, multiplying row i by $\lambda \neq 0$ is equivalent to multiplying A in front by an invertible matrix whose determinant is λ.

This time the matrix you need is just I_n with row i multiplied by λ, and its inverse is I_n with row i multiplied by $1/\lambda$.

Lastly, consider adding λ times row i to row $j \neq i$. This does not change the determinant. Note that

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c\lambda & b + d\lambda \\ c & d \end{pmatrix},$$

$$\begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} = 1, \quad \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}.$$

Given any $n \times n$ matrix A, adding λ times row i to row $j \neq i$ is equivalent to multiplying A in front by an invertible matrix whose determinant is 1.

The matrix you need is just I_n with λ inserted at an appropriate point (try an example), and its inverse is the same matrix, but with λ replaced by $-\lambda$.

The facts in §14.7 can be summarised as follows: *applying each of the three basic row operations to a square matrix* A *multiples* $\det A$ *by some non-zero real number* c*, and is equivalent to multiplying* A *in front by an invertible matrix whose determinant is* c*.*

Now remember Theorem 4.5. This said that
if A *is an* $n \times n$ *square matrix then there are two possibilities:*
(i) A *can be reduced via row operations to an echelon form matrix* A' *with a row of* 0*s;*
(ii) A *can be reduced via row operations to the identity matrix* I_n*.*

We can now express this as follows.

if A is an $n \times n$ square matrix then there are invertible $n \times n$ matrices $E_1, \ldots, E_n$ of the form in §14.7 such that

$$A' = E_n \ldots E_2 E_1 A$$

either has a row of 0s or is I_n. Also

$$\det A' = \det(E_n) \ldots \det(E_2) \det(E_1) \det A.$$

Proof that $\det(AB) = \det A \det B$ for any $n \times n$ matrix B.

Suppose first that $\det A = 0$. Then $\det A' = 0$. So we must be in the case where A' has a row of zeros. So $A'B$ has a row of zeros (because of the way matrix multiplication works). So $\det(A'B) = 0$. But

$$AB = E_1^{-1} E_2^{-1} \ldots E_n^{-1} A'B,$$

where multiplying by any of these E_j^{-1} just consists of applying one row operation, each of which multiplies the determinant by a non-zero real number. So $\det(AB) = 0$.

Now suppose $\det A \neq 0$. Then $\det A' \neq 0$ so A' must be I_n. So

$$I_n = E_n \ldots E_2 E_1 A.$$

But these E_j have the property that $E_j C$ has determinant $\det E_j \times \det C$ for any $n \times n$ matrix C. So

$$\begin{aligned} 1 &= \det I_n = \det(E_n) \det(E_{n-1} \ldots E_1 A) \\ &= \det(E_n) \det(E_{n-1}) \det(E_{n-2} \ldots E_1 A) \\ &= \ldots \\ &= \det(E_n) \ldots \det(E_2) \det(E_1) \det A, \end{aligned}$$

which implies that

$$\det(E_n) \ldots \det(E_2) \det(E_1) = \frac{1}{\det A}.$$

Also we have

$$B = I_n B = E_n \dots E_2 E_1 A B$$

and

$$\det B = \det(E_n) \dots \det(E_2) \det(E_1) \det AB = \frac{\det AB}{\det A}.$$

Proof that A^{-1} exists precisely when $\det A \neq 0$, and is given by the Gauss-Jordan method.

We saw that if A^{-1} exists then $\det A \neq 0$.

Now suppose $\det A \neq 0$. Then as in the previous proof we have $A' = I_n$ and

$$I_n = E_n \dots E_2 E_1 A.$$

Now each of these E_j has an inverse matrix, so

$$A = E_1^{-1} E_2^{-1} \dots E_n^{-1} I_n = E_1^{-1} E_2^{-1} \dots E_n^{-1}.$$

If we apply the *same* row operations to I_n as we applied to get A' from A then we get a matrix

$$D = E_n \dots E_2 E_1 I_n = E_n \dots E_2 E_1.$$

But then

$$AD = E_1^{-1} E_2^{-1} \dots E_n^{-1} E_n \dots E_2 E_1 = I_n,$$

$$DA = E_n \dots E_2 E_1 E_1^{-1} E_2^{-1} \dots E_n^{-1} = I_n,$$

so $D = A^{-1}$ as claimed.

Proof that a square matrix A and its transpose have the same determinant

When $\det A = 0$ we know that A^{-1} does not exist, so A^T cannot have an inverse matrix, because $A^T B = BA^T = I_n$ would give $B^T A = AB^T = I_n$ and B^T would be the inverse of A, which is impossible. So $\det(A^T) = 0$.

When $\det A \neq 0$ we again use the fact that there are invertible matrices E_j as in §14.7 such that

$$I_n = A' = E_n \dots E_1 A, \quad I_n = I_n^T = A^T E_1^T \dots E_n^T,$$

and so

$$1 = \det E_n \dots \det E_1 \det A, \quad 1 = \det A^T \det(E_1^T) \dots \det(E_n^T).$$

But if you look at the matrices E_j which arise in §14.7, it is easy to check that they all satisfy $\det(E_j) = \det(E_j^T) \neq 0$. So $\det A = \det(A^T)$.

Suppose we have a square matrix

$$E = \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ e_{21} & e_{22} & \dots & e_{2n} \\ \vdots & & & \\ e_{n1} & e_{n2} & \dots & e_{nn} \end{pmatrix}$$

with the property that in each row the sum of the moduli of the terms is less than 1 i.e.

$$\sum_{j=1}^{n} |e_{ij}| < 1$$

for every i.

Take any column vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Then $E^m X$ tends to the zero vector as $m \to \infty$.

To justify this assertion, take a real number $c < 1$ with

$$\sum_{j=1}^{n} |e_{ij}| \leq c < 1$$

for every i.

For any vector

$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, we let $\|Y\| = \max\{|y_i|\}$

(so $\|Y\|$ is the largest of the moduli of the entries of Y).

We claim that $\|EY\| \leq c\|Y\|$. To see this, write

$$EY = Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

and suppose that $\|Z\| = |z_k|$.

Now z_k is formed from the kth row of E and the vector Y as follows:

$$z_k = \sum_{j=1}^{n} e_{kj} y_j.$$

So

$$\begin{aligned} \|Z\| \;=\; |z_k| &\le \sum_{j=1}^{n} |e_{kj} y_j| = \sum_{j=1}^{n} |e_{kj}| \cdot |y_j| \\ &\le \; \sum_{j=1}^{n} |e_{kj}| \|Y\| \le c\|Y\| \end{aligned}$$

as claimed.

Now we get

$$\|EX\| \leq c\|X\|, \quad \|E^2X\| \leq c\|EX\| \leq c^2\|X\|,$$

which gives

$$\|E^3X\| \leq c\|E^2X\| \leq c^3\|X\|$$

and so on, so that $\|E^mX\| \leq c^m\|X\| \to 0$ as $m \to \infty$. So E^mX must be tending to the zero vector as $m \to \infty$.