Postgraduate notes on complex analysis

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For Hong, Helen and Natasha
Preface

These notes originated from a set of lectures on basic results in Nevanlinna theory and their application to ordinary differential equations in the complex domain, given at the Christian-Albrechts-Universität zu Kiel in December 1998. Over the years additional topics have been added, such as some elements of potential theory which are of use in value distribution theory, including the important technique of harmonic measure. Analytic continuation and singularities of the inverse function are also discussed, and the various themes are brought together in the Denjoy-Carleman-Ahlfors theorem and a recent theorem of Bergweiler and Eremenko concerning asymptotic values of entire and meromorphic functions.

The aim has been to develop in a single set of notes some of the key concepts and methods of function theory, in a form suitable for a postgraduate student starting out in the area. The notes have drawn on many sources, and these are indicated in the course of the development.

I would like to thank several people for drawing my attention to numerous obscurities and typos in earlier versions of these notes. These include my PhD students James Hinchliffe, Guy Kendall, Eleanor Lingham, Abdullah Alotaibi, Rob Trickey, Dan Nicks, Matt Buck and Asim Asiri, as well as Professor Christian Berg of the University of Copenhagen, who used parts of these notes in a graduate lecture course.

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Chapter 1

Some topics from real analysis

This chapter contains a number of topics from real analysis. They have nothing in particular in common except that they all play a useful role in various aspects of function theory.

1.1 Convex functions

The property of convexity plays an important role in function theory because a number of key quantities associated with entire, meromorphic and subharmonic functions turn out to be convex functions of $\log r$.

A good reference for this section is Chapter 5 of Royden’s book [61], which along with Rudin’s classic text [62] will be the main source for measure theoretic results used in these notes.

The real-valued function $f$ is convex on the open interval $I = (p,q)$, $-\infty \leq p < q \leq \infty$, if

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) \quad \text{for} \quad p < a < x < b < q.$$  

This says that the graph of $f$ over the closed interval $[a,b]$ lies on or below the straight line from $(a,f(a))$ to $(b,f(b))$. Rearranging, we find that

$$C(x,a) = f(x) - f(a) \leq \frac{f(b) - f(a)}{b-a} (x-a) \quad \text{for} \quad a < x < b.$$  

(1.1)

Keeping $a$ fixed in (1.1) we get $C(x,a) \leq C(b,a)$ for $a < x < b$. So $C(x,a)$ is non-decreasing on $(a,q)$ and the right derivative

$$f'_R(a) = \lim_{x \to a^+} C(x,a) \leq C(b,a)$$

exists, with $f'_R(a) < \infty$ for every $a \in I$. Next, keeping $b$ fixed in (1.1) we find that $C(b,x)$ is non-decreasing on $(p,b)$ and the left derivative

$$f'_L(b) = \lim_{x \to b^-} C(x,b) = \lim_{x \to b^-} C(b,x) \geq C(b,a)$$

exists and satisfies $f'_L(b) > -\infty$ for all $b \in I$. Moreover, (1.1) gives $C(x,a) \leq C(b,x)$ for $a < x < b$ and so $f'_R(a) \leq f'_L(b)$ for $a < b$. Now let $a \to x-, b \to x+$ in (1.1), which gives

$$f'_L(x) \leq f'_R(x).$$

Thus both left and right derivatives are real-valued non-decreasing functions, and $f$ is continuous on $I.$
Fix $n \in \mathbb{N}$. If
\[ f'_L(x) < f'_R(x) - 1/n \] (1.2)
then for $y > x$ we have $f'_L(x) < f'_L(y) - 1/n$. Hence on any closed interval $[a, b] \subseteq I$ there are finitely many points $x$ satisfying (1.2), because if $x_1, \ldots, x_m$ are such points with $a \leq x_1 < x_2 < \ldots < x_m \leq b$ then
\[ f'_L(b) - f'_L(a) \geq \sum_{j=2}^{m} (f'_L(x_j) - f'_L(x_{j-1})) \geq \frac{m-1}{n}. \]

Thus there exists a countable set $J$ such that on the complement $I \setminus J$ we have $f'_L = f'_R$. It follows that $f$ is differentiable on $I \setminus J$, and $f'$ is non-decreasing on $I \setminus J$.

### 1.2 The growth of real functions

#### 1.2.1 $O$ and $o$ notation

Let $s(r), g(r)$ be functions defined on $[a, \infty)$, with $s(r)$ complex-valued and $g(r)$ real and positive. We say that $s(r) = O(g(r))$ as $r \to \infty$ if there exist constants $K, L$ such that $|s(r)| \leq Kg(r)$ for all $r \geq L$. Thus, for example, $(r^2 + 3)\sin r = O(r^2)$ as $r \to \infty$. We write $s(r) = o(g(r))$ as $r \to \infty$ if $s(r)/g(r) \to 0$: for example $\log r = o(r)$.

We can also use this notation when $r$ tends to a finite limit, for example, $r^2 + 3r = O(r)$ as $r \to 0+$, and for sequences, such as $2^n = o(n!)$ as $n \to \infty$.

#### 1.2.2 lim sup and lim inf

Let $s(r)$ be a real-valued function defined on $[a, \infty)$. For each $r \geq a$, define
\[ T_r = \{ s(t) : t \geq r \}. \]
 Obviously $T_r \subseteq T_u$ if $r \geq u \geq a$. Next define, for each $r \geq a$,
\[ p(r) = p_s(r) = \inf T_r, \quad q(r) = q_s(r) = \sup T_r. \]

Here we use the convention that if a set is not bounded above then its sup is $+\infty$, while if a set is not bounded below then its inf is $-\infty$. We obviously now have
\[ p(r) \leq s(r) \leq q(r). \] (1.3)

Also $p(r)$ is a non-decreasing function, and $q(r)$ is a non-increasing function.

We define the ”limsup” and ”liminf” of $s(r)$ by
\[ \tau = \limsup_{r \to \infty} s(r) = \lim_{r \to \infty} q(r), \quad \mu = \liminf_{r \to \infty} s(r) = \lim_{r \to \infty} p(r). \]

Obviously $\mu \leq \tau$, and we obtain the following properties of $\tau$ and $\mu$.

(i) The limit $\lim_{r \to \infty} s(r)$ exists, with value $L$ (possibly $\pm \infty$), if and only if $\tau = \mu = L$.

**Proof.** Suppose $s(r)$ has limit $L$. Assume first that $-\infty \leq y < L$. Then for large $t$ we have $s(t) > y$. So $q(r) \geq p(r) \geq y$ for all large $r$, and so $\tau \geq \mu \geq y$. Similarly, if $y > L$ we get $y \geq \tau \geq \mu$. 

Conversely, suppose that $\tau = \mu = L$. Then $p(r)$ and $q(r)$ tend to $L$ as $r \to \infty$ and, by (1.3), so does $s(r)$.

(ii) Suppose $h < \tau$. Then for all large $r$ we have $h < q(r)$ and so we can find $t \geq r$ with $s(t) > h$. Hence there exists a sequence $r_n \to \infty$ with $s(r_n) > h$.

(iii) Suppose $H > \tau$. Then $s(r) \leq q(r) < H$ for all large $r$.

Obviously properties (ii) and (iii) determine $\tau$ uniquely.

(iv) If $h < \mu$ then $s(r) > h$ for all large $r$. If $H > \mu$ there exists a sequence $r_n \to \infty$ with $s(r_n) < H$.

These are proved in the same way as (ii), (iii), or using:

(v) We have
$$\limsup(-s(r)) = -\liminf s(r).$$

This is easy, since $q_s(r) = -p_s(r)$ etc.

1.2.3 The order of a function

Let $s(r)$ be a non-negative real-valued function defined on $[a, \infty)$. The order of $s(r)$ is
$$\rho_s = \limsup_{r \to \infty} \frac{\log^+ s(r)}{\log r},$$
in which
$$\log^+ x = \max\{\log x, 0\}. \tag{1.4}$$

If $\rho_s < K < \infty$ then for all large enough $r$ we have $\log^+ s(r) < K \log r$ and so $s(r) < r^K$.

1.2.4 Lemma

Suppose that $s(r), S(r)$ are non-negative real-valued functions defined on $[a, \infty)$ and that there exist $A, B, C, D \geq 1$ such that
$$S(r) < As(Br)(\log r)^C$$
for $r > D$. Then $\rho_S \leq \rho_s$.

Proof. Assume $\rho_s < K < \infty$, since if $\rho_s = \infty$ there is nothing to prove. For large $r$ we then have
$$\log^+ S(r) \leq \log^+ A + \log^+ s(Br) + C \log \log r < K \log Br + o(\log r)$$
and so $\rho_S \leq K$.

1.2.5 Borel’s lemma

Let $A > 1$. Let the function $T : [r_0, \infty) \to [1, \infty)$ be continuous from the right and non-decreasing. Then
$$T(r + 1/T(r)) \leq AT(r) \tag{1.5}$$
for all $r > r_0$ outside a set $E$ of linear measure at most $\frac{A}{A-1}$.  

\label{1.5}
Proof. Let \( r_1 \) be the infimum of those \( r > r_0 \) (if any) for which (1.5) is false, and set \( r'_1 = r_1 + 1/T(r_1) \). Continue this as follows: if \( r_1, \ldots, r_n \) have been defined, put \( r'_n = r_n + 1/T(r_n) \) and let \( r_{n+1} \) be the infimum of \( r > r'_n \) for which (1.5) fails.

If \( n \geq 1 \) and \( r_n \) exists then, by the definition of \( r_n \) as an infimum, there exists a sequence \( s_j \to r_n + \) such that (1.5) fails, i.e.

\[
T(s_j + 1/(T(s_j))) > AT(s_j).
\]

Since \( T(r) \) is non-decreasing and continuous from the right, while \( s_j \to r_n + \), this gives

\[
T(s_j + 1/T(r_n)) \geq T(s_j + 1/(T(s_j))) > AT(s_j), \quad T(r'_n) = T(r_n + 1/T(r_n)) \geq AT(r_n).
\]

If, in addition, \( r_{n+1} \) exists then \( T(r_{n+1}) \geq T(r'_n) \geq AT(r_n) \).

We identify three cases. The first is that \( r_1 \) does not exist, in which case \( E \) is empty and there is nothing more to prove. The second is that \( r_1, \ldots, r_n \) exist, but (1.5) holds for all \( r > r'_n \). In this case, \( E \) is contained in the union of the intervals \( [r_m, r'_m] \) \( (m = 1, \ldots, n) \) since, by the definition of the \( r_m \), (1.5) holds for \( r'_m < r < r_{m+1} \).

Thus

\[
\int_E dr \leq \sum_{m=1}^n (r'_m - r_m) = \sum_{m=1}^n T(r_m) - 1 \leq \sum_{m=1}^n A^{1-m}T(r_1)^{-1} \leq \frac{A}{A-1}.
\]

The final case is that in which the sequence \( r_n \) is infinite. In this case \( r_n \to \infty \), for otherwise

\[
r_n \to r^* \in (r_0, \infty), \quad r_n < r'_n \leq r_{n+1}, \quad r'_n \to r^*,
\]

and

\[
1/T(r^*) \leq 1/T(r_n) = r'_n - r_n \to 0,
\]

which is impossible. As in the second case we get

\[
\int_E dr \leq \sum_{m=1}^\infty (r'_m - r_m) \leq \sum_{m=1}^\infty A^{1-m}T(r_1)^{-1} \leq \frac{A}{A-1}.
\]

1.3 Some results on certain integrals

1.3.1 The Riemann-Stieltjes integral

See Apostol’s book [3, Ch. 7] for details of the Riemann-Stieltjes integral. Let \( f \) and \( h \) be real-valued functions on the interval \( I = [a, b] \). Let \( P = \{t_0, t_1, \ldots, t_n\} \) be a partition of \( [a, b] \). This means that \( a = t_0 < t_1 < \ldots < t_n = b \); the \( t_j \) are then called vertices of \( P \). By a Riemann-Stieltjes sum, we mean

\[
S(P, f, h) = \sum_{k=1}^n f(s_k)(h(t_k) - h(t_{k-1})),
\]

in which \( t_{k-1} \leq s_k \leq t_k \). The case \( h(x) = x \) gives the standard Riemann sums of ordinary integration.

We say that the Riemann-Stieltjes integral

\[
\int_a^b f(x)dh(x)
\]

exists and equals \( L \in \mathbb{R} \) if the following is true. To each \( \varepsilon > 0 \) corresponds a partition \( P_0 \) of \( I \) such that \( |S(P, f, h) - L| < \varepsilon \) for every refinement \( P \) of \( P_0 \) (this means that each vertex of \( P_0 \) is a vertex of \( P \)), regardless of how the \( s_k \) are chosen.
In particular, the integral exists if \( f \) is continuous and \( h \) is monotone \([3, \text{p.159}]\). Further, if
\[
\int_{a}^{b} f(x)dh(x)
\]
exists then so does
\[
\int_{a}^{b} h(x)df(x)
\]
and they satisfy the integration by parts formula \([3, \text{p.144}]\)
\[
\int_{a}^{b} f(x)dh(x) = f(b)h(b) - f(a)h(a) - \int_{a}^{b} h(x)df(x). \tag{1.6}
\]

The following lemma concerning the interplay between sums and Riemann-Stieltjes integrals is useful in Nevanlinna theory.

**1.3.2 Lemma**

Let \(-\infty < a = t_0 < t_1, \ldots < t_m = b < \infty\). Let the real-valued functions \( f \) and \( h \) be such that:

(i) \( f \) is continuous on \([a, b]\);

(ii) \( h(x) \) is non-decreasing on \([a, b]\) and constant on each interval \([t_{j-1}, t_j]\), \( j = 1, \ldots, m \).

Then
\[
\int_{a}^{b} f(x)dh = I = \sum_{j=1}^{m} f(t_j)(h(t_j) - h(t_{j-1})).
\]

**Proof.** Let \( \varepsilon > 0 \) and choose \( \delta > 0 \) such that \( \delta(h(b) - h(a)) < \varepsilon \). Next, choose \( \eta > 0 \) such that \( |f(x) - f(y)| < \delta \) for \( a \leq x < y \leq b \), \( y - x < \eta \), which is possible since \( f \) is uniformly continuous on \([a, b]\). Fix a partition \( P_0 \) of \([a, b]\) such that (a) each \( t_j \) is a vertex of \( P_0 \) and (b) the distance between successive vertices of \( P_0 \) is less than \( \eta \).

Now let \( P \) be any refinement of \( P_0 \). Then properties (a) and (b) holds with \( P_0 \) replaced by \( P \). For \( j = 1, \ldots, m \) let \( x_j \) be the greatest vertex of \( P \) in \([a, t_j]\). Then \( t_{j-1} \leq x_j < t_j \). By property (ii), any Riemann-Stieltjes sum using the partition \( P \) has the form
\[
S(P, f, h) = \sum_{j=1}^{m} f(s_j)(h(t_j) - h(x_j)) = \sum_{j=1}^{m} f(s_j)(h(t_j) - h(t_{j-1})),
\]
where \( x_j \leq s_j \leq t_j \), because all other subintervals contribute nothing to \( S(P, f, h) \). But then, since \( h \) is non-decreasing,
\[
|S(P, f, h) - I| \leq \sum_{j=1}^{m} |f(s_j) - f(t_j)|(h(t_j) - h(t_{j-1})) < \delta \sum_{j=1}^{m} (h(t_j) - h(t_{j-1})) = \delta(h(b) - h(a)) < \varepsilon.
\]

**1.3.3 Lemma**

Let \( g(r) \) be a non-negative measurable function on \([0, \infty)\), with \( \int_{0}^{r} g(t)dt < \infty \) for every finite \( r > 0 \). Let \( h \) be the non-decreasing function
\[
h(r) = \int_{0}^{r} g(t)dt.
\]
Let \( f \) be real-valued and continuous on \([a, \infty)\). Then for each real \( r > a \) the Riemann-Stieltjes integral
\[
\int_a^r f(t)dh(t)
\]
and the Lebesgue integral
\[
\int_a^r f(t)g(t)dt
\]
are equal.

**Proof.** Let \( r > a \) and \( \varepsilon > 0 \) and take \( \delta > 0 \) with
\[
\delta \int_a^r g(t)dt < \varepsilon.
\]
Pick \( \eta > 0 \) so that \(|f(x) - f(y)| < \delta\) for \( a \leq x < y \leq r, \ y - x < \eta \). Fix a partition \( P_0 \) of \([a, r]\) such that the distance between successive vertices of \( P_0 \) is less than \( \eta \). Let \( P \) be a refinement of \( P_0 \), with vertices \( a = t_0 < t_1 < \ldots < t_n = r \). Let \( t_{k-1} \leq s_k \leq t_k \). The corresponding Riemann-Stieltjes sum \( S(P, f, h) \) is given by
\[
S(P, f, h) = \sum_{k=1}^n f(s_k)(h(t_k) - h(t_{k-1})) = \sum_{k=1}^n f(s_k) \int_{t_{k-1}}^{t_k} g(t)dt.
\]
Hence
\[
S(P, f, h) - \int_a^r f(t)g(t)dt = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} (f(s_k) - f(t))g(t)dt
\]
has modulus at most
\[
\int_a^r \delta g(t)dt < \varepsilon.
\]

**1.3.4 Lemma**

If \( h > 0 \) on \([0, 2\pi]\) and \( h \) and \( \log h \) are integrable,
\[
\frac{1}{2\pi} \int_0^{2\pi} \log h(t)dt \leq \log \left( \frac{1}{2\pi} \int_0^{2\pi} h(t)dt \right).
\]
This says that the average of \( \log h \) is not more than the log of the average of \( h \). To prove the lemma we set
\[
m = \frac{1}{2\pi} \int_0^{2\pi} h(t)dt, \quad g(t) = h(t) - m > -m.
\]
Then
\[
\frac{1}{2\pi} \int_0^{2\pi} g(t)dt = m - m = 0.
\]
Also
\[
h = m(1 + g/m), \quad \log h(t) = \log m + \log(1 + g(t)/m) \leq \log m + g(t)/m,
\]
using the fact that \( \log(1 + x) \leq x \) for \( x \geq 0 \), which holds since \( p(x) = \log(1 + x) - x \) has \( p'(x) < 0 \) for \( x > 0 \) and \( p'(x) > 0 \) for \( -1 < x < 0 \).
We now get
\[
\frac{1}{2\pi} \int_0^{2\pi} \log h(t)dt \leq \log m + \frac{1}{2\pi} \int_0^{2\pi} (g(t)/m)dt = \log m.
\]
This proves the lemma, which is a special case of Jensen’s inequality.
1.4 The density of sets

Let $E$ be a measurable subset of $[0, \infty)$. The following quantities give some idea of how large and widely spread the set $E$ is \[12, 37\]. First, we set $\chi_E(t)$ to be 1 if $t$ is in $E$, and 0 otherwise, and $\chi$ is then a measurable function. We define the upper and lower linear density of $E$ by

$$D_E = \overline{\text{dens}}(E) = \limsup_{r \to \infty} \frac{\int_0^r \chi_E(t) dt}{r}, \quad d_E = \underline{\text{dens}}(E) = \liminf_{r \to \infty} \frac{\int_0^r \chi_E(t) dt}{r}.$$ 

Obviously $0 \leq d_E \leq D_E \leq 1$, and if $E$ has finite measure then $D_E = 0$. It is also easy to see that

$$D_E = 1 - d_F, \quad d_E = 1 - D_F,$$

where $F = [0, \infty) \setminus E$.

Next we define the upper and lower logarithmic densities, by

$$LD_E = \log \overline{\text{dens}}(E) = \limsup_{r \to \infty} \frac{\int_1^r \chi_E(t) dt}{\log r}, \quad ld_E = \log \underline{\text{dens}}(E) = \liminf_{r \to \infty} \frac{\int_1^r \chi_E(t) dt}{\log r}.$$ 

Again, it is obvious that $0 \leq ld_E \leq LD_E \leq 1$.

1.4.1 Example

Let $r_n = e^{s_n}, n \geq 1$, and let $E$ be the union of the intervals $[r_n, er_n]$. Then $d_E = 0, D_E > 0, ld_E = LD_E = 0$.

Proof. Let $s_n = er_n$. Then

$$\int_0^{s_n} \chi_E(t) dt \geq \int_{r_n}^{s_n} dt = (e-1)r_n = (1-1/e)s_n,$$

and so $D_E \geq 1 - 1/e$. However,

$$\int_0^{r_n} \chi_E(t) dt \leq \int_0^{s_{n-1}} \chi_E(t) dt \leq s_{n-1} = o(r_n),$$

which gives $d_E = 0$.

Suppose now that $r$ is large, with $r_n \leq r < r_{n+1}$. Then

$$\int_1^r \chi_E(t) \frac{dt}{t} \leq \sum_{j=1}^n \int_{r_j}^{s_j} \chi_E(t) \frac{dt}{t} = \sum_{j=1}^n \int_{r_j}^{s_j} \frac{dt}{t} = n = \log \log r_n \leq \log \log r.$$ 

So $LD_E = 0$.

1.4.2 Theorem

Let $E$ be a measurable subset of $[0, \infty)$. Then $0 \leq d_E \leq ld_E \leq LD_E \leq D_E$.

Proof. We only need to prove that $LD_E \leq D_E$, because with $F = [0, \infty) \setminus E$ we get

$$d_E = 1 - D_F \leq 1 - LD_F = ld_E.$$

There is nothing to prove if $D_E = 1$ so assume that $D_E < K < 1$. Then

$$h(r) = \int_1^r \chi_E(t) dt \leq \int_0^r \chi_E(t) dt < Kr$$
for all large \( r \). So there exists \( C > 0 \) with \( h(r) < C + Kr \) for all \( r \geq 1 \). Lemma 1.3.3 and the integration by parts formula (1.6) for Riemann-Stieltjes integrals now give, for large \( r \),

\[
\int_1^r \chi_E(t) \frac{dt}{t} = \int_1^r \frac{1}{t} dh(t) = \frac{h(r)}{r} + \int_1^r \frac{h(t)}{t^2} dt,
\]

which is at most

\[
\frac{C}{r} + K + \int_1^r \frac{C}{t^2} + \frac{K}{t} dt \leq K \log r + O(1).
\]

Thus \( LD_E \leq K \).

### 1.5 Upper semi-continuity

Let \( X \) be a metric space. A function \( u : X \to [-\infty, \infty) \) is called upper semi-continuous if the following is true: for every real \( t \) the set \( \{ x \in X : u(x) < t \} \) is open. Obviously if \( X = \mathbb{R}^n \) then every upper semi-continuous function \( u \) is (Borel) measurable.

#### 1.5.1 Theorem

Let \( X \) be a metric space, with metric \( d \), and suppose that \( u : X \to [-\infty, M] \) is upper semi-continuous for some \( M \in \mathbb{R} \). Then there exist continuous functions \( u_n : X \to \mathbb{R} \) with \( u_1 \geq u_2 \geq u_3 \geq \ldots \geq u \), such that \( u_n(x) \to u \) pointwise on \( X \).

**Proof.** This proof is from [59]. If \( u \equiv -\infty \) just take \( u_n = -n \). Now assume that \( u \not\equiv -\infty \) and for \( x \in X \) and \( n \in \mathbb{N} \) put

\[
u_n(x) = \sup\{u(y) - nd(x, y) : y \in X\}.
\]

Then clearly

\[
u_n(x) \in (-\infty, M].
\]

To prove that \( u_n \) is continuous we must estimate \( u_n(x) - u_n(x') \), so assume without loss of generality that \( u_n(x) \geq u_n(x') \). Take \( \delta > 0 \). Then the definition of \( u_n \) gives \( y \) with \( u(y) - nd(x, y) > u_n(x) - \delta \). Then

\[
u_n(x) - \delta - u_n(x') < u(y) - nd(x, y) - (u(y) - nd(x', y)) = nd(x', y) - nd(x, y) \leq nd(x, x').
\]

Since \( \delta \) may be chosen arbitrarily small it follows that \( u_n(x) - u_n(x') \leq nd(x, x') \), and so each \( u_n \) is continuous. Clearly \( u_1 \geq u_2 \geq \ldots \), and choosing \( y = x \) shows that \( u_n \geq u \). Note that we have not yet used the fact that \( u \) is upper semi-continuous.

To show that \( u_n(x) \to u(x) \), take \( t \in \mathbb{R} \) with \( u(x) < t \), and using the fact that \( u \) is upper semi-continuous take \( r > 0 \) such that

\[
sup\{u(y) : y \in D(x, r)\} < t.
\]

Now

\[
u_n(x) \leq \max\{\sup\{u(y) : y \in D(x, r)\}, \sup\{u(y) : y \in X\} - nr\} \leq \max\{t, M - nr\}.
\]

We thus have \( u_n(x) \leq t \) for large \( n \).

**Exercise:** if \( u(0) = 1 \) and \( u(x) = 0 \) for all real \( x \neq 0 \), determine \( u_n(x) \) for each \( x \). Do the same for \( v = -u \) (which is not upper semi-continuous).
1.5.2 Lemma

Let the function \( u \) be upper semi-continuous on a domain containing the compact subset \( K \) of \( \mathbb{C} \). Then \( u \) has a maximum on \( K \).

Proof. Let \( S \) be the supremum of \( u(z) \) on \( K \), and take \( z_n \in K \) such that \( u(z_n) \to S \). We may assume that \( z_n \) converges, and the limit \( w \) is in \( K \) since \( K \) is closed and bounded. But then \( u(w) \geq S \), because if \( u(w) < t < S \) then we get \( u(z) < t \) near \( w \) and hence \( u(z_n) < t \) for all large \( n \). We also have \( u(w) \leq S \), by the definition of \( S \), and so \( u(w) = S \).
Chapter 2

Entire functions

2.1 The growth of entire functions

2.1.1 Notation

For $z_0 \in \mathbb{C}$ and $r > 0$ the open Euclidean disc and circle of centre $z_0$ and radius $r$ will be denoted by
\[ D(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \}, \quad S(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \}, \]
respectively. If $z_0 \in \mathbb{C}^* = \mathbb{C} \cup \{ \infty \}$ then $D_q(z_0, r)$ is the spherical disc
\[ D_q(z_0, r) = \{ z \in \mathbb{C}^* : q(z, z_0) < r \}. \]

2.1.2 The maximum modulus

Let $f$ be entire (i.e. an analytic function from the complex plane into itself). Let $r > 0$ and define
\[ M(r, f) = \max\{|f(z)| : |z| = r\}. \tag{2.1} \]

By the maximum principle, we have
\[ M(r, f) = \max\{|f(z)| : |z| \leq r\}, \]
from which it follows immediately that $M(r, f)$ is non-decreasing. Note also that if $0 < r < s$ and $M(r, f) = M(s, f)$ we can choose $z$ with $|z| = r$ and $|f(z)| = M(r, f)$. Thus $|f(w)| \leq |f(z)|$ for all $w$ in $D(0, s)$ and so $f$ is constant, again by the maximum principle, since $|f|$ has a local maximum.

Hence $M(r, f)$ is strictly increasing if $f$ is non-constant.

For an entire function $f$, we now define the order (of growth) $\rho$ of $f$ by
\[ \rho = \rho(f) = \limsup_{r \to \infty} \frac{\log (\log M(r, f))}{\log r}, \]
in which $\log^+ x$ is defined by (1.4).

Example 1:

Let $f(z) = a_n z^n + \ldots + a_0$ be a polynomial in $z$. For $|z| \geq 1$ we have $|f(z)| \leq c|z|^n$, $c = \sum_{j=0}^{n} |a_j|$. Thus $\log M(r, f) \leq n \log r + \log c \leq (n + 1) \log r$ for $r \geq 1 + c$, and so $\log \log^+ M(r, f) \leq \log \log r + O(1)$ as $r \to \infty$, and $\rho = 0$. 
Example 2:
Let \( f(z) = \exp(z^n) \), with \( n \) a positive integer. Then \( \log M(r, f) = r^n \) and \( \rho = n \).

Example 3:
Let \( f(z) = \exp(\exp(z)) \). Then \( \log M(r, f) = e^r \) and \( \rho = \infty \).

### 2.2 Wiman-Valiron theory

The Wiman-Valiron theory is concerned with determining the local behaviour of an entire function from its power series. The main references for this subject are [35], from which this chapter will draw extensively, and [69]. First, if

\[
P(z) = a_n z^n + \ldots + a_0, \quad a_n \neq 0,
\]

is a polynomial of positive degree \( n \), and if \( z \) and \( z_0 \) are large, then we have

\[
P(z) \sim \left( \frac{z}{z_0} \right)^n P(z_0) \quad \text{and} \quad \frac{P'(z)}{P(z)} \sim \frac{n}{z}.
\]

If \( P \) is replaced by a non-polynomial entire function \( f \) then it is clear from Picard’s theorem that no such asymptotic relation can hold for all large \( z \) and \( z_0 \), but the aim of the Wiman-Valiron theory is to obtain comparable estimates when \( z \) is close to \( z_0 \) and \( |f(z_0)| \) is close to \( M(|z_0|, f) \). Let

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{2.2}
\]

be a transcendental entire function (here “transcendental” means “not a rational function”). Thus \( a_k \neq 0 \) for infinitely many \( k \).

#### 2.2.1 The maximum term

We define the **maximum term** \( \mu(r, f) \) as follows. For each \( r \geq 0 \) let

\[
\mu(r) = \mu(r, f) = \max\{|a_k| r^k : k = 0, 1, 2, \ldots\}. \tag{2.3}
\]

This \( \mu(r, f) \) is well-defined, because for fixed \( r \) the terms \( |a_k| r^k \) tend to 0 as \( k \to \infty \). Obviously \( \mu(0) = |a_0| \). Since \( f \) is non-constant there exists \( k > 0 \) with \( a_k \neq 0 \) and so we have \( \mu(r) \geq |a_k| r^k > 0 \) for \( r > 0 \), as well as

\[
\lim_{r \to \infty} \mu(r, f) = \infty.
\]

The first step is an initial comparison between the growth rates of \( M(r, f) \) and \( \mu(r, f) \).

#### 2.2.2 Lemma

For \( r > 0 \) we have

\[
\mu(r, f) \leq M(r, f) \leq 2 \mu(2r, f). \tag{2.4}
\]

Further, the orders of the functions \( \log^+ M(r, f) \) and \( \log^+ \mu(r, f) \) are equal, these being defined by

\[
\rho_f = \limsup_{r \to \infty} \frac{\log^+ \log^+ M(r, f)}{\log r}, \quad \rho_\mu = \limsup_{r \to \infty} \frac{\log^+ \log^+ \mu(r, f)}{\log r}.
\]
Proof. The first inequality of (2.4) comes from Cauchy’s integral formula, since for $k \geq 0$ we have

$$|a_k| = \frac{|f^{(k)}(0)|}{k!} = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} \, dz \right| \leq \frac{1}{2\pi} (2\pi r) \frac{M(r, f)}{r^{k+1}} = \frac{M(r, f)}{r^k}.$$ 

The second inequality is proved as follows. For $k \geq 0$ we have

$$|a_k|(2r)^k \leq \mu(2r, f), \quad |a_k|r^k \leq 2^{-k} \mu(2r, f),$$

and so

$$M(r, f) \leq \sum_{k=0}^{\infty} |a_k|r^k \leq \sum_{k=0}^{\infty} 2^{-k} \mu(2r, f) = 2\mu(2r, f).$$

The last assertion of the lemma now follows from (2.4) and Lemma 1.2.4.

2.2.3 Lemma

$\mu(r, f)$ is continuous and non-decreasing on $[0, \infty)$, and there exists $R \geq 0$ such that $\mu(r)$ is strictly increasing on $[R, \infty)$.

Proof. By the definition (2.3) of $\mu$ and Lemma 2.2.2 we have $\mu(0) = |a_0| \leq \mu(r) \leq M(r, f) \to |a_0|$ as $r \to 0+$, and so $\mu(r)$ is continuous as $r \to 0+$. Now choose $m > 0$ with $a_m \neq 0$. If $r_0 > 0$ then there exists $k_0 > m$ such that $|a_k|(2r_0)^k < |a_m|r_0^m$ for $k > k_0$. So for $r_0 \leq r \leq 2r_0$ we have

$$\mu(r) \geq |a_m|r^m \geq |a_m|r_0^m$$

and so

$$\mu(r) = \max\{|a_k|r^k : 0 \leq k \leq k_0\}.$$ 

So on $[r_0, 2r_0]$ our $\mu(r)$ is the maximum of finitely many continuous functions and so continuous.

If $0 \leq r < s < \infty$ take $n$ such that $\mu(r) = |a_n|r^n$. Then

$$\mu(s) \geq |a_n|s^n \geq |a_n|r^n = \mu(r),$$

so $\mu(r)$ is non-decreasing. Now take $R \geq 0$, so large that $|a_m|R^m \geq |a_0|$ for some $m > 0$ with $a_m \neq 0$. Then for $R \leq r < s < \infty$ we have $|a_m|r^m \geq |a_0|$ and so $\mu(r) = |a_n|r^n$ for some $n > 0$ with $a_n \neq 0$, which gives strict inequality in (2.5).

2.2.4 The central index

For $r > 0$ and $\mu(r)$ as above, we define the central index $\nu(r) = \nu(r, f)$ (also called $N(r)$) to be the largest $k$ for which $|a_k|r^k = \mu(r, f)$. Note that if $a_0 = 0$ then $\nu(0)$ is not defined, whereas if $a_0 \neq 0$ then $\nu(0) = 0$.

Observe further that if $r > 0$ then $\mu(r) > 0$, and that if $k \neq n$ with $a_k a_n \neq 0$ then $|a_k|r^k = |a_n|r^n$ for exactly one positive value of $r$. Thus there are only countably many values of $r$ for which there does not exist a unique $n$ with $|a_n|r^n = \mu(r)$.

2.2.5 Example

For $f(z) = e^z$ and $f(z) = \sin z$, determine $\mu(r)$ and $\nu(r)$ (hint for $e^z$: consider those $r$ for which $|a_k|r^k = |a_{k+1}|r^{k+1}$). Use Stirling’s formula to compare $M(r, f)$ with $\mu(r)$. 

2.2. WIMAN-VALIRON THEORY

2.2.6 Lemma

The central index \( \nu(r) \) is non-decreasing on \((0, \infty)\), and \( \nu(r) \to \infty \) as \( r \to \infty \). Also, \( \nu(r) \) is continuous from the right, i.e., for each \( s > 0 \),

\[
\lim_{r \to s^+} \nu(r) = \nu(s).
\]

Proof. Suppose first that \( 0 < r < s \) and \( \nu(r) = N \). If \( N = 0 \) then obviously \( \nu(s) \geq \nu(r) \). Now suppose that \( N > M \in \{0, 1, 2, \ldots \} \). Then we have

\[
|a_N|r^N \geq |a_M|r^M, \quad |a_N|s^N = |a_N|r^N \left( \frac{s}{r} \right)^N \geq |a_M|r^M \left( \frac{s}{r} \right)^N \geq |a_M|s^M,
\]

and so \( \nu(s) \geq N \).

Now let \( P > 0 \) and choose \( k \geq P \) be such that \( a_k \neq 0 \). Then if \( m < k \) we have \( |a_m|r^m < |a_k|r^k \) for all large \( r \), and so \( \nu(r) \geq k \geq P \) for all large \( r \). This says precisely that \( \nu(r) \) tends to \( \infty \).

Now we prove that \( \nu(r) \) is continuous from the right. Let \( s > 0 \) and put \( N = \nu(s) \). Take \( k_0 > N \) such that \( |a_k|(2s)^k < \mu(s) \) for \( k > k_0 \) (this is possible since the terms \( |a_k|(2s)^k \) tend to 0). Then

\[
\mu(s) \leq \mu(r, f) = \max\{|a_k|r^k : 0 \leq k \leq k_0\}
\]

for \( s \leq r \leq 2s \). But \( N \) is the largest \( k \) for which \( |a_k|r^k = \mu(s) \), so that \( |a_k|r^k < \mu(s) \) for \( k > N \). By continuity there exists \( \delta \) with \( 0 < \delta < s \) such that

\[
|a_k|r^k < |a_N|s^N = \mu(s)
\]

for \( s \leq r \leq s + \delta \) and for \( N < k \leq k_0 \). By the choice of \( k_0 \), we now have \( |a_k|r^k < |a_N|r^N \) for \( s \leq r \leq s + \delta \) and for all \( k > N \). Hence \( \nu(r) = N \) for \( s \leq r \leq s + \delta \). A similar argument shows that \( \nu(r) \) is continuous as \( r \to 0^+ \) if \( a_0 \neq 0 \).

2.2.7 Lemma

The unbounded integer-valued function \( \nu(r) \) has the following property. There exists a strictly increasing sequence \( r_n \to \infty \), with \( r_0 = 0 \), such that \( \nu(r) \) is constant on \((r_0, r_1)\) and on \([r_n, r_{n+1})\), for each \( n \geq 1 \). Also if \( 0 < s < r \) then

\[
\log \mu(r) = \log \mu(s) + \int_s^r \frac{\nu(t) \, dt}{t}.
\]

(2.6)

For large \( r \) we have

\[
\log^+ \mu(r) < \nu(r) \log r + O(1)
\]

(2.7)

and

\[
\nu(r) \log 2 \leq \log^+ \mu(2r), \quad \nu(r) \log r \leq \log^+ \mu(r^2).
\]

(2.8)

The orders of growth of \( \log^+ \mu(r) \) and \( \nu(r) \) are the same i.e.

\[
\lim_{r \to \infty} \sup \frac{\log^+ \log^+ \mu(r)}{\log r} = \lim_{r \to \infty} \sup \frac{\log^+ \nu(r)}{\log r}.
\]

(2.9)

Proof. We just set \( r_0 = 0 \), and let \( r_n, n \geq 1 \), be the points in \((0, \infty)\) at which \( \nu(r) \) is discontinuous. Here we note that if \( r > 0 \) and \( \nu(r) = N \), the function \( \nu \) cannot have more than \( N \) discontinuities in
(0, r). Since \( \nu(r) \) is continuous from the right and integer-valued, it must be constant on \( (r_0, r_1) \) and \( [r_n, r_{n+1}) \).

Now suppose that \( \nu(r) = N \) for \( r_n < r < r_{n+1} \). Then on this interval we have \( \mu(r) = |a_N|r^N \) and so
\[
\frac{d \log \mu(r)}{d \log r} = N. \tag{2.10}
\]
Since \( \mu(r) \) is continuous we get
\[
\log \mu(b) - \log \mu(a) = \int_a^b \frac{\nu(t)}{t} \, dt
\]
for \( r_n \leq a \leq b \leq r_{n+1} \). Adding these gives (2.6).

To prove (2.7) and (2.8), choose \( s \geq 1 \), so large that \( \mu(s) \geq 1 \). Then for \( r \geq s \) we have, since \( \nu(t) \) is non-decreasing,
\[
\log \mu(r) \leq \log \mu(s) + \nu(r) \int_s^r \frac{dt}{t} \leq \log \mu(s) + \nu(r) \log r,
\]
which gives (2.7). We also have
\[
\log \mu(2r) \geq \int_r^{2r} \frac{\nu(t)}{t} \, dt \geq \nu(r) \int_r^{2r} \frac{dt}{t} = \nu(r) \log 2
\]
and
\[
\log \mu(r^2) \geq \int_r^{r^2} \frac{\nu(t)}{t} \, dt \geq \nu(r) \int_r^{r^2} \frac{dt}{t} = \nu(r) \log r.
\]
This proves (2.8), the second inequality of which gives
\[
\frac{\log \mu(r)}{\log r} \geq \frac{\nu(r^{1/2})}{2} \rightarrow \infty
\]
as \( r \rightarrow \infty \). Finally, (2.9) follows from (2.7), (2.8) and Lemma 1.2.4.

2.2.8 Lemma

Let \( \varepsilon > 0 \). Then
\[
N(r) = \nu(r) \leq (\log \mu(r))^{1+\varepsilon} \leq (\log M(r, f))^{1+\varepsilon} \tag{2.11}
\]
for all \( r \geq 1 \) outside a set \( E \) of finite logarithmic measure, i.e.
\[
\int_{[1, \infty) \cap E} \frac{dt}{t} < \infty.
\]

Proof. Choose \( s \geq 1 \) with \( \mu(s) > 1 \) and let \( F \) be the set of \( r \geq s \) for which (2.11) fails. Then, for \( R > s \), integration of (2.10) gives
\[
\int_{[1, R] \cap F} \frac{dt}{t} \leq \int_{[s, R]} \frac{N(t) \, dt}{t(\log \mu(t))^{1+\varepsilon}} = \frac{1}{\varepsilon} \left( \frac{1}{(\log \mu(s))^{\varepsilon}} - \frac{1}{(\log \mu(R))^{\varepsilon}} \right). \]

Letting \( R \rightarrow \infty \) then shows that \( F \) has finite logarithmic measure, and so has \( E \), since \( E \setminus F \) is bounded.
2.2.9 The comparison sequences

Let \((\alpha_n)\) and \((\rho_n)\) be sequences such that

\[
\alpha_n > 0, \quad 0 < \rho_0 < \frac{\alpha_0}{\alpha_1}, \quad \frac{\alpha_{n-1}}{\alpha_n} < \rho_n < \frac{\alpha_n}{\alpha_{n+1}} \quad \text{for} \quad n \geq 1.
\]

(2.12)

Note that suitable sequences \((\alpha_n)\) and \((\rho_n)\) will be constructed subsequently.

2.2.10 Lemma

Let \(f\) be a transcendental entire function with \(a_0 \neq 0\) in (2.2), and assume that the sequence \((\rho_n)\) is bounded above in §2.2.9. A real number \(r > 0\) will be called normal for \(f\) with respect to the sequences \((\alpha_n)\) and \((\rho_n)\) if there exists an integer \(N \geq 0\) with

\[
|a_n|r^n \leq |a_N|r^N\frac{\alpha_n(\rho_N)^n}{\alpha_N(\rho_N)^N} \quad \text{for all} \quad n \geq 0.
\]

(2.13)

Then there exists an exceptional set \(E_0\) of finite logarithmic measure such that every \(r \geq 1\) with \(r \notin E_0\) is normal, and satisfies (2.13) with \(N = N(r)\).

Proof. It follows from (2.12) that

\[
\frac{\alpha_n}{\alpha_N} < (\rho_N)^{N-n} \quad \text{for} \quad n, N \geq 0, \ n \neq N.
\]

(2.14)

For if \(n < N\) then

\[
\frac{\alpha_n}{\alpha_N} = \frac{\alpha_n}{\alpha_{n+1}} \cdots \frac{\alpha_{N-1}}{\alpha_N} < \rho_{n+1} \cdots \rho_N \leq (\rho_N)^{N-n},
\]

while \(n > N\) gives

\[
\frac{\alpha_n}{\alpha_N} = \frac{\alpha_{N+1}}{\alpha_N} \cdots \frac{\alpha_{n}}{\alpha_{n-1}} < \frac{1}{\rho_{N}} \cdots \frac{1}{\rho_{n-1}} \leq \frac{1}{(\rho_N)^{n-N}}.
\]

This proves (2.14), which now implies in particular that if (2.13) holds then

\[
|a_n|r^n < |a_N|r^N \quad \text{for} \quad n \neq N
\]

and so \(N = N(r) = \nu(r)\).

We assert that there exists a non-decreasing sequence \((s_n)\) with limit \(\infty\) and with the following properties: (i) we have \(s_0 = 0\); (ii) if \(s_n < s_{n+1}\) then \(N(r) = n\) on \([s_n, s_{n+1})\). To see this, observe first that \(N(0) = 0\) (because \(a_0 \neq 0\)) and that \(N(r)\) is non-decreasing and continuous from the right, and integer-valued. So let

\[
0 = p_0 < p_1 < \ldots
\]

be the values taken by \(N(r)\), and let \(t_k = \min\{t \geq 0 : N(t) = p_k\}\), which exists because \(N(r)\) is continuous from the right. So we set \(s_0 = 0\) and then \(s_1 = \ldots = s_{p_1} = t_1\), and \(s_{p_1+1} = \ldots = s_{p_2} = t_2\) and so on.

Now we claim that

\[
\left|\frac{a_n}{a_0}\right| \leq \frac{1}{s_1 \ldots s_n} \quad \text{for} \quad n > 0.
\]

(2.15)

We prove (2.15) by induction. For \(0 \leq t < s_1\) we have \(N(t) = 0\) and so \(|a_1|t \leq |a_0|\), which gives (2.15) for \(n = 1\) on letting \(t \to s_1-\). Now let \(n > 1\) and let \(m\) be the largest integer such that \(s_m < s_n\). Then on \([s_m, s_n)\) we have

\[
N(r) = m \quad \text{and} \quad |a_n|r^n \leq |a_m|r^m.
\]
Let \( r \to s_n \). If \( m = 0 \) then \( s_1 = \ldots = s_n \) and we have
\[
|a_n| \leq \frac{|a_0|}{(s_n)^n} = \frac{|a_0|}{s_1 \ldots s_n}
\]
as required. On the other hand if \( m > 0 \) then we may assume by the induction hypothesis that (2.15) holds with \( n \) replaced by \( m \) and we get
\[
|a_n| \leq |a_m|(s_n)^{m-n} \leq \frac{|a_0|}{s_1 \ldots s_m} \frac{1}{(s_n)^{m-n}} = \frac{|a_0|}{s_1 \ldots s_n}.
\]
This proves (2.15).

It follows from (2.12) that, for \( n \geq 1 \),
\[
\frac{\alpha_n}{\alpha_0} = \frac{\alpha_n}{\alpha_{n-1}} \ldots \frac{\alpha_1}{\alpha_0} > \frac{1}{\rho_n \ldots \rho_1}.
\]
Combining (2.15) and (2.16) then gives, for \( n \geq 1 \),
\[
\left| \frac{\alpha_n}{\alpha_n} \right|^{1/n} \leq \left( \frac{|a_0| \rho_1}{\alpha_0 s_1} \ldots \frac{\rho_n}{s_n} \right)^{1/n}.
\]
Now we use the fact that \((\rho_m)\) is assumed to be bounded above, from which it follows that if \( T > 1 \) then \( s_m > T \rho_m \) for all \( m > M \), say. This in turn gives, by (2.17),
\[
\left| \frac{\alpha_n}{\alpha_n} \right|^{1/n} \leq \left( \frac{|a_0| \rho_1}{\alpha_0 s_1} \ldots \frac{\rho_n}{s_n} \right)^{1/n} \frac{1}{T^{(n-M)/n}} \leq \frac{2}{\sqrt{T}}
\]
for all large enough \( n \). Hence
\[
\lim_{n \to \infty} \left| \frac{\alpha_n}{\alpha_n} \right|^{1/n} = 0,
\]
and so if we set
\[
F(z) = \sum_{n=0}^{\infty} A_n z^n, \quad A_n = \left| \frac{\alpha_n}{\alpha_n} \right|
\]
then \( F \) is an entire function.

The point now is to deduce properties of \( f \) from those of \( F \). Suppose that \( \rho > 0 \) and that \( M = \nu(\rho, F) \). Then for all \( n \neq M \) we have, by (2.14) and (2.18),
\[
\frac{|a_n| (\rho M)^n}{|A_n| (\rho M)^n} = \frac{\alpha_n A_n \rho^n (\rho M)^n}{\alpha M A_M \rho^M (\rho M)^M} \leq \left( \frac{\alpha_n}{\alpha_M} \right) (\rho M)^{n-M} < 1.
\]
This implies that \( N(r) = \nu(r, f) = M \) for \( r = \rho M \), and also that \( r \) is normal for \( f \) (with \( M \) taking the role of \( N \) in (2.13)).

Since \( A_0 \neq 0 \) we can define a sequence \((S_n)\) for the function \( F \), exactly as we defined \((s_n)\) for \( f \). If we now have \( \nu(\rho, F) = n \) on \((S_n, S_{n+1})\) then we have \( \nu(r, f) = n \) on \( I_n = (S_n \rho_n, S_{n+1} \rho_n) \), and every \( r \) in the interval \( I_n \) is normal for \( f \). We also have \( S_{n+1} \rho_n < S_{n+1} \rho_{n+1} \), by (2.12). Hence all non-normal \( r \) for \( f \) lie in the union of the intervals \([S_{n+1} \rho_n, S_{n+1} \rho_{n+1}]\), each of which has logarithmic measure
\[
\log \frac{\rho_{n+1}}{\rho_n}.
\]
Since \((\rho_n)\) is bounded above, these logarithmic measures have finite sum, and so the lemma is proved.
2.2. WIMAN-VALIRON THEORY

2.2.11 Construction of the sequences \((\alpha_n)\) and \((\rho_n)\)

Choose \(\sigma \in (1, 2)\), and set
\[
\alpha(t) = \int_0^t \beta(s) \, ds, \quad (2.20)
\]
where
\[
\beta(s) = -1 \quad (0 \leq s \leq 1), \quad \beta(s) = -\frac{1}{s^\sigma} \quad (1 \leq s < \infty). \quad (2.21)
\]
Then \(\alpha(t)\) is a negative, strictly decreasing function on \((0, \infty)\), with a finite limit as \(t \to \infty\). Set
\[
\alpha_n = \exp \left( \int_0^n \alpha(t) \, dt \right), \quad \rho_n = \exp(-\alpha(n)). \quad (2.22)
\]
Since \(\alpha(t)\) is bounded below on \((0, \infty)\), the sequence \((\rho_n)\) is bounded above. It is obvious that \(\alpha_n > 0\).

To check the remaining conditions of (2.12) we note that, for \(n \geq 1\),
\[
\log \frac{\alpha_n}{\alpha_{n-1}} = \int_{n-1}^n \alpha(t) \, dt > \int_{n-1}^n \alpha(n) \, dt = \alpha(n) = \log \frac{1}{\rho_n},
\]
and also that, this time for \(n \geq 0\),
\[
\log \frac{1}{\rho_n} = \alpha(n) > \int_n^{n+1} \alpha(t) \, dt = \log \frac{\alpha_{n+1}}{\alpha_n}.
\]
This shows that sequences with the required properties do exist.

2.2.12 Lemma

The construction of §2.2.11 gives, for \(n, N \geq 0\), and \(k = n - N \neq 0\),
\[
\frac{\alpha_n(\rho_N)^n}{\alpha_N(\rho_N)^N} \leq \exp \left( -\frac{k^2}{2(N + |k|)^\sigma} \right). \quad (2.23)
\]

**Proof.** For \(n \neq N\) we have, on integrating by parts,
\[
\frac{\alpha_n(\rho_N)^n}{\alpha_N(\rho_N)^N} = \exp \left( \int_N^n \alpha(t) \, dt \right) \exp(-\alpha(N)(n-N)) = \exp \left( \int_N^n (\alpha(t) - \alpha(N)) \, dt \right) = \exp \left( \int_N^n (n-t)\beta(t) \, dt \right).
\]
If \(n > N\) then, since \(-\beta(t)\) is positive and non-increasing,
\[
-\int_N^n (n-t)\beta(t) \, dt \geq -\beta(n) \int_N^n (n-t) \, dt = \frac{(n-N)^2}{2n^\sigma} = \frac{k^2}{2(N + |k|)^\sigma}.
\]
On the other hand, if \(n < N\) then, again since \(-\beta(t)\) is positive and non-increasing,
\[
-\int_N^n (n-t)\beta(t) \, dt = \int_n^N (t-n)(-\beta(t)) \, dt \geq -\beta(N) \int_n^N (t-n) \, dt = \frac{(N-n)^2}{2N^\sigma} \geq \frac{k^2}{2(N + |k|)^\sigma}.
\]
CHAPTER 2. ENTIRE FUNCTIONS

2.2.13 Lemma

Let \(1 < \sigma < 2\) and let \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) be a transcendental entire function with central index \(N(r)\) and maximum term \(\mu(r)\). Then for all large \(r\) outside a set of finite logarithmic measure we have, with \(N = N(r)\),

\[
\frac{|a_{N+k}|r^{N+k}}{\mu(r)} \leq \exp \left( -\frac{k^2}{2(N+|k|)\sigma} \right).
\]

(2.24)

Proof. Obviously there is nothing to prove if \(a_0 \neq 0\). Then we take the sequences \((\alpha_n)\) and \((\rho_n)\) and the set of non-normal \(r\) has finite logarithmic measure. Moreover if \(r\) is normal then combining (2.13) with (2.23) gives, with \(n = N + k\) and \(k \neq 0\),

\[
|a_n|r^n \leq \mu(r) \frac{\alpha_n(\rho_N)^n}{\alpha_N(\rho_N)^N} \leq \mu(r) \exp \left( -\frac{k^2}{2(N+|k|)\sigma} \right).
\]

(2.25)

Now suppose that \(a_0 = 0\). Then we may write \(f(z) = z^p g(z)\) for some \(p > 0\), where \(g(z) = \sum_{k=0}^{\infty} c_k z^k\) is entire and \(g(0) = c_0 \neq 0\). It is then easy to see that \(c_n = a_{n+p}\) and \(\mu(r) = r^p \mu(r,g)\), while \(N(r) = \nu(r,g) + p\). Hence, for all large \(r\) outside a set of finite logarithmic measure, writing \(\nu = \nu(r,g)\) and using (2.25) with \(f\) replaced by \(g\) gives

\[
\frac{|a_{N+k}|r^{N+k}}{\mu(r)} = \frac{|c_{\nu+k}|r^{\nu+k}}{\mu(r,g)} \leq \exp \left( -\frac{k^2}{2(\nu+|k|)\sigma} \right) \leq \exp \left( -\frac{k^2}{2(N+|k|)\sigma} \right).
\]

2.2.14 Comparison between \(\nu(r,f)\) and \(\nu(r,f')\)

It is convenient to consider \(g(z) = z f'(z) = \sum_{k=1}^{\infty} k a_k z^k\), and obviously \(\nu(r,g) = \nu(r,f') + 1\). Now fix \(\varepsilon > 0\), and suppose that \(r\) is large and lies outside the exceptional set \(E\) of Lemma 2.2.13, and set \(N = \nu(r,f)\). Then for \(n \leq N\) we have

\[
n|a_n|r^n \leq N|a_n|r^n \leq N|a_N|r^N
\]

and so \(\nu(r,g) \geq N = \nu(r,f)\). Now take \(n = N + k\) with \(k \geq \varepsilon N\). Then \(N + k \leq k(1 + 1/\varepsilon)\) and Lemma 2.2.13 gives

\[
n|a_n|r^n \leq (N + k)|a_N|r^N \exp \left( -\frac{k^2}{2(N+|k|)\sigma} \right) \leq c_1 k \exp(-c_2 k^{2-\sigma}) N|a_N|r^N,
\]

where the positive constants \(c_1\) and \(c_2\) are independent of \(r\). If \(N\) is large then so is \(k\), and thus \(n|a_n|r^n < (1/2)N|a_N|r^N\) for \(n \geq (1 + \varepsilon)N\), which forces \(\nu(r,g) \leq (1 + \varepsilon)N\).

We conclude that

\[
\nu(r,f') \sim \nu(r,f) \quad \text{as } r \to \infty \text{ with } r \notin E, \text{ where } \int_{E} dt/t < \infty.
\]

(2.26)

2.2.15 Lemma

Let \(\alpha > 0\). Then

\[
N(r \exp(N(r)^{-\alpha})) < (1 + \alpha)N(r)
\]

(2.27)

for all \(r \geq 1\) outside a set of finite logarithmic measure.

Proof. Choose \(R \geq 1\) with \(N(R) \geq 1\) and set

\[
s = \log r, \quad M(s) = N(r)^{\alpha} = N(e^s)^{\alpha}
\]
for \( s \geq S = \log R \). Then \( M(s) \) is non-decreasing and continuous from the right, and \( M(s) \geq 1 \) for \( s \geq S \). Choose \( A > 1 \) with \( A^{1/A} < 1 + \alpha \). The Borel lemma 1.2.5 gives

\[
N(r \exp(N(r)^{-\alpha}))^\alpha = M(\log(r \exp(N(r)^{-\alpha}))) = M(s + 1/M(s)) \leq AM(s) = AN(r)^\alpha
\]

for \( s \geq S \) outside a set \( E_0 \) of finite measure. The corresponding exceptional set of \( r \) is just

\[
F_0 = \{e^s : s \in E_0\}
\]

and satisfies

\[
\int_{F_0} \frac{dr}{r} = \int_{E_0} ds < \infty.
\]

### 2.2.16 Estimates for sums of terms in the power series

Let \( 1 < \sigma < 2 \) and let \( \sigma < 2\tau < 2 \). Let \( q \) be a non-negative integer, and let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be a transcendental entire function with maximum term \( \mu(r) \) and central index \( N(r) \). We will estimate

\[
\sum_{|n-N(r)| \geq N(r)^\tau} n^q |a_n| \mu^n
\]

for \( \rho \) close to \( r \).

In order to do this, let \( r \) lie outside the exceptional sets of Lemmas 2.2.13 and 2.2.15, taking \( \alpha = 1/4 \) in the latter. Note that the union \( E^* \) of these exceptional sets has finite logarithmic measure (and does not depend on \( q \)). Write

\[
N = N(r), \quad \mu_0(\rho) = |a_N| \mu^N, \quad (2.28)
\]

where

\[
|\log(\rho/r)| \leq N^{-\tau}. \quad (2.29)
\]

We use \( c_1, c_2, \ldots \) to denote positive constants which do not depend on \( r \) or \( \rho \) (although in general they will depend on \( f, \sigma, \tau \) and \( q \)). Write

\[
\rho_1 = r \exp(N(r)^{-1/4}), \quad M = N(\rho_1), \quad N \leq M \leq \frac{5N}{4}, \quad (2.30)
\]

in which the last inequality follows from Lemma 2.2.15.

Then for \( r \) large enough, not in \( E^* \), and \( n > 2N \) we have, by (2.28), (2.29) and (2.30), the inequality \( n - M \geq n - 5N/4 \geq c_1 n \) and the estimates

\[
\frac{|a_n| \mu^n}{\mu_0(\rho)} = \frac{|a_n| \mu^n}{a_N \mu^N} = \frac{|a_n| \mu^n}{a_M \mu^M} \frac{|a_M| \mu^M}{a_N \mu^N} = \frac{|a_n| \rho_1^n}{a_M \rho_1^M} \left( \frac{\rho}{\rho_1} \right)^{n-M} \frac{|a_M| \rho^n}{a_N \mu^N} \left( \frac{\rho}{\rho_1} \right)^{M-N} \leq \left( \frac{\rho}{\rho_1} \right)^{n-M} \left( \frac{\rho}{\rho_1} \right)^{M-N} \leq \exp((M-n)N^{-1/4} + (n-N)|\log(\rho/r)|) \leq \exp(-c_2 n N^{-1/4} + n N^{-1/4}) \leq \exp(-c_2 n N^{-1/4}), \quad (2.31)
\]

using the fact that \( \tau > 1/2 \). Thus we have

\[
\sum_{n>2N} n^q |a_n| \mu^n \leq \mu_0(\rho) \sum_{n>2N} n^q t^n, \quad t = \exp(-c_2 n N^{-1/4}) < 1, \quad (2.32)
\]
for \( \rho \) satisfying (2.29).

Now, since \( N \) is large,

\[
\sum_{n>2N} n^q t^n < \sum_{n>N} n^q t^n = t^N \sum_{k=1}^{\infty} (N + k)^q t^k = t^N \sum_{k=1}^{\infty} (1 + N/k)^q k^q t^k \leq 2 t^N N^q \sum_{k=1}^{\infty} k^q t^k.
\]

But repeated differentiation of the geometric series shows that the power series \( \sum_{k=1}^{\infty} k^q t^k \) may be written as a linear combination of

\[
\frac{1}{1 - t}, \frac{1}{1 - t^2}, \ldots, \frac{1}{(1 - t)^{q+1}},
\]

with constant coefficients, independent of \( r \) and \( \rho \). Since \( 0 < t < 1 \) this gives

\[
\frac{1}{1 - t} = \frac{\exp(c_2 N^{-1/4})}{\exp(c_2 N^{-1/4}) - 1} \leq c_3 N^{1/4}
\]

and

\[
\sum_{n>2N} n^q t^n \leq c_4 t^N N^q \frac{1}{(1 - t)^{q+1}} \leq c_5 t^N N^q (q+1)/4.
\]

On recalling (2.32) we therefore have, for \( r \not\in E^* \) large enough,

\[
\sum_{n>2N} n^q |a_n| \rho^n \leq \mu_0(\rho) c_5 t^N N^q N^{(q+1)/4} = c_5 \mu_0(\rho) \exp(-c_2 N^{3/4} + c_6 \log N)
\]

\[
\leq \mu_0(\rho) \exp(-c_7 N^{3/4}). \quad (2.33)
\]

We consider next those \( n \) satisfying

\[
0 \leq n = N + p \leq 2N, \quad |p| \geq N^\tau.
\]

For these \( n \) and for \( \rho \) satisfying (2.29) we have, by Lemma 2.2.13 and the fact that \( 2\tau > \sigma \) gives \( \sigma - \tau < \tau \),

\[
n^q |a_n| \rho^n \frac{\mu_0(\rho)}{a_n r^N} = n^q \left( \frac{\rho}{r} \right)^p |a_n| \rho^n \frac{r^N}{a_n r^N}
\]

\[
\leq (2N)^q \left( \frac{\rho}{r} \right)^p \exp(-p^2 / 2(N + |p|)^\sigma)
\]

\[
\leq (2N)^q \exp(|p| N^{-\sigma} - p^2 / 2(2N)^\sigma)
\]

\[
= (2N)^q \exp(|p| N^{-\sigma} (N^{\sigma - \tau} - c_8 |p|))
\]

\[
\leq (2N)^q \exp(|p| N^{-\sigma} (o(N^\tau) - c_8 |p|))
\]

\[
\leq (2N)^q \exp(-c_9 p^2 N^{-\sigma})
\]

\[
\leq (2N)^q \exp(-c_9 N^{2\tau - \sigma}) = (2N)^q \exp(-c_9 N^{2\varepsilon}),
\]

where \( 2\varepsilon = 2\tau - \sigma > 0 \). Hence we get, for \( r \not\in E^* \) large, and for \( \rho \) satisfying (2.29),

\[
\sum_{n \leq 2N, |n - N| \geq N^\tau} n^q |a_n| \rho^n \frac{\mu_0(\rho)}{a_n r^N} \leq (2N)^q+1 \exp(-c_9 N^{2\varepsilon})
\]

\[
= \exp(-c_9 N^{2\varepsilon} + c_{10} \log N) \leq \exp(-N^{3\varepsilon/2}). \quad (2.34)
\]

Combining (2.33) with (2.34) then gives the following fundamental lemma.
Let 1 < σ < 2τ < 2. Then there exists δ > 0 with the following property. Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be a transcendental entire function and let \( q \) be a non-negative integer. Then for all \( r \geq 1 \) outside a set \( E_1 \) of finite logarithmic measure we have, with the notation

\[
N = \nu(r, f), \quad \mu_0(\rho) = |a_N|\rho^N,
\]

the estimate

\[
\sum_{|n-N| \geq N^r} n^q |a_n|\rho^n \leq \mu_0(\rho) \exp(-N^\delta) \quad \text{for} \quad |\log(\rho/r)| \leq N^{-\tau}.
\]

We also obtain another comparison between the maximum modulus and the maximum term. It follows using Lemma 2.2.8 that, for \( r \) outside a perhaps larger set of finite logarithmic measure,

\[
\sum_{|n-N| \geq N^r} n^q |a_n|r^n \leq \mu_0(r) = \mu(r, f), \quad N = N(r) \leq (\log \mu(r, f))^2
\]

and so

\[
\mu(r, f) \leq M(r, f) \leq \sum_{k=0}^{\infty} |a_k|r^k \leq 3N(r)^r \mu(r, f) \leq 3\mu(r, f)(\log \mu(r, f))^2.
\]

**2.2.18 A lemma concerning polynomials**

Let \( \lambda, \delta \) and \( \varepsilon \) be positive real numbers, and let \( j \in \{0, 1\} \). Let

\[
P(z) = \alpha_m z^m + \ldots + \alpha_0
\]

be a polynomial of degree at most \( m \). Then for \( R \geq r > 0 \) we have

\[
|P^{(j)}(z)| \leq e^{j} \left( \frac{m}{r} \right)^j \left( \frac{R}{r} \right)^{m-j} M(r, P)
\]

for \( |z| \leq R \). Further, if \( m^\varepsilon > e^{j}/(\delta \lambda) \) and \( |z_0| = r > 0 \) and \( |P(z_0)| \geq \lambda M(r, P) \), then

\[
|P(z) - P(z_0)| < \delta |P(z_0)| \quad \text{for} \quad |z - z_0| \leq \frac{r}{m^{1+\varepsilon}}.
\]

**Proof.** We first prove (2.38) for \( j = 0 \). Let

\[
M = M(r, P), \quad Q(z) = \frac{P(z)r^m}{z^m} = r^m \alpha_m + \ldots.
\]

Then \( Q(z) \) is analytic for \( r \leq |z| \leq \infty \), with \( Q(\infty) = r^m \alpha_m \). We also have \( |Q(z)| \leq M \) on \( |z| = r \), and so the maximum principle implies that \( |Q(z)| \leq M \) for \( |z| \geq r \). In particular, \( M(R, P) \leq (R/r)^m M \), which gives (2.38) for \( j = 0 \), using the maximum principle again.

Next, we consider the case \( j = 1 \). Let \( |z| \leq R \), and put \( h = R/m \). Then (2.38) for \( j = 0 \) and Cauchy’s integral formula lead to

\[
|P'(z)| = \left| \frac{1}{2\pi i} \int_{|u-z|=h} \frac{P(u)}{(u-z)^2} \, du \right|
\]

\[
\leq \frac{1}{h} \max \{|P(u)| : |u - z| = h\} \leq \frac{1}{h} M(R + h, P)
\]

\[
\leq \frac{M}{h} \left( \frac{R+h}{r} \right)^m = \frac{M}{h} \left( \frac{R}{r} \right)^m \left( 1 + \frac{1}{m} \right)^m
\]

\[
= \frac{mM}{r^m} \left( 1 + \frac{1}{m} \right)^m \leq e \left( \frac{m}{r} \right) \left( \frac{R}{r} \right)^{m-1},
\]
since $1 + 1/m < e^{1/m}$. This proves (2.38) for $j = 1$.

To prove (2.39) let $S = r(1 + m^{−1−ε})$. Then for $z$ as in (2.39) we have

$$|P'(z)| \leq M(S, P') \leq M(r, P) e\left(\frac{m}{r}\right) \left(\frac{S}{r}\right)^{m-1} \leq e^2 m M(r, P) \leq \frac{e^2 m M(r, P)}{r}.$$ 

Hence we obtain, for such $z$,

$$|P(z) − P(z_0)| = \left|\int_{z_0}^{z} P'(t) \, dt\right| \leq \frac{r}{m^{1+ε}} \frac{e^2 m M(r, P)}{r} \leq \frac{e^2 M(r, P)}{m^{ε}} \leq \frac{e^2 |P(z_0)|}{λ m^{ε}} < δ|P(z_0)|$$

by the lower bound on $m$.

2.2.19 The main estimates at points near to the maximum modulus

Let $f(z) = \sum_{k=0}^{∞} a_k z^k$ be a transcendental entire function, let $1 < σ < 2τ < 2$ and let $δ$ and the exceptional set $E_1$, which has finite logarithmic measure, be as in Lemma 2.2.17. Choose $λ \in (0, 1/2]$ and let $ε$ be small and positive. In addition let $τ < γ < 1$.

Let $r \notin E_1$ be large and set

$$N = ν(r, f), \quad k = [N^τ], \quad N^0(ρ) = |a_N|ρ^N, \quad (2.40)$$

where $[x]$ denotes the greatest integer not exceeding $x$. Then Lemma 2.2.17 implies that

$$\sum_{|n−N|>k} n|a_n|ρ^n \leq μ_0(ρ) \exp(−N^δ) \quad (2.41)$$

for

$$|\log(ρ/r)| \leq N^{−γ}. \quad (2.42)$$

Note that for $ρ$ satisfying (2.42) we have

$$|k \log(ρ/r)| \leq N^{−γ} = o(1), \quad \left(\frac{ρ}{r}\right)^k \sim 1, \quad (2.43)$$

as $r \to \infty$ with $r \notin E_1$.

Write

$$f(z) = \sum_{n=N−k}^{N+k} a_n z^n + φ(z) = z^{N−k}P(z) + φ(z), \quad (2.44)$$

where $P$ is a polynomial of degree at most $m = 2k$. The aim will be to show that, for appropriate choice of $z$, the remainder term $φ(z)$ is relatively small and the polynomial $P(z)$ does not vary too much, so that $f(z)$ is essentially controlled by the monomial $z^{N−k}$. To this end we apply Lemma 2.2.18 to $P$ and $P'$, with

$$R = r \exp(N^{−γ}),$$

to get

$$M(R, P) \leq \left(\frac{R}{r}\right)^m M(r, P) \leq M(r, P) \exp(2N^{−γ}) \sim M(r, P),$$

$$M(R, P') \leq e\left(\frac{2k}{r}\right) \left(\frac{R}{r}\right)^{m−1} M(r, P) < \frac{12k M(r, P)}{r}.$$ \quad \text{(2.45)}
For $|z| = \rho$ satisfying (2.42), the estimate (2.41) and the relation (2.44) imply that
\[ f(z) = z^{N-k} P(z) + o(\mu_0(\rho)) = z^{N-k} P(z) + o(M(\rho, f)), \] (2.46)
from which it follows easily that
\[ M(r, f) \sim r^{N-k} M(r, P). \] (2.47)

Now choose $z_0$ with
\[ |z_0| = r, \quad |f(z_0)| \geq 2\lambda M(r, f). \] (2.48)
Then (2.46) gives
\[ f(z_0) \sim z_0^{N-k} P(z_0), \quad |f(z_0)| \sim r^{N-k}|P(z_0)|, \] (2.49)
and hence, using (2.47),
\[ |P(z_0)| \sim r^{k-N}|f(z_0)| \geq 2\lambda r^{k-N} M(r, f) \geq \lambda M(r, P). \] (2.50)

For $|z| = \rho$ satisfying (2.42) we may now write, using the first relation of (2.46), as well as (2.43) and (2.49),
\[ \frac{f(z)}{z^N} = z^{-k} P(z) + o(|a_N|) = z^{-k} P(z) + o(r^{-N} M(r, f)) \]
\[ = z^{-k} P(z) + o(r^{-N} |f(z_0)|) = z^{-k} P(z) + o(r^{-k} |P(z_0)|) \]
\[ = z^{-k} P(z) + o(|P(z_0)|). \] (2.51)

For $\rho$ satisfying (2.42) we deduce, using (2.43) and (2.45), that
\[ M(\rho, f) \leq \rho^{N-k}(M(\rho, P) + o(|P(z_0)|)) = (1 + o(1))\rho^{N-k} M(r, P) \]
\[ \sim \left( \frac{\rho}{r} \right)^{N-k} M(r, f) \sim \left( \frac{\rho}{r} \right)^N M(r, f). \] (2.52)

Next, consider $z$ satisfying
\[ |\log(z/z_0)| \leq N^{-\gamma}. \] (2.53)
For such $z$ we have
\[ |k \log(z/z_0)| = o(1), \quad \left( \frac{z}{z_0} \right)^k \sim 1, \quad |z - z_0| = O(rN^{-\gamma}) = o \left( \frac{r}{m^{1+\varepsilon}} \right), \] (2.54)
since $\varepsilon$ is small. Thus for $z$ satisfying (2.53) we have $P(z) \sim P(z_0)$ by Lemma 2.2.18 and so (2.49), (2.51) and (2.54) give
\[ f(z) \sim z^{N-k} P(z_0) \sim \left( \frac{z}{z_0} \right)^{N-k} f(z_0) \sim \left( \frac{z}{z_0} \right)^N f(z_0), \] (2.55)
which is the main estimate of the Wiman-Valiron theory.

In particular, if we choose $z_0$ such that $|z_0| = r$ and $|f(z_0)| = M(r, f)$ then for $z$ satisfying (2.53) and $|z| = \rho$ we get
\[ |f(z)| \geq (1 - o(1)) \left( \frac{\rho}{r} \right)^N M(r, f) \]
and so (2.52) now becomes, for $\rho$ satisfying (2.42),
\[ M(\rho, f) \sim \left( \frac{\rho}{r} \right)^N M(r, f). \] (2.56)
The next step is to estimate $f'(z)$. For $|z| = \rho$ as in (2.42), the function $\phi(z)$ of (2.44) satisfies, by (2.41),
\[
|\phi'(z)| = \left| \sum_{|n-N| > k} n a_n z^{n-1} \right| \leq \frac{\mu_0(\rho) \exp(-N^4)}{\rho}.
\] (2.57)
Differentiating (2.44) thus gives, for $|z| = \rho$ satisfying (2.42),
\[
f'(z) = (N - k) z^{N-k-1} P(z) + z^{N-k} P'(z) + \phi'(z)
\]
and hence, using (2.45) and (2.50),
\[
\frac{f'(z)}{z^N} = (N - k) z^{N-k-1} P(z) + z^{N-k} P'(z) + o(\rho^{-1} a_N |\rho|^N)
\]
and hence, using (2.45) and (2.50),
\[
\frac{f'(z)}{z^N} = (N - k) z^{N-k-1} P(z) + z^{N-k} P'(z) + o(\rho^{-1} |\rho|^N)
\]
and so (2.55) and (2.58) lead to
\[
f'(z) \sim z^{N-k-1} N P(z_0) \sim \frac{N}{z} f(z) \sim \frac{N}{z} \left( \frac{z}{z_0} \right)^N f(z_0).
\] (2.60)
Again, if we choose $z_0$ such that $|z_0| = r$ and $|f(z_0)| = M(r, f)$ then we obtain a lower bound for $M(\rho, f')$ and (2.59) becomes, for $|z| = \rho$ satisfying (2.42), using (2.56),
\[
M(\rho, f') \sim \frac{N}{\rho} \left( \frac{\rho}{r} \right)^N M(r, f) \sim \frac{N}{\rho} M(\rho, f).
\] (2.61)
It follows from (2.61) that the method may be extended to handle a finite number of higher derivatives as follows. Since $z_0$ satisfies (2.48), we obtain, using (2.60) and (2.61),
\[
|f'(z_0)| \geq (1 - o(1)) \left( \frac{N}{r} \right) 2 \lambda M(r, f) \geq (2 \lambda - o(1)) M(r, f').
\]
If $\tau < \gamma' < \gamma$ then, provided $r$ lies outside a set of finite logarithmic measure, we have $\nu(r, f') \sim N$ by (2.26) and
\[
\frac{f''(z)}{f'(z)} \sim \frac{\nu(r, f')}{z} \sim \frac{N}{z}
\]
for $|\log(z/z_0)| \leq \nu(r, f')^{-\gamma'}$ and hence for $|\log(z/z_0)| \leq N^{-\gamma}$. Similarly, for these $r$ and for $\rho$ satisfying (2.42) we get $M(\rho, f'') \sim (N/\rho) M(\rho, f')$, and the whole process may be repeated a finite number of times.
Thus we have proved:
2.2.20 The main theorem of the Wiman-Valiron theory

Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be a transcendental entire function, and let \( 1/2 < \gamma < 1 \) and \( 0 < \kappa \leq 1 \). Let \( q \) be a positive integer. Then there exists a set \( E_2 \subset [1, \infty) \), of finite logarithmic measure, such that, if \( |z_0| = r \in [1, \infty) \setminus E_2 \) and \( |f(z_0)| \geq \kappa M(r, f) \) then

\[
\frac{f(z)}{f(z_0)} \sim \left( \frac{z}{z_0} \right)^N \quad \text{for} \quad |\log(z/z_0)| \leq N^{-\gamma}
\]

and \( j = 1, \ldots, q \), where \( N = \nu(r, f) \). Furthermore, for \( |\log(\rho/r)| \leq N^{-\gamma} \) we have

\[
M(\rho, f^{(j)}) \sim \frac{N^j}{\rho^j} M(\rho, f), \quad M(\rho, f) \sim \left( \frac{\rho}{r} \right)^N M(r, f)
\]

for \( j = 1, \ldots, q \).

The condition on \( \gamma \) is essentially best-possible. The Weierstrass \( \sigma \)-function has zeros at the points \( m + n\omega \), where \( \omega \) is a fixed non-real complex number and \( m \) and \( n \) are any integers. This function has order 2, and therefore so has \( N(r) \). Now on the region \( |\log(z/z_0)| \leq N^{-\gamma} \) we may write

\[
z = z_0 e^\zeta, \quad |\zeta| \leq N^{-\gamma}, \quad |z - z_0| = |z_0||e^\zeta - 1| \sim |z_0||\zeta|,
\]

and so this region has diameter roughly \( rN(r)^{-\gamma} \). If it were possible to take \( \gamma < 1/2 \) then this diameter would be large, and our Wiman-Valiron region would contain a disc of centre \( z_0 \) and large radius compared to \( 1 + |\omega| \). But such a disc must contain a zero of the \( \sigma \)-function.

2.3 Exercises

1. Let \( f \) be a transcendental entire function. Prove that

\[
\max \{ \Re f(z) : |z| = r \} \sim M(r, f)
\]

as \( r \to \infty \) outside a set of finite logarithmic measure.

2. Prove that every non-constant solution of

\[
y^{(4)} + zy' - z^4 y = 0
\]

has order 2 (every solution is entire: see the chapter on differential equations).

3. Let \( P \) and \( Q \) be non-constant polynomials. Prove that the differential equation

\[
2yy'' - (y')^2 + P(z)y + Q(z) = 0
\]

has no transcendental entire solutions.

2.4 Coefficients and the order of growth

Let \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) be a transcendental entire function. We may then prove that the order of \( g \) is

\[
\rho = \frac{1}{\sigma}, \quad \text{where} \quad \sigma = \liminf_{n \to \infty} \frac{-\log|b_n|}{n \log n},
\]
with the convention that $1/0 = \infty$.

(i) To prove that $\rho \leq 1/\sigma$ assume WLOG that $\sigma > \beta > 0$. Hence

$$-\log |b_n| \geq \beta n \log n, \quad |b_n| \leq n^{-\beta n},$$

for all sufficiently large $n$.

Let $r$ be large. Evidently if $n^\beta \geq r$ then $n$ is large and

$$|b_n|r^n \leq n^{-\beta n}r^n \leq 1.$$

Hence

$$\mu(r) \leq \max_{n \leq r^{1/\beta}} |b_n|r^n.$$

But we can assume WLOG that $|b_n| \leq 1$ for all $n$ (why?) and so

$$\mu(r) \leq r^{1/\beta} = \exp(r^{1/\beta} \log r),$$

which gives $\rho \leq 1/\beta$. Fill in the details.

(ii) To prove that $\rho \geq 1/\sigma$ assume WLOG that $\rho < \tau < \infty$. Let $n$ be large and $r = n^{1/\tau}$. Then $r$ is large and

$$|b_n|r^n \leq \mu(r) \leq \exp(r^{1/\tau}) = e^n$$

which gives

$$\log |b_n| \leq n - n \log r = n - \frac{n \log n}{\tau} = -(1 + o(1)) \frac{n \log n}{\tau}$$

and so

$$\frac{-\log |b_n|}{n \log n} \geq \frac{1}{\tau} - o(1)$$

as $n \to \infty$. Again fill in the details.
Chapter 3

Nevanlinna theory

3.1 Introduction

The standard reference for this is Hayman’s text [32], but this chapter will borrow several ideas from the excellent book by Jank and Volkmann [47].

A meromorphic function is one analytic function divided by another i.e. \( f = g/h \), where \( g \) and \( h \) are analytic, and \( h \neq 0 \). A good example is \( f(z) = \tan z \), which has poles (i.e. \( f(z) = \infty \)) wherever \( \cos z = 0 \).

The multiplicity (or order) is defined as follows. Suppose \( g \) is analytic at \( a \), with \( g(a) = 0 \). If \( g \neq 0 \), then the Taylor series of \( g \) about \( a \) has a first non-zero coefficient, say

\[
g(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \ldots, \quad a_j = \frac{g^{(j)}(a)}{j!}, \quad a_m \neq 0.
\]

We say that \( g \) has a zero of multiplicity \( m \) at \( a \). If \( g(a) \neq 0 \), we can think of this as a zero of multiplicity 0. Now consider \( g/h \). If

\[
g(z) = a_m(z-a)^m + \ldots, \quad h(z) = b_n(z-a)^n + \ldots,
\]
as \( z \to a \), with \( a_m b_n \neq 0 \), then

\[
f(z) = \frac{g(z)}{h(z)} = (z-a)^{m-n} \left( \frac{a_m + \ldots}{b_n + \ldots} \right) = (z-a)^{m-n}H(z), \quad H(a) = \frac{a_m}{b_n}
\]

near \( a \). Here \( H \) is analytic at \( a \). If \( m > n \) then \( f(a) = 0 \) (zero of multiplicity \( m - n \)). If \( m < n \) then \( f(a) = \infty \) (pole of multiplicity \( n - m \)).

Example: show that

\[
f(z) = \frac{z}{\sin^2 z}
\]
has a simple pole at 0 and double poles at \( z = k\pi, \ k \in \mathbb{Z} \setminus \{0\} \).

We have seen that the non-decreasing function \( \log^+ M(r, f) \) measures the growth of an entire function \( f \). The central idea of Nevanlinna theory is to develop an analogue for meromorphic functions, and to this end Nevanlinna introduced his characteristic function \( T(r, f) \).

3.2 Nevanlinna theory: the first steps

We begin with:
3.2.1 Poisson’s formula for the logarithm

Let $0 < R < \infty$ and let $E = \{ z \in \mathbb{C} : |z| \leq R \}$. Let $g$ be meromorphic on a domain containing $E$, with no zeros or poles in $D(0, R)$. Let the distinct zeros and poles of $g$ on the circle $S(0, R)$ be $\zeta_1, \ldots, \zeta_q$. Then an analytic branch $U$ of $\log g$ may be defined on a simply connected domain containing $E \setminus \{ \zeta_1, \ldots, \zeta_q \}$ and, for $|a| < R$,

$$U(a) = \frac{1}{2\pi} \int_0^{2\pi} U(Re^{i\phi}) \frac{R^2 - |a|^2}{|Re^{i\phi} - a|^2} d\phi. \tag{3.1}$$

**Proof.** The first assertion is true since there exists $R' > R$ such that $g$ is meromorphic in $D(0, R')$ with no zeros or poles in $R < |z| < R'$. Now let $|a| < R$. Let $\delta$ be small and positive and let $\Gamma_{\delta}$ be the circle $S(0, R)$ described once counter-clockwise, except that each $\zeta_j$ (if there are any) is avoided by instead describing clockwise an arc $\omega_j$ of the circle $S(\zeta_j, \delta)$. The resulting curve $\Gamma_{\delta}$ then goes once counter-clockwise around $a$, since $\delta$ is small. Set

$$V(w) = U(w) \left( \frac{R^2 - |a|^2}{R^2 - \bar{a}w} \right).$$

Then $V$ is analytic on and inside $\Gamma_{\delta}$ and so Cauchy’s integral formula gives

$$U(a) = V(a) = \frac{1}{2\pi i} \int_{\Gamma_{\delta}} \frac{V(w)}{w - a} dw = \frac{1}{2\pi} \int_{\Gamma_{\delta}} U(w) \left( \frac{w}{w - a} \right) \left( \frac{R^2 - |a|^2}{R^2 - \bar{a}w} \right) \frac{dw}{iw}. \tag{3.2}$$

But there exist non-zero constants $a_j$ and integers $m_j$ such that

$$g(w) \sim a_j (w - \zeta_j)^{m_j} \quad \text{and} \quad U(w) = \pm m_j \log \frac{1}{|w - \zeta_j|} + O(1) \quad \text{as} \quad w \to \zeta_j, \ w \in D(0, R).$$

In particular the argument of $g(w)$ remains bounded as $w \to \zeta_j$ in $D(0, R)$, and

$$U(w) = O \left( \log \frac{1}{\delta} \right)$$

on $\omega_j$, for small $\delta$. Hence the contribution to the integral in (3.2) from each circular arc $\omega_j$ tends to 0 as $\delta \to 0$, so that writing $w = Re^{i\phi}$ gives

$$U(a) = \frac{1}{2\pi} \int_{S(0, R)} U(w) \left( \frac{w}{w - a} \right) \left( \frac{R^2 - |a|^2}{R^2 - \bar{a}w} \right) \frac{dw}{iw} = \frac{1}{2\pi} \int_0^{2\pi} U(w) \left( \frac{1}{w - a} \right) \left( \frac{R^2 - |a|^2}{\bar{w}w - \bar{a}} \right) d\phi = \frac{1}{2\pi} \int_0^{2\pi} U(w) \left( \frac{1}{w - a} \right) \left( \frac{R^2 - |a|^2}{\bar{w}w - \bar{a}} \right) d\phi,$$

and (3.1) follows.
3.2.2 The Poisson-Jensen formula

Let $R$ be finite and positive and let $f$ be meromorphic and not identically zero in $|z| \leq R$. Let the zeros and poles of $f$ in $0 < |z| < R$ be $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ respectively, in each case with repetition according to multiplicity. Assume that near the origin $f(z)$ is given by

$$f(z) = cz^d(1 + o(1)) \quad \text{as} \quad z \to 0,$$

with $d$ an integer and $c$ a non-zero constant: this says that $cz^d$ is the first term of the Laurent series of $f$ valid in some annulus $0 < |z| < s_0$. Then

$$g(z) = f(z) \frac{R^d}{z^d} \prod_{j=1}^{m} \left( \frac{R - a_j}{R^2 - a_j z} \right)^{-1} \prod_{k=1}^{n} \left( \frac{R - b_k}{R^2 - b_k z} \right) \quad (3.3)$$

is meromorphic on $|z| \leq R$, and analytic and non-zero in $|z| < R$. Moreover, $|g(z)| = |f(z)|$ on $|z| = R$. Taking real parts in the Poisson formula 3.2.1 gives, for $u = \log |g|$ and $z = re^{i\theta}$ with $\theta$ real and $0 \leq r < R$,

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} u(Re^{i\phi}) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\theta - \phi)} d\phi. \quad (3.4)$$

But for $|w| = R$ we have $u(w) = \log |g(w)| = \log |f(w)|$, and using (3.3) this gives the Poisson-Jensen formula: if $z = re^{i\theta}, |z| < R$ and $f(z) \neq 0, \infty$ then

$$\log |f(z)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\phi})| \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\theta - \phi)} d\phi + d \log |z/R| +$$

$$+ \sum_{j=1}^{m} \log \left| \frac{R(z - a_j)}{R^2 - a_j z} \right| - \sum_{k=1}^{n} \log \left| \frac{R(z - b_k)}{R^2 - b_k z} \right|. \quad (3.5)$$

Here the $a_j$ and $b_k$ are the zeros and poles of $f$ in $0 < |z| < R$. In particular, letting $z \to 0$ we have Jensen’s formula

$$\log |c| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\phi})| d\phi + \sum_{j=1}^{m} \log \frac{|a_j|}{R} - \sum_{k=1}^{n} \log \frac{|b_k|}{R} - d \log R. \quad (3.6)$$

Of course, $c = f(0)$ if $f(0) \neq 0, \infty$.

3.2.3 The Nevanlinna functionals

We retain the notation used in the Poisson-Jensen formula. Let $n(r) = n(r, f)$ denote the number of poles of $f$ in $|z| \leq r$, counting multiplicity, and let $\mu(t) = n(t) - n(0)$. Then, using Lemma 1.3.2 and the integration by parts formula (1.6) for Riemann-Stieltjes integrals we obtain

$$\sum_{k=1}^{n} \log \frac{R}{|b_k|} = \int_{0}^{R} \log \frac{R}{t} d\mu(t) = - \int_{0}^{R} (n(t) - n(0)) d \left( \log \frac{R}{t} \right) = \int_{0}^{R} (n(t) - n(0)) \frac{dt}{t}. \quad (3.7)$$

Here the first formula follows by writing the sum as a Riemann-Stieltjes integral as in Lemma 1.3.2. Alternatively, we can prove by elementary means that

$$\sum_{k=1}^{n} \log \frac{R}{|b_k|} = \int_{0}^{R} (n(t) - n(0)) \frac{dt}{t}. \quad (3.8)$$
Indeed, if \( f \) has \( p \) poles on \(|z| = \rho\) these contribute \( p \) to \( n(t) - n(0) \) for \( \rho \leq t \leq R \) and so \( p \log R/\rho \) to the integral and this gives us (3.8).

Now write
\[
N(R, f) = \int_0^R (n(t, f) - n(0, f)) \frac{dt}{t} + n(0, f) \log R
\]
(3.9)
and
\[
m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(Re^{i\phi})| d\phi,
\]
(3.10)
where \( \log^+ x \) is defined by (1.4) and satisfies
\[
\log x = \log^+ x - \log^+ \frac{1}{x}, \quad x > 0.
\]
(3.11)
Using (3.7), (3.9), (3.10) and (3.11), Jensen’s formula (3.6) becomes
\[
\log |c| = m(R, f) + N(R, f) - m(R, 1/f) - N(R, 1/f).
\]
(3.12)
Here \( m(R, f) \) is called the proximity function (Schmiegungsfunktion) and \( N(R, f) \) the (integrated) counting function (Anzahlfunktion). The Nevanlinna characteristic is
\[
T(R, f) = m(R, f) + N(R, f),
\]
(3.13)
and the Jensen formula (3.12) can now be written
\[
\log |c| = T(R, f) - T(R, 1/f).
\]
(3.14)

3.2.4 Examples

(i) Let \( F(z) = P(z)/Q(z) \) be a rational function, in which \( P \) and \( Q \) are polynomials, of degrees \( p, q \) respectively, and with \( Q \neq 0 \). We can assume that \( P \) and \( Q \) have no common zeros. Then \( Q(z) = 0 \) has \( q \) roots, counting multiplicities, and so
\[
N(r, F) = q \log r + O(1)
\]
for large \( r \). Also, as \( z \to \infty \) we have \( F(z) = dz^{p-q}(1 + o(1)) \) for some constant \( d \neq 0 \), and so
\[
\log |F(z)| = (p - q) \log |z| + O(1), \quad z \to \infty,
\]
from which
\[
m(r, F) = \max \{(p - q), 0\} \log r + O(1), \quad r \to \infty.
\]
This gives
\[
T(r, F) = \max \{p, q\} \log r + O(1), \quad r \to \infty.
\]

(ii) Let \( f(z) = e^z \). Show that \( T(r, f) = m(r, f) = r/\pi \) for \( r > 0 \).

(iii) Show that
\[
T(r, \cos z) \leq T(r, e^{iz}) + T(r, e^{-iz}) + O(1) = 2r/\pi + O(1).
\]
By considering the range \(|\text{Im } z| \geq 100\), show that \( T(r, \cos z) \geq 2r/\pi - O(1) \). Estimate \( m(r, \sec z) \) and \( N(r, \sec z) \).
(iv) Let $f$ be meromorphic in the plane and, with $k$ a positive integer, define $g(z) = f(z^k)$. Prove that

$$n(r,g) = kn(r^k,f), \quad N(r,g) = N(r^k,f), \quad m(r,g) = m(r^k,f), \quad T(r,g) = T(r^k,f).$$

Show also that $T(r,f^k) = kT(r,f)$ and that, if $a$ is a non-zero constant and $f(0) \neq \infty$, then $T(r,f(az)) = T(|a|r,f)$.

(v) Show that if $P(z) = az^k + \ldots$ is a polynomial of degree $k$ then $T(r, e^{P}) \sim |a| r^k/\pi$ as $r \to \infty$ (hint: consider first the case $P(z) = z^k$).

### 3.2.5 Properties of the characteristic

Suppose that $f, f_1, f_2$ are meromorphic and non-constant. Then

$$T(R, f_1 f_2) \leq T(R, f_1) + T(R, f_2), \quad T(R, f_1 + f_2) \leq T(R, f_1) + T(R, f_2) + \log 2. \quad (3.15)$$

These follow easily from the inequalities

$$\log^+ xy \leq \log^+ x + \log^+ y, \quad \log^+ (x+y) \leq \log^+ (2 \max\{x, y\}) \leq \log^+ x + \log^+ y + \log 2, \quad x, y > 0,$n

and the fact that a pole of $f_1 f_2$ or $f_1 + f_2$ can only arise at a pole of $f_1$ or $f_2$, and has multiplicity not greater than the sum of the multiplicities for $f_1$ and $f_2$.

### 3.2.6 Comparing $T(r, f)$ and $\log M(r, f)$

Let $f$ be analytic in $|z| \leq R$. If $0 < r < R$ then

$$T(R, f) \leq \log^+ M(R, f), \quad \log M(r, f) \leq \left(\frac{R+r}{R-r}\right) T(R, f).$$

The first inequality is obvious, since $\log^+ |f(z)| \leq \log^+ M(R, f)$ on $|z| = R$. To prove the second, we take $z$ with $|z| = r$ and $|f(z)| = M(r, f)$, and we apply the Poisson-Jensen formula, using the fact that the contribution from the zeros of $f$ is non-positive, and the inequality

$$R^2 + r^2 - 2 R r \cos t \geq (R-r)^2, \quad R > r \geq 0, \quad t \in \mathbb{R}.$$

This relation shows that for entire functions $T(r, f)$ and $\log^+ M(r, f)$ are comparable.

### 3.2.7 A useful inequality

If $0 < r < R$ then

$$N(R, f) = \int_0^R \frac{(n(t, f) - n(0, f))}{t} dt + n(0, f) \log R$$

$$\geq \int_r^R \frac{(n(t, f) - n(0, f))}{t} dt + n(0, f) \log R$$

$$\geq \int_r^R \frac{(n(r, f) - n(0, f))}{t} dt + n(0, f) \log R$$

$$= (n(r, f) - n(0, f)) \log \frac{R}{r} + n(0, f) \log R$$

$$= n(r, f) \log \frac{R}{r} + n(0, f) \log r.$$
3.2.8 Lemma

Let $f$ be meromorphic in $\mathbb{C}$, and not a rational function. Then

$$\frac{T(r, f)}{\log r} \to \infty$$

as $r \to \infty$.

Proof. Note that we saw in Examples 3.2.4, (i) that if $f$ is a rational function then $T(r, f) = O(\log r)$ as $r \to \infty$.

Suppose then that $f$ is meromorphic and non-constant in the plane, and that $T(r_n, f) = O(\log r_n)$ through some sequence $r_n \to \infty$. Now the inequality 3.2.7 gives, with $r^2 = r_n$ and $r_n$ large,

$$C \log r > T(r^2, f) \geq N(r^2, f) \geq n(r, f) \log r$$

so $f$ has finitely many poles. Hence there exists a polynomial $P$ such that $g = P f$ is entire, and $T(r_n, g) = O(\log r_n)$. Hence §3.2.6 gives

$$\log M(s_n, g) \leq 3T(2s_n, g) \leq C_1 \log r_n, \quad s_n = r_n/2,$$

so there exists an integer $M > 0$ such that $|g(z)| \leq (s_n)^M$ on the circles $|z| = s_n \to \infty$. Thus Cauchy’s integral formula shows us that $g^{(M)}$ is bounded and so constant, and $g$ is a polynomial.

3.2.9 An alternative proof of Jensen’s formula

Let the function $f$ be meromorphic in $|z| < R$ and for simplicity assume that $f(0) \neq 0, \infty$. For $0 \leq r < R$ set

$$I(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta.$$

Here $I(0) = \log |f(0)|$ and it is not hard to see that $I(r)$ is continuous. Now suppose that $f$ has neither zeros nor poles on the circle $|z| = s \in (0, R)$. Then setting $\tau = \log |z|$ and writing $\log f$ locally as a function of $\tau + i\theta$ gives

$$sI'(s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \log |f|}{\partial \tau} (se^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \arg f}{\partial \theta} (se^{i\theta}) \, d\theta = n(s, 1/f) - n(s, f).$$

Dividing by $s$ and integrating from 0 to $r$ then yields

$$m(r, f) - m(r, 1/f) = I(r) = I(0) + N(r, 1/f) - N(r, f) = \log |f(0)| + N(r, 1/f) - N(r, f).$$

3.3 Nevanlinna’s first fundamental theorem

3.3.1 First fundamental theorem

For non-constant meromorphic $f$ and $a \in \mathbb{C}$ we have

$$m(R, 1/(f - a)) + N(R, 1/(f - a)) = T(R, f) + O(1). \quad (3.16)$$

For if $a$ is a finite complex number we have by (3.15), as $R \to \infty$,

$$T(R, f - a) \leq T(R, f) + O(1), \quad T(R, f) \leq T(R, f - a) + O(1)$$
and so \( T(R, f) = T(R, f - a) + O(1) \).

This is an equidistribution theorem: if \( f \) is meromorphic and non-constant in \( \mathbb{C} \) then by Example 3.2.4 (i) and Lemma 3.2.8 the characteristic \( T(R, f) \) tends to infinity as \( R \) tends to infinity. Hence either \( f \) takes the value \( a \) very often (so that \( N \) is large) or \( f \) is close to \( a \) on part of the circle \( |z| = R \).

A good example is \( f(z) = e^z \). Then \( m(r, f) = m(r, 1/f) = r/\pi \), while \( N(r, 1/f) = N(r, f) = 0 \). Also \( m(r, 1/(f - 1)) \) is small, but \( f \) has a lot of 1-points.

For brevity we write
\[
m(r, e^z) = m(r, f) = m(r, \infty), \quad N(r, e^z) = N(r, f) = N(r, \infty).
\]

### 3.3.2 More examples

(i) Show that if \( T \) is a Möbius transformation and \( g = T(f) \) then
\[
T(r, g) = T(r, f) + O(1), \quad r \to \infty.
\]

Deduce that \( T(r, \tan z) = (2r/\pi) + O(1) \) (Hint: write \( \tan z \) in terms of \( e^{2iz} \)).

Illustrate the first fundamental theorem by looking at \( m(r, \tan z), N(r, \tan z), N(r, 1/\tan z) \).

(ii) Show that \( f(z) = e^{2z} - e^z \) has, as \( r \to \infty \),
\[
\begin{align*}
N(r, \infty) &= N(r, f) = 0, \\
m(r, \infty) &= m(r, f) \sim m(r, e^{2z}) = 2r/\pi, \\
N(r, 0) &= N(r, 0, e^z - 1) = r/\pi + O(1), \\
m(r, 0) &\sim m(r, e^{-z}) = r/\pi, \\
m(r, a) &= O(1), \quad N(r, a) = 2r/\pi + O(1), \quad (a \in \mathbb{C} \setminus \{0\}).
\end{align*}
\]

### 3.3.3 An application of the first fundamental theorem: a lemma of Clunie

Let \( f \) be transcendental meromorphic and let \( g \) be entire. Then
\[
T(r, g) = o(T(r, f \circ g)) \quad \text{as} \quad r \to \infty.
\]

**Proof.** Choose \( a \in \mathbb{C} \) such that \( a \) is not a critical value of \( h = f \circ g \) and \( f \) has infinitely many \( a \)-points \( w_1, w_2, \ldots \). Fix \( N \in \mathbb{N} \) and choose \( C, \delta > 0 \) such that
\[
|w - w_j| < \delta \quad \text{implies that} \quad |f(w) - a| \leq C|w - w_j| \quad (j = 1, \ldots, N).
\]

This gives
\[
\sum_{j=1}^{N} m(r, w_j, g) \leq m(r, a, h) + O(1)
\]
and
\[
\sum_{j=1}^{N} N(r, w_j, g) \leq N(r, a, h).
\]
Adding and applying the first fundamental theorem yields
\[ \sum_{j=1}^{N} T(r, w_j, g) \leq T(r, a, h) + O(1), \quad NT(r, g) \leq T(r, h) + O(1), \]
so that \( T(r, g) = o(T(r, f \circ g)) \).

### 3.4 Cartan’s formula and the growth of the characteristic function

#### 3.4.1 Cartan’s formula

We saw earlier that if \( f \) is entire then \( \log^+ M(r, f) \) is a non-decreasing function, and the aim of this section is to show that \( T(r, f) \) is non-decreasing.

Let \( f \) be non-constant and meromorphic in \(|z| < R\), with \( f(0) \) finite. Let \( r \in (0, R) \) and assume that the number of points on \(|z| = r\) at which \(|f(z)| = 1\) is finite (in particular, this will always be true unless \( f \) is a rational function: see Lemma 3.9.1). Now Jensen’s formula applied to the function \( a - z \) gives
\[
\frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{it}| ds = \log^+ |a|
\]
(3.18) for a complex number \( a \). Thus
\[
m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| dt = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(re^{it}) - e^{is}| ds dt.
\]
(3.19)

Let
\[
\phi(s, t) = \log |f(re^{it}) - e^{is}|, \quad \phi^+(s, t) = \max\{\phi(s, t), 0\}, \quad \phi^-(s, t) = \max\{-\phi(s, t), 0\}.
\]
Then \( \phi = \phi^+ - \phi^- \). Also, the Fubini-Tonelli theorem gives
\[
I_1 = \int_0^{2\pi} \int_0^{2\pi} \phi^+(s, t) ds dt = \int_0^{2\pi} \int_0^{2\pi} \phi^+(s, t) dt ds
\]
and
\[
I_2 = \int_0^{2\pi} \int_0^{2\pi} \phi^-(s, t) ds dt = \int_0^{2\pi} \int_0^{2\pi} \phi^-(s, t) dt ds.
\]
But, by (3.15),
\[
0 \leq I_1 \leq \int_0^{2\pi} 2\pi (\log^+ |f(re^{it})| + \log 2) dt \leq 4\pi^2 (m(r, f) + \log 2).
\]
Thus \( I_1 \) is finite. Also Jensen’s formula gives, since \( m(r, g) \leq T(r, g) \) and \( f(0) - e^{is} \neq 0 \) for almost all \( s \),
\[
I_2 \leq 2\pi \int_0^{2\pi} m\left(r, \frac{1}{f - e^{is}}\right) ds
\]
\[
\leq \int_0^{2\pi} 2\pi (T(r, f - e^{is}) - \log |f(0) - e^{is}|) ds
\]
\[
\leq 4\pi^2 T(r, f) + 4\pi^2 \log 2 - 4\pi^2 \log^+ |f(0)| < \infty,
\]
using (3.15) and (3.18) again. Thus (3.19) and Jensen’s formula give

\[
m(r, f) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi^+(s, t) - \phi^-(s, t) ds dt
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log |f(re^{it}) - e^{is}| dt ds
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{is}) - N(r, f) + \log |f(0) - e^{is}| ds
\]

and so

\[
m(r, f) = \log^+ |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{is}) ds - N(r, f).
\]  \hspace{1cm} (3.20)

We thus obtain Cartan’s formula: for \( f(0) \) finite we have

\[
T(r, f) = \log^+ |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} N(r, e^{is}) ds,
\]  \hspace{1cm} (3.21)

for \( r \) in \((0, R)\). To obtain an analogue of (3.21) when \( f(0) = \infty \) we just apply (3.20) to \( 1/f \).

We proceed to differentiate (3.21). Let \( r \) be such that the equation \( |f(z)| = 1 \) has finitely many solutions \( z \) on \( |z| = \rho \), for all \( \rho \) close to \( r \) (this is true for all but at most one \( r \) in \((0, R)\)). Let \( 0 < s < r \). Then there exists a constant \( C_1 \) such that, for all \( \rho \) close to \( r \) we have

\[
T(\rho, f) = C_1 + \frac{1}{2\pi} \int_0^{2\pi} N(\rho, e^{it}) - N(s, e^{it}) dt
\]

and so

\[
T(\rho, f) = C_1 + \frac{1}{2\pi} \int_0^{2\pi} \int_\rho^s n(r, e^{it}) \frac{dr}{s} dt.
\]

Since the integrand is non-negative we can reverse the order of integration to get

\[
T(r, f) = C_1 + \frac{1}{2\pi} \int_s^\rho \int_0^{2\pi} n(r, e^{it}) dt \frac{dr}{s}.
\]

But we saw above that

\[
\frac{1}{2\pi} \int_0^{2\pi} n(s, e^{it}) dt
\]

is continuous at \( r \), and so

\[
\frac{dT}{dr} = \frac{1}{2\pi} \int_0^{2\pi} n(r, e^{it}) dt,
\]

which is the differentiated Cartan formula.

In particular \( T(r, f) \) is an increasing convex function of \( \log r \) i.e.

\[
P(s) = T(e^s, f)
\]

satisfies

\[
P(a) \leq P(s) \leq P(a) \frac{(b - s)}{(b - a)} + P(b) \frac{(s - a)}{(b - a)}, \quad a < s < b.
\]

This is because \( P'(t) \) is non-decreasing, so that

\[
\frac{P(s) - P(a)}{s - a} = \frac{1}{(s - a)} \int_a^s P'(t) dt \leq \frac{1}{(b - s)} \int_s^b P'(t) dt = \frac{P(b) - P(s)}{b - s}.
\]
3.4.2 The order of a meromorphic function

If \( f \) is meromorphic on \( \mathbb{C} \) we define the order \( \rho(f) \) and lower order \( \mu(f) \) by

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \mu(f) = \liminf_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.
\]

We now have two definitions for the order of growth of an entire function \( h \). However, since §3.2.6 gives

\[
T(r, h) \leq \log^+ M(r, h) \leq 3T(2r, h),
\]

Lemma 1.2.4 tells us that both give the same value \( \rho \).

3.5 The logarithmic derivative

The key to Nevanlinna’s methods is an estimate for \( m(r, f'/f) \) when \( f \) is meromorphic. This leads to the second fundamental theorem, which is a strong generalization of Picard’s theorem, and to a host of further results. The treatment here will follow the approach of Jank and Volkmann [47].

The Poisson formula (3.4) may be differentiated to give a formula for the derivative \( g'/g \) of \( \log g \). Here we write \( u(z) = \log |g(z)| \) as the real part of

\[
I(z) = \left( \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\phi}) Re^{i\phi} + z \frac{d\phi}{Re^{i\phi} - z} \right).
\]

Hence \( \log g - I \) is constant on \( |z| < R \). Writing \( f'/f \) in terms of \( g'/g \) and using the fact that \( |f| = |g| \) on \( |z| = R \) we obtain, for \( |z| = r < R \),

\[
\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{j=1}^{m} \left( \frac{\alpha_j}{R^2 - \alpha_j z} + \frac{1}{z - a_j} \right) - \sum_{k=1}^{n} \left( \frac{b_k}{R^2 - b_k z} + \frac{1}{z - b_k} \right) + \frac{d}{z},
\]

and so

\[
\frac{f'(z)}{f(z)} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \frac{2Re^{i\phi}}{(Re^{i\phi} - z)^2} d\phi + \sum_{j=1}^{m} \left( \frac{\alpha_j}{R^2 - \alpha_j z} + \frac{1}{z - a_j} \right) - \sum_{k=1}^{n} \left( \frac{b_k}{R^2 - b_k z} + \frac{1}{z - b_k} \right) + \frac{d}{z}. \tag{3.22}
\]

Now for \( |z| = r < R \) and \( |A| \leq R \) we have

\[
\frac{1}{z - A} + \frac{\overline{A}}{R^2 - \overline{A}z} = \frac{1}{z - A} \left( 1 + \frac{\overline{A}(z - A)}{R^2 - \overline{A}z} \right). \tag{3.23}
\]

Since \( |A| \leq R \) and since

\[
w = \frac{R(z - A)}{R^2 - \overline{A}z}
\]

has modulus 1 when \( |z| = R \), the term in parentheses in (3.23) has modulus at most 2. Using

\[
|\log x| = \log^+ x + \log^+ \frac{1}{x}
\]

we now get, for \( |z| = r < R \),

\[
\left| \frac{f'(z)}{f(z)} \right| \leq (m(R, f) + m(R, 1/f)) \frac{2R}{(R - r)^2} + 2 \sum \left( \frac{1}{|z - A|} \right) + \frac{|d|}{r}, \tag{3.24}
\]

with the sum over all zeros and poles \( A \) of \( f \) in \( 0 < |\zeta| < R \), repeated according to multiplicity.

This formula can be used to give pointwise estimates for \( f'/f \) (see §3.7). We will show that it leads to a very strong estimate for \( m(r, f'/f) \).
3.5.1 Estimates for the proximity function of a logarithmic derivative

Let \( f \) be non-constant and meromorphic in \( |z| \leq R \), and let \( 0 < r < R \), such that \( f \) has no zeros or poles on \( |z| = r \). Set \( S = (R + r)/2 \). Assume for now that \( f(0) \neq 0, \infty \), and replace \( R \) by \( S \) in (3.24), to give

\[
1 + \left| \frac{f'(z)}{f(z)} \right| \leq 1 + \left[ \frac{2S}{(S-r)^2} (m(S, f) + m(S, 1/f)) \right] + \sum \left( \frac{1}{|z-A|} \right) + \sum \left( \frac{1}{|z-A|} \right).
\]

Here each sum is over all zeros and poles \( A \) of \( f \) in \( 0 < |\zeta| < S \), repeated according to multiplicity. Using the formula

\[
\left( \sum_{k=1}^{n} x_k \right)^{1/2} \leq \sum_{k=1}^{n} x_k^{1/2}, \quad x_k \geq 0,
\]

which is proved simply by squaring both sides, then yields

\[
\left( 1 + \left| \frac{f'(z)}{f(z)} \right| \right)^{1/2} \leq I(z, S) = 1 + \left[ \frac{2S}{(S-r)^2} (m(S, f) + m(S, 1/f)) \right]^{1/2} + 2 \sum \frac{1}{|z-A|^{1/2}}. \tag{3.25}
\]

But, in view of the fact that \( \log^+ x \leq \log(1+x) \) for \( x \geq 0 \), (3.25) gives

\[
m(r, f'/f) \leq \frac{1}{2\pi} \int_0^{2\pi} \log \left( 1 + \left| \frac{f'(re^{it})}{f(re^{it})} \right| \right) \, dt \leq \frac{2}{2\pi} \int_0^{2\pi} \log I(re^{it}, S) \, dt. \tag{3.26}
\]

Now Lemma 1.3.4 and (3.26) lead to

\[
m(r, f'/f) \leq 2 \log X, \quad X = \frac{1}{2\pi} \int_0^{2\pi} I(re^{it}, S) \, dt. \tag{3.27}
\]

Recalling (3.25) delivers next

\[
X \leq 1 + \left[ \frac{2S}{(S-r)^2} \left( 2T(S, f) + \log^+ \frac{1}{|f(0)|} \right) \right]^{1/2} + 2 \sum I_A, \quad I_A = \frac{1}{2\pi} \int_0^{2\pi} |re^{it} - A|^{-1/2} \, dt. \tag{3.28}
\]

To estimate \( I_A \), we write

\[
I_A = r^{-1/2} J_D, \quad J_D = \frac{1}{2\pi} \int_0^{2\pi} |e^{it} - D|^{-1/2} \, dt, \quad D = A/r. \tag{3.29}
\]

To obtain an upper bound for \( J_D \), there is no loss of generality in assuming that \( D \) is real and positive. Thus

\[
J_D = \frac{1}{2\pi} \int_0^{2\pi} (1 + D^2 - 2D \cos t)^{-1/4} \, dt = (1 + D^2)^{-1/4} \frac{1}{2\pi} \int_0^{2\pi} (1 - u \cos t)^{-1/4} \, dt,
\]

in which

\[
u = \frac{2D}{1 + D^2} \leq 1.
\]

This gives

\[
J_D \leq \frac{1}{2\pi} \int_0^{2\pi} (1 - u \cos t)^{-1/4} \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} (1 - |\cos t|)^{-1/4} \, dt = \gamma,
\]

in which \( \gamma \) is some fixed positive number, independent of \( r, R \) and \( f \).
Thus (3.28) and (3.29) combine to deliver
\[ X \leq 1 + \left[ \frac{2S}{(S-r)^2} \left( 2T(S, f) + \log^+ \frac{1}{|f(0)|} \right) \right]^{1/2} + 2r^{-1/2} \gamma(n(S, f) + n(S, 1/f)). \] (3.30)

But the inequality from §3.2.7 gives
\[ N(R, f) \geq n(S, f) \log \frac{R}{S}. \]
Since
\[ \log \frac{R}{S} = \log \left( 1 + \frac{R-S}{S} \right) \geq \frac{(R-S)/S}{1+(R-S)/S} = \frac{R-S}{R}, \]
we get
\[ n(S, f) + n(S, 1/f) \leq \frac{R}{R-S} (2T(R, f) + \log^+ |1/f(0)|). \]
Thus (3.27) and (3.30) and the inequality
\[ \log^+ \sum_{k=1}^{n} x_k \leq \log^+(n \max\{x_k\}) \leq \log n + \log^+ (\max\{x_k\}) \leq \log n + \sum_{k=1}^{n} \log^+ x_k, \quad x_k > 0, \]
imply that there are positive absolute constants \( C_j \) such that
\[
m(r, f'/f) \leq C_1 + C_2 \log^+ T(R, f) + C_3 \log^+ \log^+ \frac{1}{|f(0)|} + C_4 \log^+ R + C_5 \log^+ \frac{1}{r} + C_6 \log^+ \frac{1}{R-r}. \] (3.31)

An analogous formula when \( f(0) = 0, \infty \) is easy to obtain. If \( f(z) = cz^d(1 + o(1)) \) as \( z \to 0 \), we write \( f(z) = cz^d h(z) \) so that \( h(0) = 1 \). Now we need only use the fact that
\[
\frac{f'(z)}{f(z)} = \frac{d}{z} + \frac{h'(z)}{h(z)}, \quad \left| \frac{f'(z)}{f(z)} \right| \leq \left| \frac{h'(z)}{h(z)} \right| + \left| \frac{d}{z} \right|
\]
and
\[ T(r, h) \leq T(r, f) + T(r, 1/cz^d) \leq T(r, f) + d \log r + O(1). \]

3.5.2 The lemma of the logarithmic derivative

Let \( f \) be non-constant and meromorphic in the plane. Then there are positive constants \( C_j \) such that we have
\[ m(r, f'/f) \leq C_1 \log r + C_2 \log T(r, f) \] (3.32)
as \( r \) tends to \( \infty \) outside a set of finite measure.

To prove this, choose \( R = r + 1/T(r) \) in (3.31), and apply the Borel lemma 1.2.5.

Note that this estimate is only needed for transcendental \( f \). If \( f \) is a rational function then \( f'(z)/f(z) \to 0 \) as \( z \to \infty \) so \( m(r, f'/f) = 0 \) for large \( r \).

If \( f \) has finite order we have \( m(r, f'/f) = O(\log r T(2r, f)) = O(\log r) \) with no exceptional set (just take \( R = 2r \)).

We write \( S(r, f) \) for any term which is \( O(\log^+(r T(r, f))) \) outside some set \( E^* \) of finite measure. Note that if \( f \) is not a rational function then \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) with \( r \not\in E^* \).
3.6. THE SECOND FUNDAMENTAL THEOREM

3.5.3 Theorem

We have $T(r, f') \leq T(r, f) + \overline{N}(r, f) + S(r, f)$.

Here $\overline{N}(r, f)$ counts poles of $f$, but without regard to multiplicity. The proof is easy. We have

$$N(r, f') \leq N(r, f) + \overline{N}(r, f), \quad m(r, f') \leq m(r, f) + m(r, f'/f).$$

In particular, if $f$ has finite order, then

$$T(r, f') \leq 2T(r, f) + O(\log r). \quad (3.33)$$

3.5.4 Lemma

If $f$ is non-constant and meromorphic in the plane, then $\rho(f') \leq \rho(f)$.

If $\rho(f) = \infty$, this is obvious. If $\rho(f) < \infty$, then we just use Lemma 1.2.4 and (3.33). In fact, the two orders are the same, but it is much harder to prove that $\rho(f) \leq \rho(f')$.

3.6 The second fundamental theorem

Let $f$ be again non-constant and meromorphic in the plane, and let $a_1, \ldots, a_q$ be $q$ distinct finite complex numbers. Let

$$H = \sum_{j=1}^{q} \frac{1}{f - a_j}. \quad (3.34)$$

Take a small positive $\varepsilon$, so small that $|w - a_j| < \varepsilon$ implies that $|w - a_k| > \varepsilon$ for $j \neq k$. If $|f(z) - a_j| < \varepsilon$ we then have

$$\frac{1}{|f(z) - a_j|} \leq |H(z)| + \frac{q - 1}{\varepsilon}$$

and so

$$\log^+ \frac{1}{|f(z) - a_j|} \leq \log^+ |H(z)| + O(1),$$

while if $|f(z) - a_j| \geq \varepsilon$ then obviously

$$\log^+ \frac{1}{|f(z) - a_j|} \leq \log \frac{1}{\varepsilon}.$$ 

Since the sets $E_j = \{ z : |f(z) - a_j| < \varepsilon \}$ are pairwise disjoint it follows that

$$\sum_{j=1}^{q} m(r, a_j, f) \leq \sum_{j=1}^{q} \left[ \frac{1}{2\pi} \int_{[0,2\pi] \cap E_j} \log^+ \frac{1}{|f(re^{i\phi}) - a_j|} \, d\phi + \log \frac{1}{\varepsilon} \right]$$

$$\leq \sum_{j=1}^{q} \frac{1}{2\pi} \int_{[0,2\pi] \cap E_j} \log^+ |H(re^{i\phi})| \, d\phi + O(1)$$

$$\leq m(r, H) + O(1) = m(r, f'H/f') + O(1)$$

$$\leq m(r, f'H) + m(r, 1/f') + O(1)$$

$$\leq m(r, 1/f') + S(r, f),$$
since \( f' H \) is a sum of logarithmic derivatives and \( T(r, f - a_j) \leq T(r, f) + O(1) \). Here
\[
\begin{align*}
m(r, 1/f') &= T(r, 1/f') - N(r, 1/f') \\
&= T(r, f') - N(r, 1/f') + O(1) \quad \text{(by Jensen’s formula)} \\
&= m(r, f') + N(r, f') - N(r, 1/f') + O(1).
\end{align*}
\]

Moreover,
\[
m(r, f') = m(r, f \cdot f'/f) \leq m(r, f) + m(r, f'/f).
\]

Also, since \( \overline{N}(r, f) \) counts each pole exactly once, we have
\[
N(r, f') = N(r, f) + \overline{N}(r, f) = 2N(r, f) - [N(r, f) - \overline{N}(r, f)].
\]

Thus
\[
m(r, f) + \sum_{j=1}^{q} m(r, a_j, f) \leq m(r, f) + m(r, 1/f') + S(r, f)
\]
\[
\leq 2m(r, f) + N(r, f') - N(r, 1/f') + S(r, f)
\]
\[
\leq 2m(r, f) + 2N(r, f) - [N(r, f) - \overline{N}(r, f)] - N(r, 1/f') + S(r, f)
\]
\[
= 2T(r, f) + S(r, f) - N_1(r, f), \tag{3.35}
\]

in which
\[
N_1(r, f) = N(r, f) - \overline{N}(r, f) + N(r, 1/f') \geq 0.
\]

This term \( N_1(r, f) \) counts the multiple points of \( f \) in the following sense. The function \( f \) is one-one on some neighbourhood of \( z_0 \) if and only if either \( f(z_0) \) is finite and \( f'(z_0) \neq 0 \), or \( z_0 \) is a simple pole of \( f \). Indeed, if \( f(z) \) has an \( a \)-point (a finite or infinite) of multiplicity \( p \) at \( z_0 \) then by Rouché’s theorem all values \( w \) which are sufficiently close to \( a \) are taken \( p \) times near to \( z_0 \). Thus \( z_0 \) is a multiple point of order \( p - 1 \), and contributes \( p - 1 \) to \( n_1(r, f) \).

### 3.6.1 Statement of the second fundamental theorem

From (3.35) and the fact that \( m(r, f) \geq 0 \) we obtain the second fundamental theorem: given any \( s \) distinct values \( b_j \) in \( \mathbb{C}^\ast \) (one of them is allowed to be \( \infty \) here), we have
\[
\sum_{j=1}^{s} m(r, b_j, f) \leq 2T(r, f) - N_1(r, f) + S(r, f). \tag{3.36}
\]

Adding the terms \( N(r, b_j, f) \) to both sides of (3.36) we get, by the first fundamental theorem,
\[
(s - 2)T(r, f) \leq \sum_{j=1}^{s} N(r, b_j, f) - N_1(r, f) + S(r, f).
\]

But if \( f \) has a \( b_j \)-point at \( a \), of multiplicity \( p \), then \( a \) contributes \( p \) to \( n(r, b_j, f) \) and \( p - 1 \) to \( n_1(r, f) \). Thus we get
\[
(s - 2)T(r, f) \leq \sum_{j=1}^{s} \overline{N}(r, b_j, f) + S(r, f).
\]
3.6. THE SECOND FUNDAMENTAL THEOREM

Picard’s theorem is an immediate corollary. If \( f \) is transcendental and meromorphic in the plane and takes three values \( b_j \) each only finitely often, then \( N(r, b_j, f) = O(\log r) \) for these \( b_j \). Since \( S(r, f) = o(T(r, f)) \) as \( r \) tends to infinity outside a set \( E \) of finite measure, we deduce that \( T(r, f) = O(\log r) \) for \( r \) not in \( E \), a contradiction. This proves the “great” Picard theorem. It remains only to prove that if \( f \) omits three values then \( f \) is constant (this is the “little” theorem). However, this is easy: if \( f \) is a non-constant rational function \( f = P/Q \) with \( P, Q \) polynomials having no common zero, then \( Q = 0 \) gives \( f = \infty \), while the equation \( P(z) = bQ(z) \) has solutions in \( \mathbb{C} \), for all but at most one finite \( b \).

3.6.2 The defect relation

Nevanlinna defined, for \( a \in \mathbb{C}^* \), the deficiency

\[
\delta(a, f) = \lim_{r \to \infty} \frac{m(r, a, f)}{T(r, f)} = 1 - \lim_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}
\]

as a measure of the extent to which the value \( a \) is taken rarely. The equality in (3.37) follows from the first fundamental theorem. From (3.37), we have \( 0 \leq \delta(a, f) \leq 1 \). Also (3.36) gives the defect relation

\[
\sum_{a \in \mathbb{C}^*} \delta(a, f) \leq 2.
\]

3.6.3 Examples

(i) If \( a \) is an omitted value of \( f \) then \( \delta(a, f) = 1 \). Thus the defect relation (3.38) implies Picard’s theorem.

(ii) A meromorphic function \( f \) can take a value \( a \) infinitely often, but still have \( \delta(a, f) = 1 \). For example,

\[
f(z) = e^{z^2} \tan z
\]

has \( \delta(0, f) = \delta(\infty, f) = 1 \), since

\[
T(r, \tan z) = O(r), \quad T(r, e^{z^2}) = \frac{r^2}{\pi} \leq T(r, f) + T(r, \cot z).
\]

(iii) Determine the Nevanlinna deficiencies of \( e^{2z} - e^z \) (see Examples 3.3.2).

(iv) Here we give an example of an entire function \( f \) having two finite deficient values, each with deficiency \( \frac{1}{2} \), and so sum of all deficiencies equal to 2. Set

\[
f(z) = \int_0^z e^{-t^2} dt, \quad I = \int_0^\infty e^{-t^2} dt.
\]

Here the integral \( I \) is over \([0, \infty)\), and in fact equals \( \frac{1}{2} \sqrt{\pi} \), although all we require here is that \( I \neq 0 \). Suppose first that \(|\arg z| < \pi/4\). Then Cauchy’s theorem gives

\[
f(z) = I - \int_{\gamma_z} e^{-t^2} dt,
\]

in which \( \gamma_z \) follows the (shorter) circular arc from \( z \) to \( r = |z| \), followed by the straight line from \( r \) to infinity. On \( \gamma_z \) we have

\[
|e^{-t^2}| = e^{-|t|^2} \cos(2 \arg t) \leq |e^{-z^2}|.
\]
We write
\[ e^{-t^2} = \frac{2t e^{-t^2}}{2t} \]
and integrate by parts. This gives
\[ \int_{\gamma_z} e^{-t^2} dt = \frac{e^{-z^2}}{2z} - \int_{\gamma_z} \frac{e^{-t^2}}{2t^2} dt \]
and so, as \( r = |z| \to \infty \),
\[ \left| \int_{\gamma_z} e^{-t^2} dt \right| \leq |e^{-z^2}| \left( o(1) + \int_{\gamma_z} \frac{1}{2|t|^2} |dt| \right) \leq |e^{-z^2}|. \]
Thus
\[ m(r, 1/(f - I)) \geq \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} r^2 \cos 2\theta d\theta = \frac{r^2}{2\pi}. \]
Since Taylor’s theorem gives \( f(z) = -f(-z) \) we also have
\[ m(r, 1/(f + I)) \geq \frac{r^2}{2\pi}. \]
We now estimate \( T(r, f) = m(r, f) \). For \( |\arg z| \leq \pi/4 \) or \( |\arg(-z)| \leq \pi/4 \) we have \( f(z) = O(1) \). On the other hand if \( \pi/4 < |\arg z| < 3\pi/4 \) we have \( |e^{-t^2}| \leq |e^{-z^2}| \) on the straight line from 0 to \( z \), and so \( |f(z)| \leq |ze^{-z^2}| \). Thus
\[ T(r, f) \leq O(\log r) + \frac{1}{\pi} \int_{-\pi/4}^{3\pi/4} (-r^2 \cos 2\theta) d\theta = \frac{r^2}{\pi}. \]
Exercise: generalize this to \( g(z) = \int_0^z e^{-t^q} dt \), using the fact that \( g(z e^{2\pi i/q}) = e^{2\pi i/q} g(z) \).

### 3.7 Pointwise estimates for logarithmic derivatives

#### 3.7.1 Definition

By an \( R \)-set we mean a countable union \( U \) of discs \( D(z_j, r_j) \) such that \( z_j \to \infty \) as \( j \to \infty \) and \( \sum r_j < \infty \).

#### 3.7.2 Lemma

Let \( U \) be an \( R \)-set. Let \( E \) be the set of \( r > 0 \) for which the circle \( |z| = r \) meets at least one disc of \( U \), and let \( H \) be the set of \( \theta \in [0, 2\pi) \) such that the ray \( \arg z = \theta \) meets infinitely many discs of \( U \).

Then \( E \) has finite Lebesgue measure, and \( H \) has zero Lebesgue measure.

Proof. The first assertion is easy, since the set of \( r > 0 \) for which the circle \( |z| = r \) meets \( D(z_j, r_j) \) has measure at most \( 2r_j \).

Now suppose that \( j_0 \) is large, and \( j \geq j_0 \). Then \( z_j \) is large, and \( r_j \) is small, and the disc \( D(z_j, r_j) \) subtends at the origin an angle at most \( cr_j/|z_j| \), with \( c \) a positive constant independent of \( j \) and \( j_0 \). So the measure of \( H \) is at most
\[ c \sum_{j=j_0}^{\infty} r_j/|z_j| \to 0 \quad \text{as} \quad j_0 \to \infty. \]
3.7. POINTWISE ESTIMATES FOR LOGARITHMIC DERIVATIVES

3.7.3 Lemma

Let \( f \) be a transcendental meromorphic function of order \( \rho < L < M < \infty \). Let \( z_j \) be the zeros and poles of \( f \) in \( |z| > 2 \), repeated according to multiplicity. Then the union \( U \) of the discs \( D(z_j, |z_j|^{-M}) \) is an \( R \)-set, and

\[
|f'(z)/f(z)| = o(|z|^{L+M})
\]

for all \( z \) with \( |z| \) large and \( z \notin U \). Also,

\[
\sum_{|z_j| \geq r/2} |z_j|^{-M} = o(r^{L-M}) \quad (3.39)
\]

as \( r \to \infty \).

Note that if \( f \) is a rational function, not identically zero, then \( f'(z)/f(z) = O(1/|z|) \) as \( z \to \infty \).

Proof. Let \( m(t) \) be the number of \( z_j \) in \( |z| \leq t \). Then \( m(t) \leq n(t, f) + n(t, 1/f) \). For large \( t \) we have

\[
T(t, f) = o(t^L)
\]

and so

\[
N(2t, f) + N(2t, 1/f) = o(t^L)
\]

for large \( t \). Lemma 3.2.7 now gives

\[
m(t) \leq n(t, f) + n(t, 1/f) \leq o(t^L) \quad (3.40)
\]

for \( t \) large.

We prove (3.39) first, which will then show that \( U \) is an \( R \)-set. For large \( r \) and \( R > r \) we set \( s = r/4 \) and we have

\[
\sum_{r/2 \leq |z_j| \leq R} |z_j|^{-M} \leq \int_s^{2R} t^{-M} d(m(t) - m(s)) = (m(2R) - m(s))(2R)^{-M} + M \int_s^{2R} (m(t) - m(s))t^{-M-1}dt,
\]

using integration by parts. Using (3.40) this gives

\[
\sum_{r/2 \leq |z_j| \leq R} |z_j|^{-M} \leq m(2R)(2R)^{-M} + M \int_s^{2R} m(t)t^{-M-1}dt \leq o(R^{L-M}) + M \int_s^{2R} o(t^{L-M-1})dt.
\]

Letting \( R \to \infty \) we get

\[
\sum_{r/2 \leq |z_j|} |z_j|^{-M} \leq M \int_s^{\infty} o(t^{L-M-1})dt = o(s^{L-M}) = o(r^{L-M}),
\]

which proves (3.39).

To estimate \( f'/f \), take \( z \notin U \) with \( |z| = r \) large, and use (3.24), with \( R = 2r \). Since

\[
m(2r, f) + m(2r, 1/f) \leq 2T(2r, f) + O(1)
\]

we get

\[
|f'(z)/f(z)| \leq o(r^L) + 2 \sum \frac{1}{|z - A|}
\]

with the sum over all zeros and poles \( A \) of \( f \) in \( 0 < |\zeta| < 2r \). Now if \( |A| < r/2 \) then \( |z - A| > r/2 \). On the other hand, if \( r/2 \leq |A| < 2r \) then \( A \) is one of the \( z_j \) and so \( |z - A| \geq |A|^{-M} \geq (2r)^{-M} \). Hence

\[
|f'(z)/f(z)| \leq o(r^L) + (n(2r, f) + n(2r, 1/f))(2r)^M = o(r^{L+M}).
\]
3.7.4 Lemma

Let $f$ be a transcendental meromorphic function of finite order $\rho < L < M$, and let $n$ be a positive integer. Then we can find an $R$-set $U$ of discs $D(z_j, |z_j|^{-M})$, such that for $|z|$ large and $z \notin U$ we have

$$|f^{(m+1)}(z)/f^{(m)}(z)| = o(|z|^{L+M})$$

(3.41)

for $0 \leq m \leq n - 1$.

Proof. By Lemma 3.5.4, each derivative $f^{(m)}$ has order at most $\rho$. So for each $m$ we form an $R$-set $U_j$ of discs $D(z_j, |z_j|^{-M})$ such that for $|z|$ large and $z \in U_j$ we have (3.41). Now just note that the union $U$ of these finitely many $R$-sets is an $R$-set.

By writing $f''/f = (f''/f')(f'/f)$ etc., we also have

$$|f^{(m)}(z)/f(z)| = o(|z|^{n(L+M)})$$

for $|z|$ large, with $z$ not in $U$.

3.8 Product representations

Taylor’s theorem tells us that an entire function $f$ has a power series representation $f(z) = \sum_{n=0}^{\infty} c_n z^n$: here we show that functions meromorphic in $\mathbb{C}$ can be represented as products.

3.8.1 The exponent of convergence

Let $(a_n)$ be a sequence of non-zero complex numbers, tending to infinity. For $r > 0$ let $n(r)$ be the number of $a_n$ in $|z| \leq r$, and set

$$N(r) = \int_0^r n(t) \frac{dt}{t}.$$ 

The exponent of convergence of the sequence $(a_n)$ is then defined as

$$\lambda = \limsup_{r \to \infty} \frac{\log N(r)}{\log r} = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}. \quad (3.42)$$

The equality in (3.42) follows easily from Lemma 1.2.4 and the inequalities, for large $r$,

$$N(r) \leq n(r) \log r + O(1), \quad N(2r) \geq \int_r^{2r} n(t) \frac{dt}{t} \geq n(r) \log 2.$$ 

If $q > 0$ then, assuming without loss of generality that all the $a_n$ are non-zero,

$$\sum_{|a_n| \leq r} |a_n|^{-q} = \int_0^r t^{-q} dn(t) = n(r)r^{-q} + q \int_0^r n(t)t^{-q-1}dt. \quad (3.43)$$

3.8.2 Lemma

The exponent of convergence $\lambda$ is the infimum of $q > 0$ such that $\sum |a_n|^{-q}$ converges.

Proof. Suppose first that $\lambda < p < q < \infty$. Then $n(t) < t^p$ for all large positive $t$, and so

$$n(r)r^{-q} + q \int_0^r n(t)t^{-q-1}dt$$
tends to a finite limit as \( r \to \infty \), which implies using (3.43) that \( \sum |a_n|^{-q} \) converges. Conversely, suppose that \( \sum |a_n|^{-q} = S < \infty \). Then for \( r > 0 \) we have \( n(r) \leq Sr^q \) by (3.43) and so \( \lambda \leq q \).

It is now clear that \( \sum |a_n|^{-\mu} \) converges for \( \lambda < \mu < \infty \) and diverges for \( 0 \leq \mu < \lambda \).

### 3.8.3 Weierstrass products

Define

\[
E(z, 0) = (1 - z), \quad E(z, p) = (1 - z) \exp \left( \sum_{j=1}^{p} \frac{z^j}{j} \right), \quad p \in \mathbb{N},
\]

(the Weierstrass primary factors). Then for \( |z| \leq \frac{1}{2} \) we have

\[
|\log E(z, p)| = | - z^{p+1} (p + 1)^{-1} - \ldots | \leq |z|^{p+1} + |z|^{p+2} + \ldots \leq 2|z|^{p+1}.
\]

Next, for \( |z| \geq 1 \) and \( p \geq 1 \) we have

\[
\log |E(z, p)| \leq \log(1 + |z| + |z| + \ldots + |z|^p) + \frac{|z|^p}{p}, \quad |z| \geq 1.
\]

Applying the maximum principle gives

\[
\log |E(z, p)| \leq A(p) = p + \log 2, \quad |z| \leq 1.
\]

### 3.8.4 Lemma

Let \( (a_n) \) be a non-zero sequence tending to infinity, and let \( q_n \geq 0 \) be integers such that for every positive \( r \) we have

\[
\sum \left( \frac{r}{|a_n|} \right)^{q_n+1} < \infty.
\]

Then

\[
F(z) = \prod E(z/a_n, q_n)
\]

converges, and is an entire function with zero sequence \( (a_n) \).

**Proof.** Fix \( K > 0 \). Then for \( |z| \leq K \) we have, by (3.44) and (3.47),

\[
\sum_{|a_n| \geq 2K} |\log E(z/a_n, q_n)| \leq \sum_{|a_n| \geq 2K} 2|K/a_n|^{q_n+1} < \infty.
\]

Hence

\[
\sum_{|a_n| \geq 2K} \log E(z/a_n, q_n)
\]

converges absolutely and uniformly on \( |z| \leq K \), and

\[
F(z) = \exp \left( \sum_{|a_n| \geq 2K} \log E(z/a_n, q_n) \right) \prod_{|a_n| < 2K} E(z/a_n, q_n)
\]

is analytic on \( D(0, K) \).
3.8.5 Theorem

Suppose that the non-zero sequence \((a_n)\) has finite exponent of convergence \(\lambda\), and let \(q\) be the least integer such that \(\sum |a_n|^{-q-1}\) converges. Then the product

\[ F(z) = \prod E(z/a_n, q) \]

converges in \(\mathbb{C}\), and has order \(\lambda\). Further, we have \(\log M(r, F) = o(r^{\lambda+1})\) as \(r \to \infty\).

Proof. We obviously have \(\lambda \leq q + 1\), by definition of \(\lambda\), and the fact that \(\sum |a_n|^{-\mu}\) converges for every \(\mu > \lambda\) gives \(q + 1 \leq \lambda + 1\). We note next that replacing \(q\) by \(q + 1\) in (3.43) and letting \(r \to \infty\) leads to

\[ \int_0^\infty n(t) t^{-q-2} dt < \infty \quad (3.48) \]

and so

\[ n(R) \int_R^{2R} t^{-q-2} dt \leq \int_R^{2R} n(t) t^{-q-2} dt = o(1), \]

which gives

\[ n(R) = o(R^{q+1}), \quad R \to \infty. \quad (3.49) \]

The product \(F(z)\) converges since (3.47) is satisfied for every \(r > 0\), with \(q_n = q\), and it is obvious that \(F\) has order at least \(\lambda\), by Jensen’s formula. Now suppose that

\[ q < s \leq q + 1, \quad \lim_{r \to \infty} \frac{n(r)}{r^s} = 0. \quad (3.50) \]

In particular, (3.50) is satisfied by \(s = q + 1\), by (3.49). Let \(|z| = r\) be large. Then

\[ \log |F(z)| \leq \sum \log |E(z/a_n, q)|. \]

Splitting the sum into those over (i) \(|a_n| \leq r\), (ii) \(r < |a_n| < 2r\) and (iii) \(2r \leq |a_n|\) respectively, and using (3.44), (3.45) and (3.46), we obtain

\[ \log |F(z)| \leq S_1 + S_2 + S_3, \]

in which

\[ S_1 = \sum_{|a_n| \leq r} (\log(1 + r/|a_n|) + q(r/|a_n|)^q), \]

\[ S_2 = \sum_{r < |a_n| < 2r} A(q) \leq A(q)n(2r) = o(r^s) \]

by (3.50), and

\[ S_3 = \sum_{|a_n| \geq 2r} 2|r/a_n|^{q+1}. \]

Now

\[ S_1 = \int_0^r (\log(1 + r/t) + q(r/t)^q)dn(t) = n(r)(\log 2 + q) + \int_0^r (r/t(t + r) + q^2r^q/t^{q+1})n(t)dt \]

and so

\[ S_1 < n(r)(\log 2 + q) + N(r) + q^2r^q \int_0^r n(t)/t^{q+1} dt. \]
Hence (3.50) gives
\[ S_1 < o(r^s) + O(r^q) + q^2 r^q \int_0^r o(t^{s-q-1})dt = o(r^s), \]
using the fact that \( s - q - 1 > -1 \). Next, integration by parts and (3.49) give
\[ S_3 = \int_{2r}^\infty 2(r/t)^{q+1}dn(t) \leq 2(q+1)r^{q+1} \int_{2r}^\infty n(t)t^{-q-2}dt. \]
Thus
\[ S_3 = o(r^s) : \]
to see this, if \( s = q + 1 \) we use (3.48), which tells us that the integral from \( 2r \) to \( \infty \) tends to 0, while if \( q < s < q + 1 \), then we use (3.50) and substitute \( n(t) = o(t^s) \) into the integral. Hence \( \log M(r, F) = o(r^s) \), and so \( F \) has order at most \( \lambda \); this is obvious if \( \lambda = q + 1 \), while if \( q \leq \lambda < q + 1 \) we take \( s \) with \( \lambda < s < q + 1 \).

For any function \( f \neq 0 \) meromorphic in \( \mathbb{C} \), we now define \( \lambda(f) \) to be the exponent of convergence of the zero sequence of \( f \). Obviously this is the same as the order of \( N(r, 1/f) \), and by Jensen’s formula is not greater than the order of \( f \). Similarly \( \lambda(1/f) \) is the exponent of convergence of the zeros of \( 1/f \) and so poles of \( f \).

3.8.6 Hadamard representation theorem

Let \( f \neq 0 \) be meromorphic in \( \mathbb{C} \). Then there exist entire functions \( F_1, F_2, h \) and an integer \( m \) such that \( \rho(F_1) = \lambda(f) \) and \( \rho(F_2) = \lambda(1/f) \) and \( f(z) \equiv z^m \frac{F_1(z)}{F_2(z)} e^{h(z)}. \)

Proof. Let \((a_n)\) be the sequence of zeros of \( f \) in \( 0 < |z| < \infty \), and let \((b_n)\) be the sequence of poles of \( f \) in \( 0 < |z| < \infty \), in both cases repeated according to multiplicity. Then there exist entire functions \( F_1, F_2 \), of orders \( \lambda(f), \lambda(1/f) \) respectively, such that the zero sequence of \( F_1 \) is \((a_n)\), and that of \( F_2 \) is \((b_n)\) (if either of these sequences is finite then \( F_2 \) is a finite product, while if the sequence is empty we put \( F_2 = 1 \)). We then choose an integer \( m \) so that \( f(z)z^{-m}F_2(z)F_1(z)^{-1} = g(z) \) is analytic and non-zero at 0, and it follows that \( g \) is analytic and non-zero in the plane, since all singularities of \( g \) and \( 1/g \) have been removed. Thus we may write \( g = e^h \) with \( h \) entire.

3.9 Appendix: lemmas underlying the Cartan formula

Cartan’s formula was derived in §3.4.1, and the following lemmas serve to show that certain quantities are in fact measurable functions.

3.9.1 Lemma

Let \( 0 < r < R \) and let \( f \) be a function non-constant and meromorphic on \( D(0, R) \). Assume that the circle \( |z| = r \) contains infinitely many points \( z \) with \( |f(z)| = 1 \). Then \( f \) is a rational function.

Proof. Let \( S = \{ s \in \mathbb{R} : |f(re^{is})| = 1 \} \) and let \( T \) be the set of \( t \) in \( \mathbb{R} \) such that \( t \) is a limit point of \( S \).

Suppose that \( t_0 \in T \). Then we can find \( t_n \to t_0, n \to \infty, \) with \( t_n \) real, \( t_n \neq t_0 \) and \( |f(re^{it_n})| = 1 \). Obviously \( |f(re^{it_0})| = 1 \), by continuity. For \( z \) near \( re^{it_0} \), put
\[ g(z) = \log f(z), \quad u = i \log z, \quad h(u) = g(z) = \log f(e^{-iu}). \]
Let \( u_0 = -t_0 + i \log r \). Taylor’s theorem allows us to write

\[
h(u) = \sum_{n=0}^{\infty} a_n(u - u_0)^n,
\]

with the power series absolutely convergent on an open disc \( D \) centred at \( u_0 \). Let

\[
H(u) = \sum_{n=0}^{\infty} \text{Re}(a_n)(u - u_0)^n,
\]

so that \( H \) is analytic on \( D \). Setting \( u_n = -t_n + i \log r \) we see that \( u_n - u_0 \) is real and \( H(u_n) = \text{Re}(h(u_n)) = \log |f(re^{it_n})| = 0 \), and so \( H(u) \equiv 0 \) on \( D \), by the identity theorem. So if \( s \) is real and close to \( t_0 \) then \( H(-s + i \log r) = \text{Re}(h(-s + i \log r)) = 0 \).

It follows that if \( t \in T \) then there exists \( \delta_t > 0 \) such that \(|f(re^{is})| = 1 \) for \( t - \delta_t < s < t + \delta_t \), and so \( T \) is open.

Now suppose that \( v \) is real, but not in \( T \). Then \( v \) is not a limit point of \( S \), and so there exists \( \rho_v > 0 \) such that \(|f(re^{is})| \neq 1 \) for \( v - \rho_v < s < v \) and \( v < s < v + \rho_v \). So no \( t \) in the interval \((v - \rho_v, v + \rho_v)\) is a limit point of \( S \), and so \( \mathbb{R} \setminus T \) is open.

But \( \mathbb{R} \) is connected, and \( T \) is non-empty, since \( S \cap [0,2\pi] \) is infinite by hypothesis, so that \( S \) has a limit point in the compact set \([0,2\pi]\). Thus we see that \( \mathbb{R} = T \).

We have now proved that \(|f(z)| \equiv 1 \) on the circle \(|z| = r \). Let \( a_\mu \) be the zeros of \( f \) in \(|z| < r \), and \( b_\nu \) the poles of \( f \) in \(|z| < r \), in both cases repeated according to multiplicity. For \(|a| < r \) we have

\[
|U_a(z)| = 1, \quad |z| = r, \quad U_a(z) = \frac{r(z-a)}{r^2-\bar{a}z},
\]

in which \( U_a \) is a Möbius transformation with a zero at \( a \) and a pole at \( r^2/\bar{a} \) (except that \( U_a(z) = z/r \) if \( a = 0 \)). Let

\[
F(z) = f(z) \prod_\mu U_{a_\mu}(z)^{-1} \prod_\nu U_{b_\nu}(z).
\]

Then \( F \) is meromorphic in \( D(0, R) \) and analytic and non-zero in \( D(0, r) \), with \(|F(z)| = 1 \) on \(|z| = r \). By the maximum principle applied to \( F \) and \( 1/F \), we see that \(|F(z)| \equiv 1 \) for \(|z| \leq r \). Hence \( \log F(z) \) has constant real part on \( D(0, r) \) and is constant there, by the Cauchy-Riemann equations. Thus \( F \) is constant and \( f \) is a rational function, given by

\[
f(z) = C \prod_\mu \left( \frac{r(z-a_\mu)}{r^2-\bar{a}_\mu z} \right) \prod_\nu \left( \frac{r(z-b_\nu)}{r^2-\bar{b}_\nu z} \right)^{-1}, \tag{3.52}
\]

in which \( C \) is a constant of modulus 1, and the products are over all zeros \( a_\mu \) and poles \( b_\nu \) in \(|z| < r \), in each case with repetition according to multiplicity. Notice that the zeros and poles of \( f \) in \( 0 \leq |z| < r \) determine the poles and zeros of \( f \) in \(|z| > r \).

### 3.9.2 Lemma

Suppose that \( f \) is meromorphic in \( D(0, R) \) and that \(|f(z)| = 1 \) on \(|z| = r_1 \) and \(|z| = r_2 \), where \( 0 < r_1 < r_2 < R \). Then \( f \) is constant.

**Proof.** Of all those zeros of \( f \) (if any) lying in \( 0 < |z| < r_2 \), let \( a \) be the nearest to the origin. Applying formula (3.52) with \( r = r_2 \), we see that \( f(c) = \infty, c = r_2^2/\bar{a} \). But, according to formula
(3.52) with \( r = r_1 \), the function \( f \) cannot have a pole at any \( \zeta \) with \(|\zeta| > |r_1^2/\bar{\alpha}|\). This contradiction shows that there cannot be any such \( \alpha \), and so \( f \) has no zeros, and by the same argument no poles, in \( 0 < |z| < r_2 \). Again by (3.52), \( f \) has no zeros or poles in \(|z| > r_2\) either. So \( f(z) = Dz^n \) for some constant \( D \) and integer \( n \), and the fact that \(|f(z)| = 1\) on \(|z| = r_1\) and \(|z| = r_2\) forces \( n = 0 \).

3.9.3 Lemma

Let \( 0 < r < R \) and let \( f \) be meromorphic and non-constant in \( D(0, R) \). Then there exists \( C > 0 \) such that, for all real \( t \),

\[
n(r, e^{it}) < C, \quad n(r, a) = n(r, 1/(f - a)) \text{.}
\]

Proof. Take \( r, s, S \) with \( r < s < S < R \). Choose \( z_0 \) with \(|f(z_0)| \neq 0, 1, \infty\) and with \( z_0 \) so close to 0 that the circle \(|z| = r\) lies in \( D(z_0, s) \), and such that the circle \(|z - z_0| = S\) lies in \( D(0, R) \). Let \( g(z) = f(z_0 + z) \). Then \( g \) is meromorphic on some disc \( D(0, T) \), with \( T > S \), and Lemma 3.2.7 gives, for real \( t \),

\[
n(r, 1/(f - e^{it})) \leq n(s, 1/(g - e^{it})) \leq DN(S, 1/(g - e^{it})), \quad D = (\log S/s)^{-1}.
\]

Now we just note that (again with \( t \) a real constant)

\[
N(S, 1/(g - e^{it})) \leq T(S, 1/(g - e^{it})) = T(S, g - e^{it}) - \log |g(0) - e^{it}|.
\]

This equals

\[
T(S, g - e^{it}) - \log |f(z_0) - e^{it}| \leq T(S, g) + T(S, e^{it}) + \log 2 + d = T(S, g) + \log 2 + d = C_1,
\]

with \( d \) and \( C_1 \) constants, independent of \( t \), using the fact that \(|f(z_0) - e^{it}| \geq ||f(z_0)| - 1|\).

3.9.4 Lemma

Let \( 0 < r < R \) and let \( f \) be non-constant and meromorphic on \( D(0, R) \). Then \( h(t) = n(r, e^{it}) \) and \( H(t) = N(r, e^{it}) \) are measurable functions on \( \mathbb{R} \).

Proof. The following argument (communicated to the author by Christian Berg) shows that for fixed \( r \) the function \( n(r, e^{it}) \) is measurable in \( t \). Rouché’s theorem implies that \( n_+(s, a) \) is lower semi-continuous in \( a \), where \( n_+(s, a) \) denotes the number of solutions of \( f(z) = a \) in \(|z| < s\). Hence \( n(r, a) = \lim_{s \to r^+} n_+(s, a) \) is measurable.

Now consider \( N(r, a) \) for \( a \in \mathbb{C} \). Take all zeros \( z_1, \ldots, z_m \) for \( f - a \) in \(|z| \leq r \). Assume for now that all of these zeros are simple and that \( f(0) \neq a \).

Now take a small positive \( \delta \) and let \( a_n \to a \) through a sequence. Then for large \( n \) there does not exist \( \zeta_n \) with \(|\zeta_n| \leq r \) and \(|\zeta_n - z_j| \geq \delta \) for all \( j \) and such that \( f(\zeta_n) = a_n \), since otherwise we may assume that \( \zeta_n \to \zeta \) which gives \( f(\zeta) = a \), a contradiction. So for large \( n \) there is a root \( z_{j,n} \) of \( f(z) = a_n \) near to \( z_j \), and there are no other roots of \( f(z) = a_n \) in \(|z| \leq r \). Hence, as \( n \to \infty \),

\[
N(r, a_n) = \sum_{j=1}^{m} \log^+ \frac{r}{|z_{j,n}|} \to \sum_{j=1}^{m} \log^+ \frac{r}{|z_j|} = N(r, a).
\]

This shows that, for fixed \( r \), the function \( N(r, a) \) is continuous off a finite set, and therefore measurable.
3.9.5 Lemma

Let \( f \) be non-constant and meromorphic on \( D(0, R) \). For \( 0 \leq s < R \) define

\[
\psi(s) = \int_0^{2\pi} n(s, e^{it}) dt.
\]

Suppose that \( 0 < r < R \) and that there are only finitely many \( z \) with \( |z| = r \) and \( |f(z)| = 1 \). Then \( \psi \) is continuous at \( r \).

Proof. Take \( S \) with \( r < S < R \) and take \( C \) as in Lemma 3.9.3, such that \( n(S, e^{it}) < C \) for all real \( t \). Let \( z_j = re^{it_j}, 1 \leq j \leq n \), with \( t_j \) real, be the finitely many points on \( |z| = r \) at which \( |f(z)| = 1 \). There is no loss of generality in assuming that \( 0 < t_1 < \ldots < t_n < 2\pi \), since replacing \( f(z) \) by \( f(ze^{iQ}) \), for some real \( Q \), does not change \( \psi \). Let \( \varepsilon > 0 \) and let \( \delta \) be small and positive, in particular so small that \( 2n\delta C < \varepsilon \).

Now suppose that \( t \in [0, 2\pi] \), with \( t \) not one of the \( t_j \). Then \( f(z) \neq e^{it} \) on \( |z| = r \) and so \( f(z) \neq e^{it} \) for \( |z| \) close to \( r \). Thus we can find \( \rho_t > 0 \) and \( \sigma_t > 0 \) such that \( |f(z) - e^{it}| \geq \sigma_t \) for \( r - \rho_t \leq |z| \leq r + \rho_t \). This in turn gives us \( \eta_t > 0 \) such that if \( p \) is real with \( |p - t| < \eta_t \), then \( f(z) \neq e^{ip} \) for \( r - \rho_t \leq |z| \leq r + \rho_t \).

This defines \( \rho_t > 0, \eta_t > 0 \) for \( t \in [0, 2\pi] \setminus \{t_1, \ldots, t_n\} \). For \( t = t_j \), we just set \( \rho_t = \eta_t = \delta \).

Now the intervals \( (t - \eta_t, t + \eta_t) \) cover the compact set \([0, 2\pi] \), and so we can find a finite set \( J \) such that \([0, 2\pi] \) is a subset of the union \( \bigcup_{t \in J} (t - \eta_t, t + \eta_t) \). Let \( \rho \) be the minimum of all the \( \rho_t, t \in J \).

By reducing \( \rho \) if necessary, we can assume that \( 0 < r - \rho < r + \rho < S \).

Now if \( p \) is in \([0, 2\pi] \) but not in any of the intervals \( (t_j - \delta, t_j + \delta) \), then \( p \) is in the interval \((t - \eta_t, t + \eta_t)\), for some \( t \in J \setminus \{t_1, \ldots, t_n\} \) and so, by definition of \( \eta_t \) and \( \rho \), we have \( f(z) \neq e^{ip} \) for \( r - \rho \leq |z| \leq r + \rho \). Hence \( n(s, e^{ip}) = n(r, e^{ip}) \) for \( r - \rho < s < r + \rho \).

We now see that for \( r - \rho < s < r + \rho \) we have

\[
\psi(s) - \psi(r) = I = \int_E n(s, e^{it}) - n(r, e^{it}) dt,
\]

in which

\[
E = [0, 2\pi] \cap \left( \bigcup_{j=1}^{n} (t_j - \delta, t_j + \delta) \right).
\]

Since \( |n(s, e^{it}) - n(r, e^{it})| \leq n(S, e^{it}) < C \), we get

\[
|\psi(s) - \psi(r)| \leq |I| \leq 2n\delta C < \varepsilon, \quad |s - r| < \rho.
\]
Chapter 4

Applications to differential equations

4.1 Some basic facts about linear differential equations

4.1.1 Existence-uniqueness theorem

Let \( k \geq 1 \), let \( D \) be a simply connected domain in \( \mathbb{C} \), and let \( a_0(z), \ldots, a_{k-1}(z) \) be analytic in \( D \). Let \( a \in D \) and let \( c_0, \ldots, c_{k-1} \in \mathbb{C} \). Then there exists a unique solution \( f \) of the equation

\[
w^{(k)} + \sum_{j=0}^{k-1} a_j w^{(j)} = 0, \quad (4.1)
\]
such that \( f \) is analytic in \( D \) and \( f^{(j)}(a) = c_j, \, 0 \leq j \leq k - 1 \).

**Proof.** We use the (standard) Newton-Picard successive approximations method, but some simplifications arise from the equation being linear and the Riemann mapping theorem. Once we have an analytic solution \( f \), the uniqueness is obvious. Given two such solutions \( f_1, f_2 \), we have \( (f_1 - f_2)^{(j)}(a) = 0 \) for all \( j \geq 0 \), and so \( f_1 - f_2 \equiv 0 \) on \( D \), by the identity theorem.

We write the equation (4.1) in matrix form using

\[
\begin{align*}
\varphi &= (c_0, \ldots, c_{k-1})^T, \\
w &= (w_0, \ldots, w_{k-1})^T, \\
w_j &= w^{(j)}
\end{align*}
\]
and

\[
w_0' = w_1, \ldots, w_{k-1}' = w^{(k)} = -\sum_{j=0}^{k-1} a_j w_j. \tag{4.3}
\]

Here \( T \) denotes the transpose, so that \( \varphi \) and \( w \) are column vectors. The equation (4.1) becomes

\[
w' = Aw \tag{4.4}
\]
in which \( A \) is a \( k \) by \( k \) matrix with entries 1 immediately above the main diagonal, and with last row \(-a_0, \ldots, -a_{k-1}\), and all other entries 0. The equation can now be written in integral form as

\[
w(z) = w(a) + \int_a^z A(t)w(t)dt. \tag{4.5}
\]

On \( \mathbb{C}^k \) we use the norm

\[
\|(z_1, \ldots, z_k)\| = \sum_{j=1}^k |z_j|. \tag{4.6}
\]
Define
\[ w^0(z) = 0, \quad w^1(z) = c, \ldots, \quad w^{q+1}(z) = c + \int_a^z A(t)w^q(t)dt, \quad q \geq 0. \] (4.7)

Our first step is to prove that solutions exist in a neighbourhood of \( a \).

### 4.1.2 Lemma

Suppose that \( |a_j(z)| \leq M < \infty \) on \( D(a, \delta) \subseteq D \), for \( 0 \leq j \leq k - 1 \), and take \( \rho \) with \( 0 < \rho \leq \delta \) and \( kM \rho \leq \frac{1}{2} \). Then there is a solution \( f \) of (4.1), analytic on \( D(a, \rho) \), with \( f^{(j)}(a) = c_j, 0 \leq j \leq k - 1 \).

**Proof.** For \( q \geq 0 \) set
\[ M_q = \sup \{ \| w^{q+1}(z) - w^q(z) \| : z \in D(a, \rho) \}. \] (4.8)

Then \( M_0 = \| c \| \). But (4.7) gives
\[ w^{q+2}(z) - w^{q+1}(z) = \int_a^z A(t)(w^{q+1}(t) - w^q(t))dt, \]
and each entry of \( A(t)(w^{q+1}(t) - w^q(t)) \) has modulus at most \( MM_q \) on \( D(a, \rho) \). Hence
\[ M_{q+1} \leq k\rho MM_q \leq \frac{1}{2}M_q. \]

Thus \( M_q \leq (1/2)^q \| c \| \) and the series
\[ w(z) = \sum_{j=0}^{\infty} (w^{j+1}(z) - w^j(z)) = \lim_{q \to \infty} \sum_{j=0}^{q-1} (w^{j+1}(z) - w^j(z)) = \lim_{q \to \infty} w^q(z) \]
converges absolutely and uniformly on \( D(a, \rho) \), the limit function \( \overline{w}(z) \) analytic there, by Weierstrass’ theorem. Since \( w^{q+1}(z) \) and \( w^q(z) \) both converge to \( w(z) \), we get
\[ w(z) = c + \int_a^z A(t)w(t)dt \]
and so
\[ w' = Aw, \quad w(a) = c. \]

Note that uniqueness can also be established as follows. If \( c = 0 \), then, with \( \rho \) as above and
\[ T = \sup \{ \| w(z) \| : z \in D(a, \rho) \} \]
we get \( T \leq \frac{1}{2}T \) and so \( T = 0 \).

### 4.1.3 Exercise

Write the equation \( y'' + y = 0 \) in matrix form, and find the approximants \( \overline{w}^q \) to the solution with \( c = (0, 1)^T \).
With \( \rho \) as above there exist linearly independent solutions \( f_0, \ldots, f_{k-1} \) of (4.1), analytic on \( D(a, \rho) \), such that on \( D(a, \rho) \) every solution \( f \) of (4.1) may be written as a linear combination \( f = d_0f_0 + \ldots + d_{k-1}f_{k-1} \), with constants \( d_j \). Further, any \( k + 1 \) solutions \( g_j \) are linearly dependent.

**Proof (standard).** To define \( f_j \), we let \( c_j = 1 \) and \( c_m = 0 \) for \( 0 \leq m \leq k - 1, m \neq j \). The linear independence is obvious and the representation for \( f \) follows from the uniqueness assertion earlier. Just set \( d_j = f^{(j)}(a) \).

To see that any \( k + 1 \) solutions \( g_j \) are linearly dependent, just choose \( C_j \in \mathbb{C} \) with the property that \( \sum C_j g_j^{(m)}(a) = 0 \) for \( 0 \leq m \leq k - 1 \) and it then follows that \( \sum C_j g_j(z) \) is identically zero.

**4.1.5 Continuation of the solution**

We now extend \( f \) to all of the simply connected domain \( D \). We have seen how to define \( f \) on \( D(a, \rho) \), for \( a \in D \), where \( \rho \) depends on the coefficients \( a_j \) but not on \( f \) (or \( \varphi \)).

If \( D \) is a disc \( D(0, R), 0 < R \leq \infty \), and \( \varepsilon \) is given, let \( S \) be the supremum of \( r > 0 \) such that there exists an analytic solution \( f \) on \( D(0, r) \), with \( f^{(j)}(0) = c_j, 0 \leq j \leq k - 1 \). We know that \( S \) is non-empty. Hence \( g_j \equiv g_m \) on \( D(b_j, \sigma_j) \) and so to \( D(0, S') \), where \( S' > S \). This contradiction shows that \( S = R \).

Thus we have proved the existence-uniqueness theorem for the whole plane and for a disc. Now if \( D \) is any simply connected domain, not the whole plane, and \( a \in D \), choose an analytic one-one function \( \phi \) mapping the unit disc \( D(0, 1) \) onto \( D \), with \( \phi(0) = a \), and let \( u = \psi(z) \) be the inverse function of \( z = \phi(u) \). Starting from the solution \( f \) of (4.1) in a neighbourhood of \( a \), we define \( g(u) \) near 0 by

\[
f(z) = g(u), \quad f'(z) = g'(u)\psi'(z),
\]

and we have

\[
f^{(m)}(z) = g^{(m)}(u)(\psi'(z))^m + \sum_{j=1}^{m-1} g^{(j)}(u)T_{j,m}(z),
\]

with coefficients \( T_{j,m} \) analytic in \( D \). But we can write \( z = \phi(u) \), and we have \( \psi' \neq 0 \), since \( \psi \) is one-one. We thus get an equation

\[
g^{(k)} + \sum_{j=0}^{k-1} B_j g^{(j)} = 0,
\]

satisfied by \( g \) in a neighbourhood of 0, with coefficients which are analytic in \( D(0, 1) \). Hence \( g \) extends to be analytic in \( D(0, 1) \), and \( f \) is analytic in \( D \), and solves (4.1) throughout \( D \) by the identity theorem.

In the case of a general domain \( D \), we can sometimes cover \( D \) with finitely many simply connected domains. However, it may not be possible to obtain solutions analytic in all of \( D \). For example, \( 1/z \) is analytic in \( D = \mathbb{C} \setminus \{0\} \). On any simply connected subdomain of \( D \) we can define \( w = \log z \), and \( w \) satisfies \( w'' + (1/z)w' = 0 \), but \( w \) is not analytic on \( D \).
4.1.6 Oscillation theory on the real line

Suppose that \( u \) is a real-valued solution of

\[
 u'' + A(x)u = 0,
\]

where \( A \) is a continuous real-valued function on an open interval \( I \) in \( \mathbb{R} \). Then the zeros of \( u \) in \( I \) are isolated and do not coincide with zeros of \( u' \). For if \( t \in I \) and \( u(t) = u'(t) = 0 \) then \( u \equiv 0 \) by the (real) existence-uniqueness theorem, and this will be the case if \( u \) has distinct zeros \( t_k \to t \), by continuity and Rolle’s theorem.

Given such a solution \( u \) of a homogeneous linear differential equation on an unbounded interval, an obvious and important question is whether \( u \) tends to infinity (e.g. \( e^x \) on \((0, \infty)\)) or decays to 0 (e.g. \( e^{-x} \) on \((0, \infty)\)) or is oscillatory (e.g. \( \sin x \) on \((0, \infty)\)). There are a lot of criteria for oscillation, and one which is easy to prove and quite useful is:

4.1.7 Sturm’s comparison theorem

Suppose that \( G_1, G_2 \) are continuous real-valued functions on an open interval \( I \) in \( \mathbb{R} \), and that on \( I \) the functions \( u, v \) are real-valued, not identically zero, and satisfy

\[
 u'' + G_1 u = 0, \quad v'' + G_2 v = 0.
\]

Suppose that \( x_1, x_2 \in I \) with \( x_1 < x_2 \) and \( u(x_1) = u(x_2) = 0 \) and \( u(x) \neq 0 \) on \((x_1, x_2)\), and that \( G_2(x) \geq G_1(x) \) on \([x_1, x_2]\). Then either (i) \( v \) has a zero in \((x_1, x_2)\) or (ii) on \([x_1, x_2]\) the function \( G_2 - G_1 \) vanishes identically and \( v \) is a constant multiple of \( u \).

Proof. Suppose that \( v \) has no zero in \((x_1, x_2)\); then it may be assumed that \( u(x) \) and \( v(x) \) are positive on \((x_1, x_2)\), and that \( u'(x_1) > 0, u'(x_2) < 0 \). This delivers

\[
 (u'v - uv')(x_2) = (u'v)(x_2) \leq 0, \quad (u'v - uv')(x_1) = (u'v)(x_1) \geq 0, \tag{4.9}
\]

and so

\[
 0 \geq (u'v - uv')(x_2) - (u'v - uv')(x_1) = \int_{x_1}^{x_2} (G_2(x) - G_1(x))u(x)v(x) \, dx \geq 0.
\]

Thus it must be the case that \( G_2(x) = G_1(x) \) on \([x_1, x_2]\), so that \( u'v - uv' \) is constant there, and hence identically zero by \((4.9)\).

In the complex domain, there are comparatively few such results. A good reference is [44, Ch. 8], but most result are negative, leading to zero-free regions, lower bounds for the distance between zeros etc. However, since the solutions of \((4.1)\) are analytic when the coefficients are, we can use the value distribution theory for meromorphic functions developed by Nevanlinna.

4.2 Nevanlinna theory and differential equations

In this section we describe some applications of Nevanlinna theory to the equation

\[
 w'' + A(z)w = 0, \tag{4.10}
\]

in which \( A \) is an entire function. By the existence-uniqueness theorem, all solutions are entire functions. The first result goes back to Wittich.
4.2 NEVANLINNA THEORY AND DIFFERENTIAL EQUATIONS

4.2.1 Theorem

Let $f$ be a non-trivial (i.e. not identically zero) solution of (4.10), with $A \neq 0$ entire. Then

(i) We have

$$T(r, A) = S(r, f).$$

(4.11)

(ii) If $f$ has finite order then $A$ is a polynomial.

(iii) If $c$ is a finite, non-zero complex number then

$$m(r, 1/(f - c)) = S(r, f),$$

(4.12)

so that in particular $\delta(c, f) = 0$.

Proof. To prove (i), we just write

$$-A = f''/f = (f''/f')(f'/f)$$

so that the lemma of the logarithmic derivative gives

$$T(r, A) = m(r, A) \leq S(r, f) + S(r, f') = S(r, f).$$

Also (ii) follows in the same way. Later we will see that the converse of (ii) is true.

Now that we have (i), we establish (iii) by writing

$$\frac{1}{f - c} = \frac{1}{Ac} \left( \frac{f''}{f} - \frac{f''}{f - c} \right).$$

However $\delta(0, f) = 1$ is possible. Indeed,

$$w'' - (g'' + (g')^2)w = 0$$

has the zero-free solution $f = e^g$. Consequently, in order to discuss zeros of solutions of (4.10), it is normally necessary to consider two linearly independent solutions.

Let $f_1, f_2$ be solutions of (4.10), and let $W$ be the Wronskian determinant

$$W = W(f_1, f_2) = f_1f'_2 - f'_1f_2.$$

Then $W' = 0$ and $W = c$ is a constant. It is easy to see that $c = 0$ if and only if $f_1$ and $f_2$ are linearly dependent. We say that $f_1$ and $f_2$ are normalized LI solutions if $W(f_1, f_2) = 1$.

4.2.2 A result of Bank (Crelle’s Journal, 1972)

The result of (iii) in §4.2.1 generalizes as follows. Suppose that $f$ is a transcendental meromorphic function in the plane and satisfies a $k$’th order differential equation

$$0 = \sum_{j=1}^{p} a_j f^{m_{0,j}} (f')^{m_{1,j}} \ldots (f^{(k)})^{m_{k,j}},$$

(4.13)

with meromorphic coefficients $a_j$, which are not all identically zero and satisfy $T(r, a_j) = S(r, f)$. Let $n$ be the degree of the equation (the largest of those sums $m_{0,j} + \ldots + m_{k,j}$ for which $a_j \neq 0$), and set $F = f'/f$. Then for each positive integer $k$, we can write

$$f^{(k)} = Q_k(F)f,$$

in which $Q_k(F)$ is a polynomial in $F$ and its derivatives, with constant coefficients. This is easily proved by induction, using

$$f' = Ff, \quad f'' = (F' + F^2)f,$$
and
\[ f^{(k+1)} = (Q_k(F))' f + Q_k(F) F. \]

Grouping together all terms of the same degree, we can write the equation (4.13) in the form
\[ 0 = \sum_{q=0}^{n} f^q L_q(z, F), \]

in which each \( L_q \) is a polynomial in \( F \) and its derivatives, with coefficients \( b \) satisfying \( T(r, b) = S(r, f) \).

There are now two cases.

**Case 1:** We have \( L_q \equiv 0 \) for every \( q \).

In this case for each \( q \) the equation \( 0 = L_q(z, F) \) gives a homogeneous differential equation satisfied by \( f \).

**Case 2:** Suppose \( s + 1 \) is the greatest \( q \) for which \( L_q \not\equiv 0 \). Then we divide the equation (4.14) through by \( f^s L_{s+1} \) to get an equation
\[ f = \sum_{k=0}^{s} f^{k-s} M_k, \quad M_k = -L_k/L_{s+1}, \]

where
\[ T(r, M_k) \leq O(T(r, F)) + S(r, f). \]

Hence
\[ m(r, f) \leq \sum_{k=0}^{s} m(r, M_k) + O(1) \leq O(T(r, F)) + S(r, f) \]

and
\[ N(r, f) \leq \sum_{k=0}^{s} N(r, M_k) \leq O(T(r, F)) + S(r, f). \]

This gives
\[ T(r, f) \leq O(T(r, F)) + S(r, f) \leq O(\overline{N}(r, f) + \overline{N}(r, 1/f)) + S(r, f). \] (4.15)

We illustrate this with two examples. First, if \( A \neq 0 \) is an entire function and \( c \) is a non-zero complex number then (4.10) may be written as
\[ w'' + A(w - c) = -Ac. \]

Hence if \( f \) is a non-trivial solution of (4.10) we have a non-homogeneous differential equation in \( g = f - c \) with coefficients which are small functions compared to \( g \), and so we get
\[ T(r, f) \leq T(r, g) + O(1) \leq O(\overline{N}(r, 1/g)) + S(r, g) = O(\overline{N}(r, 1/(f - c))) + S(r, f). \]

Next, suppose that \( f \) is a transcendental meromorphic solution in the plane of
\[ af'' + bf'^2 + cf^2 + Af'' + Bf' + Cf + D = 0, \]

with \( a, b, c, A, B, C, D \) rational functions. With \( F = f'/f \) we get
\[ f^2 L_2 + f L_1 + L_0 = 0, \]
in which
\[ L_2 = a(F' + F^2) + bF^2 + c, \quad L_1 = A(F' + F^2) + BF + C, \quad L_0 = D. \]

If all the \( L_q \) are identically zero, we get three homogeneous equations, namely
\[ a f f'' + bf'^2 + cf^2 = 0, \quad Af'' + Bf' + Cf = 0, \quad D = 0, \]
which in principle may be easier to solve. If some \( L_q \) fails to vanish identically, we can estimate \( T(r, f) \) in terms of \( N(r, f) \) and \( N(r, 1/f) \) using (4.15). In particular, if
\[ N(r, f) + N(r, 1/f) = S(r, f) \]
then we must have \( L_0 = L_1 = L_2 = 0 \).

### 4.2.3 The Schwarzian

For meromorphic \( U \) define
\[ S(U) = \{ U, z \} = \frac{U'''}{U'} - \frac{3}{2} \left( \frac{U''}{U'} \right)^2. \] (4.16)

Note that if \( U \) has a simple pole at \( a \) then there is a constant \( c \neq 0 \) such that
\[ U'(z) = c(z-a)^{-2} + O(1), \quad U''(z) = -2c(z-a)^{-3} + O(1), \quad U'''(z) = 6c(z-a)^{-4} + O(1), \quad z \to a. \]

Hence the only poles of \( S \) are at zeros of \( U' \) and multiple poles of \( U \), i.e. at multiple points of \( U \).

If \( U \) is the quotient \( f_1/f_2 \) of \( L_1 \) solutions of (4.10), then we have \( U' = cf_2^{-2} \) for some non-zero constant \( c \), and an easy calculation gives
\[ S(U) = 2A. \] (4.17)

Also \( U' \neq 0 \) and, since \( f_2 \) has only simple zeros, \( U \) is locally one-one.

Conversely, suppose that \( F \) is meromorphic without multiple points on a simply connected domain \( D \). Then (4.17) defines a function \( A \) analytic on \( D \), and it is easy to check that \( f_2 = (U')^{-1/2} \) is an analytic solution of (4.10) in \( D \). If we choose a second solution \( f_1 \) of (4.10) such that \( W(f_1, f_2) = -1 \) then \( U' = (f_1/f_2)' \) and \( U \) is the quotient of linearly independent solutions of (4.10).

The Schwarzian derivative plays an important role in conformal mapping. Suppose that \( U \) is meromorphic and locally one-one in the unit disc \( D(0, 1) \). If \( U \) is one-one in \( D \) then
\[ (1 - |z|^2)^2 |S(U)| \leq 6 \]
there. In the other direction,
\[ (1 - |z|^2)^2 |S(U)| \leq 2 \]
is sufficient to imply that \( U \) is one-one. Both constants are sharp and the results are due to Nehari. The first uses coefficient inequalities and the second can be proved using differential equations or quasiconformal maps. Note that if \( U = f_1/f_2 \) is one-one on \( D \) then each of \( f_1 \) and \( f_2 \) has at most one zero in \( D \).
4.2.4 The Bank-Laine product

This approach was introduced by Bank and Laine [7]. It is convenient first to note that if \( h \) and \( E \) are related by

\[
\frac{h'}{h} = \frac{1}{2} \left( \frac{E'}{E} + \frac{c}{E} \right),
\]

where \( c = \pm 1 \) is a constant, then a straightforward calculation shows that

\[
\frac{h''}{h} = -\frac{1}{4} \left( \frac{(E')^2 - 2E''E - 1}{E^2} \right).
\]

Now let \( f, g \) be LI solutions of (4.10), normalized so that \( W(f, g) = fg' - gf' = 1 \), and set

\[
U = \frac{f}{g}, \quad E = fg, \quad \frac{U'}{U} = -\frac{1}{E}.
\]

Then

\[
\frac{E'}{E} = \frac{f'}{f} + \frac{g'}{g}, \quad \frac{1}{E} = \frac{g'}{g} - \frac{f'}{f}.
\]

Solving thus gives

\[
\frac{f'}{f} = \frac{E' - 1}{2E}, \quad \frac{g'}{g} = \frac{E' + 1}{2E}.
\]

and so the identity above, with \( h = f \), yields the Bank-Laine equation

\[
4A = \frac{(E')^2 - 2E''E - 1}{E^2}.
\]

Multiplying out by \( E^2 \) and differentiating, we also have

\[
E''' + 4AE' + 2A'E = 0.
\]

Note that (4.21) appears in [46], but (4.20) does not seem to have been used before Bank and Laine.

The product \( E \) is a Bank-Laine function: this means an entire function \( E \) such that \( E = 0 \) implies \( E' = \pm 1 \). Conversely, suppose that \( E \) is a Bank-Laine function. Then \( A \) as defined by (4.20) is entire, since the numerator has at least a double zero at any zero of \( E \). Choose \( w \) with \( E(w) \neq 0 \) and define \( f \) and \( g \) near \( w \) by (4.19). Then \( f \) and \( g \) are solutions of (4.10) near \( w \) and so are entire functions. Since the Wronskian of \( f \) and \( g \) is then a constant, which has to be non-zero, and since (4.19) is unaffected if \( f \) and \( g \) are multiplied by a constant, it may be assumed that \( W(f, g) = 1 \). But then (4.19) gives

\[
\frac{1}{E} = \frac{g'}{g} - \frac{f'}{f} = \frac{1}{fg}.
\]

Thus we obtain:

4.2.5 Theorem (Bank-Laine 1982-3)

An entire function \( E \) is a Bank-Laine function if and only if \( E \) is the product of linearly independent normalized solutions of an equation (4.10) with \( A \) entire.
4.2.6 The advantages of the product

Let \( f_1, f_2 \) be normalized LI solutions of \((4.10)\), with product \( E = f_1 f_2 \). Let \( c \) denote a positive constant (not necessarily the same at every occurrence).

(i) We have

\[
T(r, A) \leq cT(r, E) + S(r, E). \tag{4.22}
\]

This follows at once from (4.20).

(ii) We have

\[
T(r, E) \leq \frac{1}{2} T(r, A) + N(r, 1/E) + S(r, E). \tag{4.23}
\]

To see this, write \( T(r, E) = m(r, 1/E) + N(r, 1/E) + O(1) \) and note from (4.20) that

\[
2m(r, 1/E) = m(r, 1/E^2) \leq T(r, A) + S(r, E).
\]

(iii) If \( A \) has finite order and the zeros of \( E \) have finite exponent of convergence, then \( E \) has finite order.

(iv) If \( E \) has finite order then \( A \) is a polynomial if and only if \( m(r, 1/E) = O(\log r) \).

4.2.7 Examples of Bank-Laine functions

(i) Let \( E = e^Q \) with \( Q \) a polynomial. Then \( E \) is a Bank-Laine function and \( A \) has the form

\[
4A = -2Q'' - (Q')^2 - e^{-2Q}.
\]

(ii) Let \( P \) be a polynomial with only simple zeros, and let \( Q \) be a non-constant polynomial, chosen using Lagrange interpolation, so that \( E = Pe^Q \) is a Bank-Laine function. Here both \( E \) and \( A \) have order equal to the degree of \( Q \).

(iii) Let \( K = (2n + 1)^2/16 \) with \( n \) a non-negative integer, and define

\[
Q(\zeta) = \sum_{m=0}^{n} a_m \zeta^m
\]

by \( a_0 = 1 \) and, with \( c = \pm i \),

\[
(4m^2 + 4m + 1 - 16K)a_m = 16c(m + 1)a_{m+1}.
\]

Then \( W(z) = Q(e^{-z/2}) \) satisfies

\[
W'' + W'(2ce^{z/2} - 1/2) + W(-K + 1/16) = 0
\]

and \( w(z) = W(z) \exp(2ce^{z/2} - z/4) \) solves

\[
w'' + (e^z - K)w = 0. \tag{4.24}
\]

We thus have linearly independent solutions whose zeros have exponent of convergence at most 1. In fact, the change of variables \( \zeta = 2e^{z/2}, u(\zeta) = w(z) \), turns (4.24) into Bessel’s equation (this is in [44]). There are quite a lot of similar examples of equations (4.10), with \( A \) a polynomial in \( e^{\alpha z} \) and \( e^{-\alpha z} \), having LI solutions with \( \lambda(f_1 f_2) \leq 1 \).
4.2.8 The Bank-Laine conjecture

It is conjectured that if \( A \) is a transcendental entire function and the equation (4.10) has linearly independent solutions \( f_1, f_2 \) with \( \lambda(f_1 f_2) < \infty \), then the order of \( A \) is either \( \infty \) or a positive integer.

It has been proved (Rossi, Shen 1986) that if \( A \) is transcendental and \( \rho(A) \leq \frac{1}{2} \) then \( \lambda(f_1 f_2) = \infty \).

4.2.9 Theorem (Bank-Laine)

Suppose that \( A \) is a transcendental entire function of order \( \rho < \alpha < \frac{1}{2} \), and that \( E = f_1 f_2 \) is the product of normalized LI solutions of (4.10). Then \( \lambda(E) = \infty \).

Proof. Suppose that \( \lambda(E) < \infty \). Then \( E \) has finite order. By Lemmas 3.7.2 and 3.7.4 there exists a constant \( M > 0 \) such that provided \( |z| \) lies outside a set of finite measure we have

\[
|E''(z)/E(z)| + |E'(z)/E(z)| \leq |z|^M.
\]  

(4.25)

The next ingredient is a classical result known as the \( \cos \pi \rho \) theorem: since \( A \) has order \( \rho < \alpha < \frac{1}{2} \) we have

\[
\log |A(z)| \log M(r, A) > \cos \pi \alpha > 0, \quad |z| = r,
\]  

(4.26)

for all \( r \) in a set \( H \) of lower logarithmic density at least \( 1 - \rho/\alpha \), so that

\[
\int_{H \cap [1, s]} \frac{dt}{t} > (1 - \rho/\alpha - o(1)) \log s, \quad s \to \infty.
\]

This gives us arbitrarily large \( r \) satisfying (4.26), such that (4.25) also holds on \( |z| = r \). Since

\[
\log r = o(T(r, A)) = o(\log M(r, A)),
\]

we deduce from (4.20) that \( E \) must be small on the whole circle \( |z| = r \), which is obviously impossible, by the maximum principle.

4.2.10 Theorem (Bank-Laine)

Suppose that \( A \) is a transcendental entire function of finite order \( \rho \), and that (4.10) has normalized LI solutions \( f_1, f_2 \) such that \( \lambda(f_1 f_2) < \rho \). Then \( \rho \) is a positive integer.

Proof. With \( E = f_1 f_2 \) we have

\[
\lambda(E) < \rho(A) \leq \rho(E) < \infty.
\]

Hence we may write \( E = \Pi e^g \) with \( \Pi \) entire of order \( \lambda(E) \) and \( g \) a polynomial, of degree \( \rho(E) \). We now have

\[
m(r, 1/E) = (1 + o(1))T(r, E)
\]

and so \( \rho(A) \geq \rho(E) \).

4.3 Polynomial coefficients

There is an extensive literature on the asymptotic behaviour of solutions of (4.1), when the \( a_j \) are polynomials or rational functions. We will describe here the solutions of

\[
w'' + b(z)w = 0, \quad (4.27)
\]

when \( b(z) \) is a rational function with \( b(z) = cz^n(1 + o(1)), z \to \infty, n \geq -1, c \neq 0 \).
4.3. POLYNOMIAL COEFFICIENTS

4.3.1 Hille’s method

Let $c > 0$ and $0 < \varepsilon < \pi$. Then there exists a constant $d > 0$, depending only on $c$ and $\varepsilon$, with the following properties.

Suppose that the function $F$ is analytic, with $|F(z)| \leq c|z|^{-2}$, in

$$\Omega = \{z : 1 \leq R \leq |z| \leq S < \infty, |\arg z| \leq \pi - \varepsilon\}. \quad (4.28)$$

Then the equation

$$u'' + (1 - F(z))u = 0 \quad (4.29)$$

has linearly independent solutions $U(z), V(z)$ satisfying

$$U(z) = e^{-iz}(1 + \delta_1(z)), \quad U'(z) = -ie^{-iz}(1 + \delta_2(z)), \quad V(z) = e^{iz}(1 + \delta_3(z)), \quad V'(z) = ie^{iz}(1 + \delta_4(z)), \quad (4.30)$$

in which

$$|\delta_j(z)| \leq \frac{d}{|z|} \quad \text{for} \quad z \in \Omega_1 = \Omega \setminus \{z : \text{Re}(z) < 0, |\text{Im}(z)| < R\}. \quad (4.31)$$

Here $\Omega_1$ can be thought of as $\Omega$ with the “shadow” of $D(0, R)$ removed.

To prove this, let $X = Se^{i\sigma}$, where $\sigma = \min\{\pi/2, \pi - \varepsilon\}$. Choose a solution $v$ of the equation

$$v'' + 2iv' - Fv = 0, \quad (4.32)$$

analytic in $\Omega$, such that $v(X) = 1, v'(X) = 0$. Set, for $z \in \Omega$,

$$L(z) = v(z) - 1 + \frac{1}{2i} \int_X^z (e^{2i(t-z)} - 1) F(t)v(t)dt, \quad (4.33)$$

the integration being independent of path in $\Omega$, by Cauchy’s theorem. Now

$$L'(z) = v'(z) - \int_X^z e^{2i(t-z)} F(t)v(t)dt, \quad (4.34)$$

and

$$L''(z) = v''(z) + 2i \int_X^z e^{2i(t-z)} F(t)v(t)dt - F(z)v(z)$$

$$= v''(z) + 2i(v'(z) - L'(z)) - F(z)v(z) = -2iL'(z).$$

Since $L(X) = L'(X) = 0$, the existence-uniqueness theorem gives $L(z) \equiv 0$ on $\Omega$.

Now let $z \in \Omega_1$. Choose the path of integration $\gamma_z$ to be the arc of the circle $|t| = S$ from $X$ clockwise to the first point $x$ of intersection of the circle $|t| = S$ with the line $\text{Im}(t) = \text{Im}(z)$, followed by the straight line segment from $x$ to $z$. Then $\text{Im}(t-z) \geq 0$ and hence $|e^{2i(t-z)}| \leq 1$ on $\gamma_z$, and this is the reason for the choice of $\Omega_1$ and $X$.

Since $L(z) = 0$, (4.33) gives

$$|v(z) - 1| \leq \int_X^z |F(t)v(t)||dt|, \quad |v(z)| \leq 1 + \int_X^z |F(t)v(t)||dt|. \quad (4.35)$$

We apply the method generally known as Gronwall’s lemma. Let $s$ denote arc length on $\gamma_z$, and parametrize $\gamma_z$ with respect to $s$. Set

$$H(s) = 1 + \int_X^{\zeta(s)} |F(t)v(t)||dt| = 1 + \int_0^s |F(\zeta(s))v(\zeta(s))||ds, \quad \zeta \in \gamma_z.$$
Then the second estimate of (4.35) gives

\[
\frac{dH}{ds} = |F(\zeta(s))v(\zeta(s))| \leq |F(\zeta(s))|H(s)
\]

so that

\[
H(s) = \frac{H(s)}{H(0)} \leq \exp \left( \int_X^s |F(t)| |dt| \right).
\]

Thus the first estimate of (4.35) becomes

\[
|v(z) - 1| \leq H(s) - 1 \leq \exp \left( \int_X^z |F(t)| |dt| \right) - 1.
\] (4.36)

Let \(d_1, d_2, \ldots\) denote positive constants depending only on \(c\) and \(\epsilon\). The circle \(|t| = S\) evidently contributes at most \(d_1 S^{-1} \leq d_1 |z|^{-1}\) to the integral in (4.36). Similarly, if \(|\arg z| \leq \pi/4\) then \(\Re(z) > 0\) and the horizontal part of \(\gamma_z\) contributes at most

\[
\int_{\Re(z)}^{\infty} \frac{c}{\pi^2} \frac{dt}{t^2} \leq \frac{d_2}{\Re(z)} \leq d_3 |z|.
\]

Finally, if \(\pi/4 \leq |\arg z| \leq \pi - \epsilon\) we write \(z = a + ib\) with \(a, b\) real and \(|b| > d_4 |z|\), and the contribution from the horizontal part of \(\gamma_z\) to the integral in (4.36) is at most

\[
\int_R \frac{c}{x^2 + b^2} \frac{dx}{dx} \leq \frac{d_5}{|b|} \leq d_6 |z|.
\]

Thus (4.36) gives

\[
|v(z) - 1| \leq \exp \left( \frac{d_7}{|z|} \right) - 1 \leq \frac{d_8}{|z|} \leq d_8,
\]

using the fact that \(R \geq 1\), and (4.34) gives

\[
|v'(z)| \leq \int_X^z |F(t)| |dt| \leq \frac{d_{10}}{|z|}.
\]

Now we need only set \(V(z) = v(z)e^{iz}\) so that \(V\) solves (4.29), by (4.32), and (4.30) for \(V\) follows at once.

To obtain \(U\), we set \(Y = \overline{X}\) and choose a solution \(u\) of

\[
u'' - 2iu' - Fu = 0,
\]

with \(u(Y) = 1, u'(Y) = 0\), and the integral equation for \(u\) is

\[
u = 1 + \frac{1}{2i} \int_Y^z (e^{-2i(t-z)} - 1)F(t)u(t)dt.
\]

The path of integration has \(\Im(t - z) \leq 0\). Finally we set \(U(z) = u(z)e^{-iz}\).
4.3. POLYNOMIAL COEFFICIENTS

4.3.2 Other regions

An almost identical argument works if $\Omega$ is replaced by

$$\{z : 1 \leq R \leq |z| \leq S < \infty, |\arg z - \pi| \leq \pi - \varepsilon\},$$

with this time

$$\Omega_1 = \Omega \setminus \{z : \text{Re}(z) > 0, |\text{Im}(z)| < R\}.$$ We may also replace $\Omega$ with an unbounded region. Suppose that $F$ is analytic, with $|F(z)| \leq c|z|^{-2}$, in

$$\Omega' = \{z : 1 \leq R \leq |z| < \infty, |\arg z| \leq \pi - \varepsilon\}.$$ We take a sequence $S_n \to \infty$, and obtain corresponding solutions $U_n, V_n$ in

$$\{z : R \leq |z| \leq S_n, |\arg z| \leq \pi - \varepsilon\} \setminus \{z : \text{Re}(z) < 0, |\text{Im}(z)| < R\}.$$ The corresponding error terms $\delta_{j,n}(z), j = 1, 2, 3, 4$, are uniformly bounded, since the constant $d$ is independent of $S$ in §4.3.1. Thus by normal families we may assume, passing to a subsequence if necessary, that the $U_n, V_n, \delta_{j,n}$ converge locally uniformly on

$$\Omega'' = \{z : 1 \leq R < |z| < \infty, |\arg z| < \pi - \varepsilon\} \setminus \{z : \text{Re}(z) \leq 0, |\text{Im}(z)| \leq \varepsilon\}.$$ The limit functions $U, V$ solve (4.30), and the corresponding $\delta_j(z)$ satisfy (4.31) on $\Omega''$.

4.3.3 Equations with a polynomial coefficient

The standard application of Hille’s method is to the equation (4.27), when $b$ is a polynomial, not identically zero. Slightly more generally, suppose that $b(z)$ is analytic in $R_0 < |z|$, with $b(z) = cz^n(1 + o(1)), z \to \infty,$ in which $c$ is a non-zero constant and $n$ is an integer not less than $-1$.

4.3.4 The case $n = -1$

If $n = -1$ it is convenient to set

$$z = u^2, \quad g(u) = f(z) = f(u^2),$$
in which $f$ is a solution of (4.27). Then $g$ solves

$$g''(u) = 2f'(u^2) + 4u^2f''(u^2) = g'(u)/u - 4u^2b(u^2)g(u)$$
and so

$$g''(u) - g'(u)/u + c(u)g(u) = 0, \quad c(u) = 4u^2b(u^2) = 4c(1 + o(1)), \quad u \to \infty.$$ Now set $h(u) = u^{-1/2}g(u) = u^{-1/2}f(u^2)$ so that $h$ satisfies

$$h''(u) + (c(u) - 3/4u^2)h(u) = 0.$$ In the equation for $h$ we have $n = 0$, and from the asymptotic behaviour of $h$ we can deduce that of $f$. We assume henceforth that $n \geq 0$. 

4.3.5 Critical rays

The critical rays are those rays \( \arg z = \theta \in \mathbb{R} \) for which

\[
\arg c + (n + 2)\theta = 0 \quad \text{(mod } 2\pi\text{)}.
\] (4.37)

Assume that \( \arg z = \theta_0 \) is a critical ray, let \( R_0 \) be large and positive, and with \( \varepsilon \) small and positive define

\[
Z = \int_{2R_0e^{i\theta_0}}^{z} b(t)^{1/2} dt = \frac{2c^{1/2}}{n+2} z^{(n+2)/2} (1 + o(1)), \quad z \to \infty, \quad |\arg z - \theta_0| \leq \frac{2\pi}{n+2} - \varepsilon.
\]

Here we are free to choose either branch of \( b(t)^{1/2} \) (each of which is of course \(-1\) times the other).

The condition (4.37) implies that \( cz^{n+2} \) is real and positive on the critical ray, and so we may choose the branch of \( b(t)^{1/2} \) in order to ensure that \( c^{1/2}z^{(n+2)/2} \) is also real and positive on \( \arg z = \theta_0 \). We assume henceforth that this has been done.

4.3.6 Lemma

Let \( R_1 \) be large and let \( \sigma \) be small and positive. Let \( V = V(z) \) satisfy

\[
V(z) = \frac{2c^{1/2}}{n+2} z^{(n+2)/2} (1 + o(1)) \quad \text{as } z \to \infty, \quad |\arg z - \theta_0| \leq \frac{2\pi}{n+2} - \tau,
\] (4.38)

where \( 0 < \tau < \sigma \). Then \( V \) is univalent on the region \( T_1 \) given by

\[
|z| > R_1, \quad |\arg z - \theta_0| < \frac{2\pi}{n+2} - \sigma,
\]

and \( V \) maps \( T_1 \) onto a region containing

\[
T_1^* = \{ w : |w| > R_1^*, |\arg w| < \pi - \sigma^* \}.
\]

Here we may take any large \( R_1^* \) and any \( \sigma^* \) with \( \sigma^* > (n + 2)\sigma / 2 \).

To prove the lemma, note first that

\[
\zeta = \frac{2c^{1/2}}{n+2} z^{(n+2)/2}
\]

(with the same choice of square roots as before) is univalent on the region \( T_2 \) given by

\[
|z| > 0, \quad |\arg z - \theta_0| < \frac{2\pi}{n+2} - \frac{\sigma}{2},
\]

and \( \zeta \) maps \( T_2 \) onto the sector

\[
T_3 = \left\{ \zeta : |\zeta| > 0, \quad |\arg \zeta| < \pi - \frac{\sigma(n+2)}{4} \right\}.
\]

But (4.38) and Cauchy’s estimate for derivatives give

\[
\frac{dV}{dz} \sim c^{1/2} z^{n/2}, \quad \frac{dV}{\zeta} = \frac{dV}{dz} \frac{dz}{d\zeta} = 1 + o(1) \quad \text{as } z \to \infty, \quad |\arg z - \theta_0| < \frac{2\pi}{n+2} - \sigma.
\]
4.3. POLYNOMIAL COEFFICIENTS

Thus if $R_1$ is large and $z_1, z_2$ are distinct and in $T_1$, we set $\zeta_j = \zeta(z_j)$, and we may integrate from $\zeta_1$ to $\zeta_2$ along a a straight line, to obtain

$$V(z_1) - V(z_2) = \int_{\zeta_2}^{\zeta_1} \frac{dV}{d\zeta} d\zeta = \int_{\zeta_2}^{\zeta_1} 1 + o(1) d\zeta = \zeta_1 - \zeta_2 + o(|\zeta_1 - \zeta_2|) \neq 0.$$  

This shows that $V$ is univalent on $T_1$. To see that $V(T_1)$ contains $T^*_1$, just take $R$ large and positive, and $\sigma'$ with $(n + 2)\sigma/2 < (n + 2)\sigma'/2 < \sigma^*$, and look at the image under $\zeta$ of

$$U_R = \{ z : R < |z| < 2R, \ |\arg z - \theta_0| < \frac{2\pi}{n + 2} - \sigma' \}.$$  

This is, for some large $S$,

$$V_R = \left\{ w : S < |w| < 2(n+2)/S, \ |\arg w| < \pi - \frac{(n + 2)\sigma'}{2} \right\}.$$  

As $z$ goes once around the boundary $\partial U_R$ we see that $\zeta$ goes once around $\partial V_R$, and $V(z)$ describes a simple closed curve $\Gamma_R$ which is close to $\partial V_R$, since $V(z) \sim \zeta$. But $V(T_1)$ is simply connected, and so the interior of $\Gamma_R$ lies in $V(T_1)$, which gives

$$\left\{ w : S(1 + \sigma) < |w| < 2(n+2)/S(1 - \sigma), \ |\arg w| < \pi - \sigma^* \right\} \subseteq V(T_1).$$  

This proves the last conclusion.

4.3.7 The Liouville transformation

Let $\delta$ be small and positive, let $R_1$ be large and write

$$W(Z) = b(z)^{1/4}w(z) \quad (4.39)$$

in which $w$ is a solution of (4.27), and $z$ lies in

$$Q_1 = \left\{ z : |z| > \frac{R_1}{4}, \ |\arg z - \theta_0| < \frac{2\pi}{n + 2} - \frac{\delta}{4} \right\}.$$  

By Lemma 4.3.6, we have, for some large $R_2$,

$$Q_2 = \left\{ w : |w| > R_2, \ |\arg w| < \pi - \left( \frac{n + 2}{4} \right) \right\} \subseteq Z(Q_1),$$  

and the same asymptotics for $Z$ show that

$$Z(S_1) \subseteq Q_2, \text{ where } S_1 = \left\{ z : |z| > R_1, \ |\arg z - \theta_0| < \frac{2\pi}{n + 2} - \delta \right\}.$$  

The equation (4.27) transforms to

$$\frac{d^2W}{dZ^2} + (1 - F_0(Z))W = 0, \quad F_0(Z) = \frac{b''(z)}{4b(z)^2} - \frac{5b'(z)^2}{16b(z)^3}, \quad (4.40)$$

and we have $|F_0(Z)| = O(|Z|^{-2})$ in $Q_2$. By §4.3.1 there exist solutions $U_1(Z), U_2(Z)$ of (4.40) satisfying (4.30) in $Q_2$ and these give principal solutions

$$u_j(z) = b(z)^{-1/4} \exp((-1)^j iZ + o(1)) \quad (4.41)$$

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of (4.27) in $S_1$. The $u_j$ are zero-free in $S_1$, but if $A, B$ are non-zero constants we show that

$$w = Au_1 - Bu_2$$

has zeros near the critical ray, as follows. Set

$$V(z) = \frac{1}{2i} \log \frac{u_2(z)}{u_1(z)}.$$

Now, $w(z) = 0$ if and only if $u_2/z = A/B$, which is the same as

$$2iV(z) = \log \frac{u_2(z)}{u_1(z)} = 2iZ + o(1) = \log(A/B) + k2\pi i,$$  \hspace{1cm} (4.42)

with $k$ an integer and any (fixed) determination of $\log(A/B)$. First of all, if $z \in S_1$ is large and $w(z) = 0$ then (4.42) gives

$$\frac{2c_1/2}{n + 2} z^{(n+2)/2(1 + o(1))} \sim V(z) \sim k\pi,$$

and in particular this leads to $\arg V(z) = o(1)$ and hence $\arg z \sim \theta_0$. Thus zeros $z$ of $w$ in $S_1$ with $|z|$ large must lie near the critical ray.

Now let $k$ be a large positive integer. Then

$$V_k = \frac{1}{2i} \log \frac{A}{B} + k\pi$$

lies near the positive real axis, and so by Lemma 4.3.6 there is a solution $z_k$ of $V(z_k) = V_k$ in $S_1$. Moreover, this $z_k$ is unique by the univalence of $V$ and $z_k$ lies near the critical ray. Now the number of these $V_k$ inside a disc of centre 0 and large radius $R$ is $(1 + o(1))R/\pi$. Hence by (4.38) the number of these zeros $z_k$ of $w$ in $|z| \leq S$ is $(1 + o(1))c_1 S^{(n+2)/2}$ as $S \to \infty$, for some positive constant $c_1$, which gives the following result \[7\].

**Theorem.** Let $b \not\equiv 0$ be a polynomial of degree $n$ and let $w$ be a solution of (4.27) with infinitely many zeros. Then

$$\lim\inf_{r \to \infty} \frac{N(r, 1/w)}{r^{(n+2)/2}} > 0.$$

### 4.4 Asymptotics for equations with transcendental coefficients

For a linear differential equation with transcendental entire coefficients it is in general much harder to obtain asymptotic representations for the solutions. However, when one coefficient is sufficiently dominant it is possible to obtain local representations for solutions with few zeros. For the case $k = 2$ it is interesting to compare the results of the next theorem with the solutions (4.41) obtained for polynomial coefficients.

#### 4.4.1 Theorem

Let $k \geq 2$ and let $A_0, \ldots, A_{k-2}$ be entire functions of finite order, with $A = A_0$ transcendental. Let $E_1$ be a subset of $[1, \infty)$, of infinite logarithmic measure, and with the following property. For each $r \in E_1$ there exists an arc

$$a_r = \{re^{it} : 0 \leq \alpha_r \leq t \leq \beta_r \leq 2\pi\}$$  \hspace{1cm} (4.43)
of the circle $S(0, r)$, such that
\begin{equation}
\lim_{r \to \infty, r \in E_1} \frac{\min \{\log |A(z)| : z \in a_r\}}{\log r} = \infty, \tag{4.44}
\end{equation}
and, if $k \geq 3$,
\begin{equation}
\lim_{r \to \infty, r \in E_1} \max \left\{ \frac{\log^+ |A_j(z)|}{\log |A(z)|} : z \in a_r \right\} = 0, \tag{4.45}
\end{equation}
for $j = 1, \ldots, k - 2$.

Let $f$ be a solution of
\begin{equation}
y^{(k)} + \sum_{j=0}^{k-2} A_j y^{(j)} = 0, \tag{4.46}
\end{equation}
with $\lambda(f) < \infty$. Then there exists a subset $E_2 \subseteq [1, \infty)$ of finite measure, such that for large $r \in E_0 = E_1 \setminus E_2$ the following is true. We have
\begin{equation}
\frac{f'(z)}{f(z)} = c_r A(z)^{1/k} - \frac{k-1}{2k} A'(z) + O(r^{-2}), \quad z \in a_r. \tag{4.47}
\end{equation}

Here $c_r$ is a constant which may depend on $r$, but satisfies $c_r^k = -1$. The branch of $A^{1/k}$ in (4.47) is analytic on $a_r$ (including in the case where $a_r$ is the whole circle $S(0, r)$).

We may summarize (4.44) and (4.45) as saying that, as $r \to \infty$ in $E_1$,
\begin{equation}
|z| + \sum_{1 \leq j \leq k-2} |A_j(z)| \leq |A(z)|^{o(1)} \tag{4.48}
\end{equation}
for $z \in a_r$. To prove the theorem, we start by writing
\begin{equation}
f = Ve^h, \quad \rho(V) < \infty, \tag{4.49}
\end{equation}
where $V$ and $h$ are entire functions. We may assume that $h' \not\equiv 0$ (if $h' \equiv 0$ then $h$ is constant and we can replace $h(z)$ by $h(z) + z$ and $V(z)$ by $V(z)e^{-z}$, which has finite order).

Now
\begin{equation}
\frac{f'}{f} = \frac{V'}{V} + h' \tag{4.50}
\end{equation}
and it is easy to prove by induction that, for $m = 1, 2, \ldots$,
\begin{equation}
\frac{f^{(m)}}{f} = (h')^m + m(h')^{m-1}\frac{V'}{V} + \frac{m(m-1)}{2}(h')^{m-2}h'' + T_{m-2}(h'), \tag{4.51}
\end{equation}
where $T_{m-2}(h')$ is a polynomial in $h'$ of degree at most $m - 2$, with coefficients which are polynomials in the logarithmic derivatives $V^{(j)}/V, h^{(j)}/h', j = 1, \ldots, m$ (for $m = 1$ we set $T_{m-2} = 0$).

Denote positive constants by $M_j$. Substituting (4.50) into (4.46) gives
\begin{equation}
(h')^k + k(h')^{k-1}\frac{V'}{V} + \frac{k(k-1)}{2}(h')^{k-2}h'' + T_{k-2}(h') + \sum_{1 \leq j \leq k-2} A_j \left( (h')^j + j(h')^{j-1}\frac{V'}{V} + \frac{j(j-1)}{2}(h')^{j-2}h'' + T_{j-2}(h') \right) + A_0 = 0. \tag{4.51}
\end{equation}
Claim 1: $h'$ has finite order (and therefore so has $h$).

To prove this suppose $|z| = r$ is large and $|h'(z)| \geq 1$, and divide (4.51) through by $h'(z)^{k-1}$. Since

$$m(r, V^{(j)}/V) = O(\log r), \quad r \to \infty,$$

for each $j \in \mathbb{N}$, and since $A_0, \ldots, A_{k-2}$ have finite order, we obtain

$$m(r, h') \leq S(r, h') + O(\log r) + O(r^{M_0})$$

outside a set $E_2$ of finite measure, giving

$$m(r, h') = O(r^{M_0}), \quad r \not\in E_2.$$

For large $r \in E_2$, choose $s \in [r, 2r] \setminus E_2$ to obtain

$$m(r, h') \leq m(s, h') = O(s^{M_0}) = O(r^{M_0}).$$

This proves Claim 1.

Since $V, h'$ and the coefficients $A_\mu$ have finite order we can use §3.7 to find points $u_m$ with $|u_m| \geq 4$ and $u_m \to \infty$ as $m \to \infty$ such that

$$\left| \frac{V^{(j)(z)}}{V(z)} \right| + \left| \frac{h^{(j)(z)}}{h'(z)} \right| + \frac{|A'_\mu(z)|}{A_\mu(z)} \leq |z|^{M_1}$$

(4.52)

for $1 \leq j \leq k$ and $0 \leq \mu \leq k - 2$ and for all large $z$ satisfying

$$z \not\in U_0 = \bigcup_{m=1}^{\infty} D(u_m, |u_m|^{-M_2}),$$

(4.53)

and this can be done so that

$$\sum_{m=1}^{\infty} |u_m|^{-M_2} < \infty.$$  

(4.54)

Let $U$ be the set obtained by doubling the radii of all the discs of $U_0$. Since the set of $r \geq 1$ such that the circle $S(0, r)$ meets the disc $D(u_m, 2|u_m|^{-M_2})$ has linear measure at most $2|u_m|^{-M_2} \leq 2$, it follows using (4.54) that there exists a set $E_2$ of finite linear measure such that for $r \not\in E_2$ the circle $S(0, r)$ meets none of the discs of $U$.

Let $E_0 = E_1 \setminus E_2$ be as in the statement of the theorem. Then $E_0$ is unbounded. Let $M_3 > 0$ be large compared to $M_1$ and $M_2$.

Claim 2: for large $r \in E_0$ and $z_0 \in a_r$ we have (4.48) and (4.52) for $z \in D(z_0, |z_0|^{-M_3})$.

To prove Claim 2, note first that if $r \in E_0$ is large then the circle $S(0, r)$ does not meet $U$, and so provided $M_3$ was chosen large enough the disc $D(z_0, |z_0|^{-M_3})$ does not meet any of the discs $D(u_m, |u_m|^{-M_2})$, so that (4.52) holds for $z \in D(z_0, |z_0|^{-M_3})$. In particular, integrating $A'_\mu/A_\mu$ shows that

$$|\log |A_\mu(z)/A_\mu(z_0)|| = \left| \int_{z_0}^{z} A'_\mu(t)/A_\mu(t) dt \right| < \ln 2$$
for \( z \in D(z_0, |z_0|^{-M_3}) \), again provided \( M_3 \) was chosen large enough, which gives

\[
|A(z)| \geq \frac{1}{2}|A(z_0)|, \quad |A_\mu(z)| \leq 2|A_\mu(z_0)|,
\]

and so (4.48) for such \( z \). This proves Claim 2.

Claim 3: for large \( r \in E_0 \) and \( z_0 \in a_r \) we have

\[
\frac{1}{2} |A(z)|^{1/k} \leq |h'(z)| \leq 2|A(z)|^{1/k}. \tag{4.55}
\]

Suppose first that \( |h'(z)| < \frac{1}{2}|A(z)|^{1/k} \). Then (4.48), (4.51) and (4.52) give

\[
|A(z)| < 2^{-k}|A(z)| + |A(z)|^{(k-1)/k}(O(|z|^{M_4}) + O(|A(z)|^{o(1)})) < 2^{-k}|A(z)| + |A(z)|^{(k-1)/k}O(|A(z)|^{o(1)}),
\]

which is clearly impossible. Now suppose that \( |h'(z)| > 2|A(z)|^{1/k} \). Then \( h'(z) \) is large and (4.48), (4.51) and (4.52) yield

\[
|h'(z)|^k < 2^{-k}|h'(z)|^k + |h'(z)|^{k-1}(O(|z|^{M_4}) + O(|A(z)|^{o(1)})) < 2^{-k}|h'(z)|^k + |h'(z)|^{k-1}O(|h'(z)|^{o(1)})
\]

which is again impossible. Claim 3 is proved.

For large \( r \in E_0 \) and \( z_0 \in a_r \) we may now define a branch of \( A(z)^{1/k} \), analytic on \( D(z_0, |z_0|^{-M_3}) \), since \( A \) is large there and so in particular non-zero.

Claim 4: we have

\[
h'(z) = c_{20}A(z)^{1/k} + O(r^{M_5}), \quad z \in D(z_0, |z_0|^{-M_3}). \tag{4.56}
\]

Here the constant \( c = c_{20} \) may depend on \( z_0 \) but satisfies \( c^k = -1 \).

To prove Claim 4 set \( u(z) = h'(z)A(z)^{-1/k} \). Dividing (4.51) through by \( A(z) \) and using (4.48), (4.52) and (4.55) we get

\[
0 = u^k + O(r^{M_5}|A(z)|^{-1/k}) + 1 = u^k + 1 + o(1).
\]

Since \( u \) is continuous on \( D(z_0, |z_0|^{-M_3}) \) there is a fixed \( c \) with \( c^k = -1 \) such that \( u = c + o(1) \) on \( D(z_0, |z_0|^{-M_3}) \), and the binomial theorem gives

\[
u = (-1 + O(r^{M_5}|A(z)|^{-1/k}))^{1/k} = c(1 + O(r^{M_5}|A(z)|^{-1/k}))
\]

from which (4.56) follows on multiplying out by \( A(z)^{1/k} \). This proves Claim 4.

For large \( r \in E_0 \) and \( z_0 \in a_r \) we now set

\[
f(z) = W(z) \exp \left( \int_{z_0}^{z} c_{20}A(t)^{1/k} dt \right), \quad \frac{f'(z)}{f(z)} = c_{20}A(z)^{1/k} + \frac{W'(z)}{W(z)}, \quad z \in D(z_0, |z_0|^{-M_3}). \tag{4.57}
\]

By (4.49), (4.52) and (4.56) we have

\[
w(z) = \frac{W'(z)}{W(z)} = O(r^{M_6}), \quad z \in D(z_0, |z_0|^{-M_3}). \tag{4.58}
\]
Now (4.58) and Cauchy’s estimate for derivatives give
\[
\begin{aligned}
w^{(j)}(z_0) &= \frac{j!}{2\pi i} \int_{|z-z_0|=\frac{1}{2}|z_0|-M_3} \frac{w(z)}{(z-z_0)^{j+1}} dz = O(r^{M_7}), \quad j = 1, \ldots, k,
\end{aligned}
\]
and so we get
\[
\frac{W^{(j)}(z_0)}{W(z_0)} = O(r^{M_8}), \quad j = 1, \ldots, k.
\]
Also, writing
\[
H(z) = \int_{z_0}^z c_{z_0} A(t)^{1/k} dt, \quad \frac{H''(z)}{H'(z)} = \frac{A'(z)}{kA(z)}
\]
gives, using (4.52),
\[
\frac{H^{(j)}(z_0)}{H'(z_0)} = O(r^{M_9}), \quad j = 1, \ldots, k.
\]
Substituting \( f = W e^H \) into (4.46) gives at \( z_0 \) (compare (4.51))
\[
(H')^k + k(H')^{k-1} \frac{W'}{W} + \frac{k(k-1)}{2} (H')^{k-2} H'' + T_{k-2}(H') + \sum_{1 \leq j \leq k-2} A_j \left( (H')^j + j(H')^{j-1} \frac{W'}{W} + \frac{j(j-1)}{2} (H')^{j-2} H'' + T_{j-2}(H') \right) + A_0 = 0,
\]
which by (4.60) we may write in the form
\[
k(H')^{k-1} \frac{W'}{W} + \frac{k(k-1)}{2} (H')^{k-2} H'' + O(r^{M_{10}})(H')^{k-2} = 0,
\]
so that
\[
\frac{W'}{W} = -\frac{k(k-1)}{2k} \frac{H''}{H'} + O(r^{-2}) = -\frac{k(k-1)}{2k^2} \frac{A'}{A} + O(r^{-2}),
\]
using (4.60) again. Substituting this estimate into (4.57) we obtain (4.47) at \( z_0 \).

We show now that we may take the same branch of \( A^{1/k} \) and the same \( k \)’th root \( c_r \) of \(-1\) for all \( z_0 \in a_r \). Suppose first that \( \beta_r - \alpha_r < 2\pi \) in (4.43). Then we may define an analytic branch of \( A(z)^{1/k} \) on a simply connected domain containing \( a_r \), since \( A(z) \) is large near \( a_r \). Then we have, for each \( z_0 \in a_r \), using (4.47), (4.48) and (4.52),
\[
\frac{f'(z_0)}{f(z_0) A(z_0)^{1/k}} = c_{z_0} + o(1)
\]
in which \( c_{z_0}^k = -1 \). Since the left hand side is continuous, we see that the root \( c_{z_0} \) is the same for all \( z_0 \in a_r \).

Suppose finally that \( 0 = \alpha_r, \beta_r = 2\pi \). Then we take a small \( \delta > 0 \) and obtain (4.47) on \( a'_r = \{ z : |z| = r, 0 \leq \arg z \leq 2\pi - \delta \} \). Here \( c = c_r \) does not depend on \( \delta \). As we then let \( \delta \to 0+ \) both sides of (4.47) are continued analytically around the circle \( S'(0, r) \) and since the left hand side is continuous and \( A(z) \) is large on \( S(0, r) \) it follows that \( A(z)^{1/k} \) must return to the same branch of \( A^{1/k} \) as we continue once around \( S(0, r) \), since otherwise it would return to the original branch of \( A^{1/k} \) multiplied by a constant \( d \neq 1 \) with \( d^k = 1 \).
Chapter 5

Meromorphic flows

5.1 Introduction

The standard application of complex analysis to (incompressible, irrotational) fluid flow on a plane domain $D$, as given in many textbooks, goes as follows. The velocity of the fluid at $z \in D$

$$\dot{z} = \frac{dz}{dt} = g(z), \quad (5.1)$$

where $g = u + iv$ is analytic on $D$ (with $u, v$ real). In this model, if $D$ is simply connected, the streamlines (trajectories) along which particles of fluid travel are found as follows. Let $G = P + iQ$ be analytic on $D$, with $G' = g$ and $Q = \text{Im} G$. Then a streamline $z(t) = x(t) + iy(t)$ is determined by writing

$$g = u + iv = G' = P_x + iQ_x = Q_y + iQ_x, \quad \frac{dQ}{dt} = Q_x x_t + Q_y y_t = Q_x u - Q_y v = vu - uv = 0,$$

so that $Q$ is constant on the streamline. Hence the trajectories in this model are determined by finding level curves of $Q$.

Consider next a meromorphic flow given by

$$\dot{z} = \frac{dz}{dt} = f(z), \quad (5.2)$$

in which the function $f$ is meromorphic on a simply connected domain $D \subseteq \mathbb{C}$ (this will be the case throughout this chapter). Suitable references for these flows include [21, 22, 26, 29, 30, 48]. A trajectory for (5.2) will mean a continuous $z(t)$, defined on some maximal open interval of $\mathbb{R}$, with

$$z = z(t) \in D, \quad \frac{dz(t)}{dt} = f(z(t)) \in \mathbb{C}.$$

It will be shown in §5.3 that for $z_0 \in D$ with $f(z_0) \neq \infty$ there exists a unique trajectory with $z(0) = z_0$, and that $z(t)$ depends continuously (and indeed analytically for fixed $t$) on $z_0$.

5.1.1 A connection between (5.1) and (5.2)

If in (5.1) we set $f(z) = 1/g(z)$ then we obtain

$$\dot{z} = \frac{dz}{dt} = g(z) = \frac{1}{f(z)} = \frac{f(z)}{|f(z)|^2}. \quad (5.3)$$
This flow is therefore linked to (5.2), insofar as at every point the direction of travel is the same, although in general the speed is not. Indeed, given a trajectory \( z(t) \) of (5.2) through the point \( z(0) \), define \( s = \phi(t) \) by

\[
\phi(0) = 0, \quad \frac{ds}{dt} = \phi'(t) = |f(z(t))|^2.
\]

Then \( \phi \) is strictly increasing, with inverse function \( t = \psi(s) \). Now set \( w(s) = z(t) \), which gives \( w(0) = z(0) \) and

\[
w'(s) = z'(t) \psi'(s) = \frac{f(z(t))}{\phi'(t)} = \frac{f(z(t))}{|f(z(t))|^2} = \frac{f(w(s))}{|f(w(s))|^2},
\]

so \( w(s) = z(t) = z(\psi(s)) \) is a trajectory of (5.3) which passes through the same points as \( z(t) \), but at different speed.

5.2 Examples

5.2.1 Example I

Let \( f(z) = z^2 \) in (5.2). Then any trajectory \( z(t) \) with \( z(0) = 1/T \neq 0 \) has

\[
\frac{1}{z(t)} = \frac{1}{z(0)} - t = T - t.
\]

If \( T \) is real and positive then \( z(t) \) is real and tends to \( +\infty \) as \( t \to T^- \), and to 0 as \( t \to -\infty \). Thus \( z(t) \) follows the positive real axis in the outward direction as \( t \) goes from \( -\infty \) to \( T \).

If \( T \) is real and negative then \( z(t) \) is real and tends to \( -\infty \) as \( t \to T^+ \), and to 0 as \( t \to +\infty \). Thus \( z(t) \) follows the negative real axis in the inward direction as \( t \) goes from \( T \) to \( +\infty \).

If \( T \) is non-real then \( z(t) \) is defined for all real \( t \), and as \( t \to \pm\infty \) we have \( 1/z(t) \to \infty \) and \( z(t) \to 0 \).

5.2.2 Example II

Suppose that \( f(z) = e^z \) in (5.2). Then integration gives

\[
e^{-z(t)} = e^{-z(0)} - t
\]

for any trajectory \( z(t) \). If \( T = e^{-z(0)} \) is real and positive (that is, if \( \Im z(0) = k2\pi \) for some integer \( k \) then \( e^{-z(t)} \to 0 \) and so \( z(t) \to \infty \) as \( t \to T^- \). In this case the trajectory moves from left to right along the horizontal line \( \Im z = k2\pi i \), and the maximal interval of definition of the trajectory is \( (-\infty, T) \).

If \( e^{-z(0)} \) is not real and positive then \( z(t) \) is defined for all \( t \in \mathbb{R} \). As \( t \to \pm\infty \), the term \( e^{-z(t)} \) tends to infinity, so that \( z(t) \) tends to infinity in the left half-plane.

5.3 Existence and uniqueness

The following standard argument shows that for \( z_0 \in D \) with \( f(z_0) \neq \infty \) there is at most one trajectory \( z(t) \) with \( z(0) = z_0 \). Take positive real numbers \( M \) and \( \delta \) such that \( \delta \) is small and \( |z - z_0| \leq \delta \) gives \( |f'(z)| \leq M \). Suppose that \( \eta > 0 \) and that \( z_1(t) \) and \( z_2(t) \) are trajectories which are both defined for \( |t| \leq \eta \), and which satisfy \( z_1(0) = z_2(0) = z_0 \). It may be assumed that \( \eta \) is so small that \( |t| \leq \eta \) gives \( |z_j(t) - z_0| \leq \delta \). Suppose that \( |t| \leq \lambda = \max\{\eta, 1/2M\} \), and that

\[
|z_1(s) - z_2(s)| \leq |z_1(t) - z_2(t)| \quad \text{for} \ |s| \leq |t|.
\]
Then
\[ |z_1(t) - z_2(t)| = \left| \int_0^t f(z_1(s)) - f(z_2(s)) \, ds \right| \]
\[ = \left| \int_0^t \int_{z_2(s)}^{z_1(s)} f'(u) \, du \, ds \right| \]
\[ \leq \int_0^t M |z_1(s) - z_2(s)| \, ds \]
\[ \leq \int_0^t M |z_1(t) - z_2(t)| \, ds \leq \frac{|z_1(t) - z_2(t)|}{2}. \]
This forces \( z_1(t) = z_2(t) \) for \( 0 \leq |t| \leq \lambda \) and repetition of the same argument shows that the trajectories \( z_j(t) \) are identical. Thus if \( f(z_0) = 0 \) then the only trajectory through \( z_0 \) is the trivial solution \( z(t) \equiv z_0 \).

When \( z_0 \in D \) and \( f(z_0) \neq 0, \infty \), the local existence and uniqueness of the trajectory through \( z_0 \) may be established by the following argument. Choose \( r > 0 \) with
\[ \left| \frac{1}{f(z)} - \frac{1}{f(z_0)} \right| \leq \frac{1}{2} \left| \frac{1}{f(z_0)} \right| \]
on \( D(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \} \). Then
\[ G(z) = \int_{z_0}^z \frac{1}{f(u)} \, du = \frac{z - z_0}{f(z_0)} + \int_{z_0}^z \frac{1}{f(u)} - \frac{1}{f(z_0)} \, du \]
(5.4)
satisfies
\[ G(z_0) = 0 \quad \text{and} \quad |G(z_1) - G(z_2)| \geq \frac{|z_1 - z_2|}{2f(z_0)} \]
on \( D(z_0, r) \). Thus \( G \) is analytic and univalent on \( D(z_0, r) \). Taking \( z_2 = z_0 \) and applying Rouché’s theorem shows that \( G(D(z_0, r)) \) contains the disc \( D(0, s) \), where \( s = r/2|f(z_0)| \). Now the local change of variables \( w = G(z) \) gives the flow \( \dot{w} = 1 \), which evidently has a unique trajectory with \( w(0) = 0 \) given by \( w(t) = t \). Hence a trajectory of (5.2) satisfying \( z(0) = z_0 \), and defined at least for \(-s < t < s\), is given uniquely by \( z(t) = G^{-1}(t) \).

### 5.4 Dependence on initial conditions

Suppose that \( z_0 \in D \) is such that the trajectory \( z(t) \) exists and is injective for \( 0 \leq t \leq A \), with \( A > 0 \) and \( z_0 = z(0) \). Then \( f \) has neither zeros nor poles on the curve \( \gamma = \{ z(t) : 0 \leq t \leq A \} \). Hence the function \( G \) of (5.4) is analytic on a simply connected domain \( \Omega \) containing \( \gamma \), and maps \( \gamma \) onto the real interval \( [0, A] \), with \( G(z(t)) = t \) and \( G(z_0) = 0 \). Here \( \Omega \) may be formed as follows: let \( \mathbb{C}^\infty = \mathbb{C} \cup \{ \infty \} \) and map the complement of \( \gamma \) on the Riemann sphere conformally to \( \{ v \in \mathbb{C}^\infty : |v| > 1 \} \) by \( v = h(z) \), so that \( \infty \) is mapped to \( \infty \). Then take \( S_1 > 1 \) such that the images under \( h \) of all zeros of \( f \) in \( D \) lie in \( X_1 = \{ v \in \mathbb{C}^\infty : |v| \geq S_1 \} \). Finally, let \( \Omega \) be the complement of the closed connected subset \( h^{-1}(X_1) \) of \( \mathbb{C}^\infty \).

The next step is to choose a sub-domain \( \Omega' \subseteq \Omega \) such that \( G \) is univalent on \( \Omega' \). By compactness and the argument of §5.3, there exists \( r > 0 \) such that, for each \( t \in [0, A] \), the function \( G \) is univalent on \( D(z(t), r) \), which lies in \( \Omega \). Uniform continuity gives \( R > 0 \) such that \( |z(t) - z(t')| < r/2 \) for all \( t, t' \in [0, A] \) with \( |t - t'| < R \), and for each \( s \in [0, A] \) there exists \( p(s) \in (0, r/2) \) with \( |G(z) - G(z(s))| < R/2 \) for all \( z \) in \( D(z(s), p(s)) \).
5.5 Re-scaling, conjugacy and simple zeros

Suppose that \( z_0 \in D \) with \( f(z_0) \neq 0, \infty \). Then for any \( a, b \in \mathbb{C} \) with \( a \neq 0 \), a re-scaled flow may be defined by
\[
    w = az + b, \quad g(w) = af(z) = af((w - b)/a), \quad \dot{w} = \dot{a} z = af(z) = g(w). \tag{5.5}
\]
Here any prescribed value may be assigned to \( w \), and any prescribed non-zero value to \( g(w) \).

Suppose next that \( f \) has a simple zero at \( z_0 \in D \). Assume without loss of generality that \( z_0 = 0 \), and set \( \alpha = f'(0) \neq 0 \). Then writing
\[
    w = \psi(z), \quad \frac{\psi'(z)}{\psi(z)} = \frac{\alpha}{f(z)} = \frac{1}{z} + \ldots, \tag{5.6}
\]
defines a conformal change of variables near the origin, and yields
\[
    \dot{w} = \psi'(z) \dot{z} = \psi'(z)f(z) = \alpha \psi(z) = \alpha w.
\]
If \( z(t) \) is a trajectory of (5.2) passing near to 0 then \( w(t) = \psi(z(t)) \) is a trajectory of
\[
    \dot{w} = \alpha w. \tag{5.7}
\]

5.5.1 The case where \( \alpha \) is real

In this case the flow (5.7) has a node at 0 (see [22]). The trajectory through a starting point \( w_0 \neq 0 \) satisfies \( w(t) = w_0 e^{\alpha t} \) and is a ray, the direction of flow determined by the sign of \( \alpha \). Thus all trajectories of (5.2) in a punctured neighbourhood of 0 flow towards, or away from, 0.

5.5.2 The case where \( \alpha \) is neither real nor purely imaginary

This case is referred to as a focus. The trajectory through a starting point \( w_0 \neq 0 \) still satisfies \( w(t) = w_0 e^{\alpha t} \), but is a spiral. All trajectories of (5.2) in a punctured neighbourhood of 0 either spiral into, or away from, the fixpoint at the origin.

A focus or node is called attracting (or a sink) if \( \Re \alpha < 0 \), and repelling (or a source) when \( \Re \alpha > 0 \).

5.5.3 The case where \( \alpha \) is purely imaginary

Here the flows (5.2) and (5.7) have a centre at 0. The trajectory of (5.7) through a starting point \( w_0 \neq 0 \) satisfies \( w(t) = w_0 e^{\alpha t} \), but this time a circle, and all trajectories of (5.2) in a punctured neighbourhood of 0 are periodic and flow around the fixpoint at the origin.
5.7 Periodic cycles and their stability

Suppose that \( z_0 \in D \) and \( f(z_0) \neq 0, \infty \) and that the trajectory through \( z_0 \) satisfies \( z_0 = z(0) = z(T) \) for some (minimal) positive \( T \). Then \( z_0 \) lies on a periodic cycle, and its trajectory describes a Jordan curve \( \Gamma \) in \( D \), which has

\[
\int_{\Gamma} \frac{1}{f(u)} \, du = T. \tag{5.9}
\]

It will be shown that if \( z_1 \) lies close enough to \( z_0 \) then \( z_1 \) also lies on a periodic cycle of period \( T \).

The following approach is used in [29, Theorem 2]. For \( z \) close to \( z_0 \) let \( \zeta_z(t) \) be the trajectory with \( \zeta_z(0) = z \). Then \( \zeta_z(T) \) depends analytically on \( z \); to see this, split \( \Gamma \) into two injective sub-trajectories, each taking time \( T/2 \) to describe, and use the method of §5.3 and the chain rule (since \( \zeta_z(T/2) \) depends analytically on \( z \)). But if \( z \) lies on \( \Gamma \) then \( \zeta_z(T) - z = 0 \). So \( \zeta_z(T) = z \) for all \( z \) close to \( z_0 \), by the identity theorem. Continuous dependence on initial conditions and (5.9) imply that the period is the same.

An alternative proof proceeds as follows. Let \( \delta \) be small and positive and take the pre-image \( L = L_\delta(z_0) \) of the real interval \([-\delta, \delta]\) under the function \( i \int_{z_0}^z 1/f(u) \, du \). Then any trajectory which meets \( L \) does so non-tangentially. For \( z \in L \), close to \( z_0 \), follow the trajectory through \( z \) until the first point \( z' \) at which it meets \( L \) again, as it must by continuous dependence on initial conditions, and suppose that \( z \neq z' \). Joining \( z' \) to \( z \) by a sub-arc of \( L \) gives a simple closed curve \( \Gamma' \) for which \( \int_{\Gamma'} 1/f(u) \, du \) is non-real. But Cauchy’s theorem gives \( \int_{\Gamma'} 1/f(u) \, du = \int_{\Gamma} 1/f(u) \, du = T \in \mathbb{R} \).

5.7.1 Periodic cycles not enclosing poles

The following argument is adapted from [21]. Suppose again that \( z_0 \in D \) and \( f(z_0) \neq 0, \infty \) and the trajectory through \( z_0 \) satisfies \( z_0 = z(0) = z(T) \) for some (minimal) positive \( T \), so that \( z(t) \) describes a Jordan curve \( \Gamma \) in \( D \) as \( t \) goes from \( 0 \) to \( T \). Assume that the interior domain of \( \Gamma \) lies in \( D \) but contains no poles of \( f \).

By (5.9) and Cauchy’s theorem, \( \Gamma \) must enclose at least one zero of \( f \), without loss of generality at \( 0 \). Let \( z = g(v) \) be the Riemann mapping from \( D(0,1) \) to the interior domain of \( \Gamma \), with \( g(0) = 0 \). Set

\[
G(z) = \exp \left( \frac{2\pi i}{T} \int_{z_0}^z \frac{1}{f(u)} \, du \right).
\]

5.6 The behaviour near poles

Suppose that \( f(z) \sim c(z - z_0)^{-m} \) as \( z \to z_0 \), for some \( c \neq 0 \) and \( m \geq 0 \). Define a conformal mapping \( w = \phi(z) \) near \( z_0 \) by writing

\[
\phi(z)^{m+1} = \int_{z_0}^z \frac{1}{f(u)} \, du = \frac{(z - z_0)^{m+1}}{(m+1)c} + \ldots.
\]

This gives

\[
(m + 1)\dot{w} = w^{-m}, \quad w^{m+1}(t) = w^{m+1}(0) + t. \tag{5.8}
\]

The equation (5.8) has \( m+1 \) disjoint trajectories tending to \( 0 \) in increasing time, determined by choosing \( w^{m+1}(0) \in (-\infty, 0) \subseteq \mathbb{R} \). Thus (5.2) has \( m+1 \) trajectories tending to \( z_0 \) in increasing time (each taking finite time to do so).

Suppose next that \( D \) contains an annulus \( R < |z| < \infty \) and \( f \) has a pole of order \( n \geq 2 \) at infinity. Setting \( w = 1/z \) gives \( \dot{w} = g(w) = -f(z)/z^2 \), so that \( g \) has a pole of order \( n-2 \) at \( 0 \) and (5.2) has \( n-1 \) trajectories tending to infinity in finite increasing time: this is a result of King and Needham [48, Theorem 5].
near $z_0$. By compactness and the discussion in §5.7, there exists $\delta_1 > 0$ such that if the distance from $w$ to $\Gamma$ is less than $\delta_1$ then $w$ lies on a periodic cycle of period $T$. Moreover, there exists $\delta_2 > 0$ such that if $|w - z_0| < \delta_2$ then the trajectory of (5.2) through $w$ always has distance less than $\delta_1$ from $\Gamma$. Take a small positive $\delta$, so small that the pre-image $L = L_\delta(z_0)$ of the real interval $(-\delta, \delta)$ under the function $i \int_{z_0}^z 1/f(u) \, du$ lies within the disc of centre $\delta_2$ and radius $z_0$.

Let $\Omega$ be the union of all trajectories which meet $L$, each of these having period $T$. Then $\Omega$ is open, and doubly connected, since any point lying between two of these periodic trajectories must also lie on a periodic trajectory, which must in turn meet $L$. The function $G$ clearly continues analytically throughout $\Omega$. Moreover, $G$ maps the trajectory through $w \in \Omega$ injectively onto a circle of centre 0, its radius determined by the real part of

$$\frac{2\pi i}{T} \int_{z_0}^w \frac{1}{f(u)} \, du,$$

and hence by the point at which the trajectory meets $L$. Thus $G$ extends to be analytic and univalent on $\Omega$, mapping $\Omega$ onto an open annulus containing the unit circle.

Moreover, $|G(g(u))|$ is defined and tends to 1 as $|u| \to 1$. If $|u_0| = 1$ then reflection gives an extension of $H = G \circ g$ to a disc $D(u_0, \sigma_0)$ with $u_0 > 0$. A compactness argument and the fact that the intersection of two discs is connected extends $H$ analytically to an annulus $\Omega_1$ given by $1/R < |u| < R$, where $R > 1$. This extension has the property that if $u^* = 1/\pi$ is the reflection of $u$ across the unit circle, then $H(u^*)$ is the reflection of $H(u)$. Thus $H$ is univalent for $1/R < |u| < R$, because it is univalent for $1/R < |u| < 1$. As $u$ crosses the unit circle, so does $H(u)$, and therefore $G^{-1}(H(u))$ crosses $\Gamma$. Hence $g$ may be extended analytically to $D(0, R)$ by writing $g = G^{-1} \circ H$ on $\Omega_1$, and $g(u)$ lies outside $\Gamma$ for $1 < |u| < R$. This property, coupled with the fact that $G^{-1} \circ H$ is univalent on $\Omega_1$, ensures that $g$ is analytic and univalent on $V = D(0, R)$.

Now consider the equation

$$\dot{v} = \frac{f(g(v))}{g'(v)} = \sigma(v) = v\rho(v) \quad (5.10)$$

on $V$. Here $\rho$ is analytic on $V$ since $f(g(0)) = 0$ and $f$ is analytic on $g(V)$ (because $\Gamma$ encloses no poles of $f$). The unit circle is a periodic trajectory of this flow, since $g(v) \in \Gamma$ for $|v| = 1$ and $z = g(v)$ gives $\dot{z} = f(z)$. This means that for $|v| = 1$ the vector $\sigma(v)$ must be perpendicular to the vector $v$, and so $\rho(v)$ must be purely imaginary. But then $\rho(v)$ is constant on $V$ by the maximum principle for harmonic functions. Thus the flow (5.10) reduces to $\dot{v} = \lambda v$, where $\rho(v) \equiv \lambda \in i\mathbb{R} \setminus \{0\}$. Using the Taylor expansion of $f$ and $g$ about 0 shows that $\lambda = f'(0)$.

If $0 < r < R$ then the circle $|v| = r$ is mapped by $z = g(v)$ to a Jordan curve $\Gamma_r$, and

$$\frac{2\pi i}{\lambda} = \int_{|v|=r} \frac{1}{\lambda v} \, dv = \int_{|v|=r} \frac{g'(v)}{f(g(v))} \, dv = \int_{\Gamma_r} \frac{1}{f(z)} \, dz.$$

Setting $r = 1$ shows that $\lambda = f'(0) = 2\pi i/T$. Thus each circle $|v| = r \in (0, 1)$ is a cycle of (5.10) with period $T = 2\pi i / f'(0)$, and every point in $g(V)$ lies on a periodic cycle of (5.2) with the same period. In particular this is true for all points inside $\Gamma$, and all points close enough to $\Gamma$. Also 0 is the only zero of $f$ in $g(V)$, because of the equation $f(g(v)) = \lambda vg'(v)$.

Now let $P$ be the union of $\{0\}$ and all periodic trajectories $\gamma$ which enclose 0 (that is, have non-zero winding number about 0) but enclose no poles of $f$. Then $P$ is open, by the above argument (or by stability of periodic cycles), and is a domain since the interior of each such $\gamma$ contains a neighbourhood of 0.

In fact, $P$ is simply connected, for the following reason. Let $\Lambda$ be a Jordan curve in $P$. For each $z \in \Lambda$ there exists a cycle $\gamma_z \subseteq P$ which encloses $z$, and if $z' \in \Lambda$ lies close enough to $z$ then $z'$ also lies inside $\Gamma_z$. Compactness gives finitely many cycles $\gamma_{z_j} \subseteq P$, each enclosing 0, such that every $z \in \Lambda$
lies inside at least one of them. But these cycles either coincide or are disjoint, and so one of them, \( \gamma \) say, must enclose all the others. But then the interior of \( \gamma \) lies in \( P \), and so does that of \( \Lambda \).

**Lemma 5.7.1 ([21])** Suppose that \( D = \mathbb{C} \) and every \( z_0 \in \mathbb{C} \setminus \{0\} \) lies on a periodic cycle enclosing 0. Then \( f(z) = \alpha z \) for some \( \alpha \in \mathbb{C} \setminus \{0\} \).

**Proof.** The above argument shows that 0 is the only zero of \( f \), and all the cycles have the same minimal period \( T \). The function

\[
F(z) = \exp \left( \frac{2\pi i}{T} \int_1^z \frac{1}{f(u)} \, du \right)
\]

is analytic on the plane, and univalent on, and so inside, each periodic cycle. Thus \( F \) is a univalent entire function and so linear, and so are \( F/F' \) and \( f \).

### 5.8 An example

Following [21], consider the flow

\[
\frac{1}{w^n} + \frac{\lambda}{w} \dot{w} = 1
\]

on \( \mathbb{C} \), where \( \lambda \in \mathbb{C} \). To determine trajectories for (5.11) set

\[
u = w^{1-n}, \quad \dot{v} = (1 - n) \dot{w},
\]

so that \( u \) satisfies, near infinity,

\[
((1 - n)u + \lambda) \frac{\dot{u}}{u} = 1 - n, \quad \left( u + \frac{\lambda}{1 - n} \right) \frac{\dot{u}}{u} = 1.
\]

Let \( R, S/R \) and \( T/S \) be large and positive and consider first a trajectory \( u(t) \) which has

\[
|u(0)| \geq T, \quad \Re \left( u(0) + \frac{\lambda}{1 - n} \right) \geq 0.
\]

Write

\[
v = u + \frac{\lambda}{1 - n} \log u \quad \text{on} \quad D_R^+ = \{ u \in \mathbb{C} : |u| > R, -\pi < \arg u < \pi \}.
\]

Then the trajectory \( u(t) \) has \( u(0) \in D_R^+ \) and \( |\arg u(0)| < \pi/2 + \delta \), where \( \delta > 0 \) can be chosen arbitrarily small, subject to \( T \) being large enough. But then \( |\arg v(0)| < \pi/2 + 2\delta \) and \( \dot{v} = 1 \), and so \( v(t) = v(0) + t \) has \( S \leq |v(t)| \to +\infty \) and \( |\arg v(t)| < \pi/2 + 2\delta \) for \( t \geq 0 \). Since \( \Re v \) is bounded above as \( u \to \partial D_R^+ \), the trajectory for \( u \) stays in \( D_R^+ \) and also tends to infinity, with \( \arg u(t) \to 0 \) as \( t \to +\infty \).

Now suppose that

\[
|u(0)| \geq T, \quad \Re \left( u(0) + \frac{\lambda}{1 - n} \right) \leq 0.
\]

This time writing

\[
v = u + \frac{\lambda}{1 - n} \log u \quad \text{on} \quad D_R^- = \{ u \in \mathbb{C} : |u| > R, 0 < \arg u < 2\pi \}
\]

gives \( \dot{v} = 1 \) again, and shows that, as \( t \to -\infty \), both \( v(t) \) and \( u(t) \) tend to infinity, with \( \arg u(t) \to \pi \).
Figure 5.1: Trajectories of (5.11) near the origin
5.9. MULTIPLE ZEROS

Now take any trajectory \( u(t) \) which has \(|\text{Im} u(0)| \geq T \). Then \( u(0) \in D^+_R \). Continue \( u(t) \) in the directions of both increasing and decreasing \( t \), as far as is possible while keeping \( u \in D^+_R \), and define \( v \) by (5.14). Since this gives \(|\text{Im} v(t)| = |\text{Im} v(0)| \geq S \), whereas \( \text{Im} v \) is bounded on \( \partial D^+_R \), this continuation never causes \( u \) to exit \( D^+_R \), and \( u(t) \rightarrow \infty \) as \( t \rightarrow \pm \infty \). Again \( \arg u(t) \rightarrow 0 \) as \( t \rightarrow +\infty \), and \( \arg u(t) \rightarrow \pi \) as \( t \rightarrow -\infty \).

In summary, \( u(t) \rightarrow \infty \) and \( w(t) \rightarrow 0 \) as \( t \rightarrow +\infty \) when (5.13) holds, and \( u(t) \rightarrow \infty \) and \( w(t) \rightarrow 0 \) as \( t \rightarrow -\infty \) when (5.15) holds. Every trajectory for (5.11) with \(|w(0)| \) small enough is such that at least one of these is satisfied, and there are infinitely many trajectories for which both hold.

Now let \( s > 0 \) be small and take a trajectory \( w(t) \) of (5.11) for which \(|w(0)| = s \), so that \(|u(0)| \) is large. If, at time \( t = 0 \),

\[
\text{Re } \left( \frac{\dot{u}}{u} \right) = \frac{d}{dt} (\log |u(t)|) \geq 0,
\]

then (5.12) implies that (5.13) holds, and so \( u(t) \rightarrow \infty \) and \( w(t) \rightarrow 0 \) as \( t \rightarrow +\infty \). Similarly, any trajectory \( w(t) \) of (5.11) which has \(|w(0)| = s \) and \(|w| \) non-decreasing at time \( t = 0 \) is such that (5.15) holds, so that \( u(t) \rightarrow \infty \) and \( w(t) \rightarrow 0 \) as \( t \rightarrow -\infty \). Therefore every trajectory of (5.11) which meets \(|w| = s \) tends to 0 as \( t \rightarrow +\infty \) or as \( t \rightarrow -\infty \) or both, depending on the sign of \( \text{Re } (\dot{u}/u) \) (and hence of \( \text{Re } (\dot{w}/w) \)).

This gives rise to “elliptic sectors” in the terminology of [22]. Divide up a neighbourhood of \( w = 0 \) into sectors on which \( \text{Im } u \) is alternately positive and negative; each is bounded by rays \( R^+, R^- \) on which \( u \) is real and positive, negative respectively. If \(|\text{Im } u(0)| \geq T \) then \( w(t) \rightarrow 0 \) as \( t \rightarrow +\infty \), with \( w(t) \rightarrow R^+ \) as \( t \rightarrow +\infty \) and \( w(t) \rightarrow R^- \) as \( t \rightarrow -\infty \). If \(|u(0)| \) is large enough but \(|\text{Im } u(0)| < T \) then one of (5.13) and (5.15) is satisfied, and \( u(0) \) is close to \( R^+ \) or \( R^- \), and the trajectory tends to zero in increasing or decreasing time.

5.9 Multiple zeros

Assume that \( f \) has a zero of multiplicity \( n \geq 2 \) at the origin. Then the following argument from [21] shows that (5.2) is conjugate near 0 to an equation of form (5.11). In a neighbourhood of 0 write

\[
\frac{1}{f(z)} = \frac{b_n}{z^n} + \ldots + \frac{b_2}{z^2} + \frac{\lambda}{z} + q(z), \quad b_n \neq 0, \quad q(0) \in \mathbb{C}, \quad (5.16)
\]

and

\[
f_1(z) = \frac{b_n}{(1-n)z^{n-1}} + \ldots + \frac{b_2}{z} + \int_0^z q(u) \, du = \frac{g_1(z)}{z^{n-1}}, \quad f_2(w) = \frac{1}{(1-n)w^{n-1}} = \frac{g_2(w)}{w^{n-1}}, \quad (5.17)
\]

so that \( f_1'(z) = 1/f(z) - \lambda/z \) and \( f_2'(w) = 1/w^n \) by (5.16). Choose \( \mu_0 \) so that

\[
\frac{g_1(0)}{g_2(0)} = b_n = \frac{1}{\mu_0^{n-1}}, \quad (5.18)
\]

using (5.17). For \( v \) near to 0 and \( \mu \) close to \( \mu_0 \), set

\[
H(v, \mu) = \frac{g_2(\mu v)}{\mu^{n-1}} + \lambda v^{n-1} \log \mu - g_1(v). \quad (5.19)
\]

Here

\[
H(0, \mu_0) = \frac{g_2(0)}{\mu_0^{n-1}} - g_1(0) = 0
\]
by (5.18), and
\[
\frac{\partial H}{\partial \mu}(0, \mu_0) = \frac{(1 - n)g_2(0)}{\mu_0^2} \neq 0.
\]
Thus the implicit function theorem (Lemma 5.11.1) gives a function \( \psi(v) \) with \( \psi(0) = \mu_0 \neq 0 \) such that \( \psi \) is analytic near 0 and satisfies there
\[
H(v, \psi(v)) = 0. \tag{5.20}
\]
Set \( \phi(v) = v\psi(v) \), so that \( \phi \) is conformal in a neighbourhood of 0, with \( \phi(0) = 0 \). Then (5.19) and (5.20) yield, near 0,
\[
0 = \frac{g_2(v\psi(v))}{\psi(v)^n} + \lambda v^{n-1} \log \psi(v) - g_1(v)
= \frac{v^{n-1}g_2(\phi(v))}{\phi(v)^n} + \lambda v^{n-1} \log \frac{\phi(v)}{v} - g_1(v),
\]
and so
\[
0 = \frac{g_2(\phi(v))}{\phi(v)^n} + \lambda \log \frac{\phi(v)}{v} - \frac{g_1(v)}{v^{n-1}},
= f_2(\phi(v)) + \lambda \log \frac{\phi(v)}{v} - f_1(v).
\]
Differentiating now yields
\[
0 = \frac{\phi'(v)}{\phi(v)^n} + \lambda \left( \frac{\phi'(v)}{\phi(v)} - \frac{1}{v} \right) - \frac{1}{f(v)} + \frac{\lambda}{v}
= \frac{\phi'(v)}{\phi(v)^n} + \lambda \frac{\phi'(v)}{\phi(v)} - \frac{1}{f(v)}.
\]
Given a trajectory \( z(t) \) of (5.2) near 0 write \( w(t) = \phi(z(t)) \) so that
\[
\dot{w} \left( \frac{1}{w^n} + \frac{\lambda}{w} \right) = \dot{z} \phi'(z) \left( \frac{1}{\phi(z)^n} + \frac{\lambda}{\phi(z)} \right) = \frac{\dot{z}}{f(z)} = 1,
\]
which makes \( w(t) \) a trajectory of (5.11).

**Lemma 5.9.1** Suppose that \( f \) has a zero of multiplicity \( n \geq 2 \) at \( z_0 \in \mathbb{C} \), and let \( \delta > 0 \). Then there exists a Jordan curve \( C \subseteq D(z_0, \delta) \) which surrounds \( z_0 \) and has the following properties: any trajectory \( z(t) \) of (5.2) which passes from outside \( C \) to inside in increasing time tends to \( z_0 \) as \( t \to +\infty \); any trajectory \( z(t) \) which passes from inside \( C \) to outside in increasing time tends to \( z_0 \) as \( t \to -\infty \). Furthermore, there exists at least one trajectory \( z(t) \) which remains inside \( C \) and tends to \( z_0 \) as \( t \to \pm \infty \).

The lemma is proved by assuming that \( z_0 = 0 \) and taking \( C \) to be the pre-image under \( \phi \) of the circle \( |w| = s \), for some small positive \( s \); the asserted properties all hold by §5.8.

### 5.10 Limit points of trajectories

**Lemma 5.10.1** Let the function \( f \) be meromorphic and non-constant on a simply connected domain \( D \subseteq \mathbb{C} \), with finitely many zeros in \( D \), or finitely many poles in \( D \). Let \( z(t) \) be a non-periodic trajectory of (5.2), with maximal interval of definition \( (a_0, b_0) \subseteq \mathbb{R} \). Suppose that \( z_0 \in D \) is a limit point of \( z(t) \) as \( t \to b_0 \), that is, there exist \( s_n \in (a_0, b_0) \) with \( s_n \to b_0 \) such that \( z(s_n) \to z_0 \). Then \( f(z_0) \in \{0, \infty\} \) and \( \lim_{t \to b_0^-} z(t) = z_0 \).
Proof. It suffices to show that $f(z_0) \in \{0, \infty\}$; once this is proved, it must be the case that $\lim_{t \to t_0^-} z(t) = z_0$, since otherwise there exists $z_0^* \in D$ with $f(z_0^*) \not\in \{0, \infty\}$ such that $z_0^*$ is a limit point of $z(t)$ as $t \to t_0$.

Assume then that $z(t)$ and $z_0$ are as in the hypotheses, but that $f(z_0) \neq 0, \infty$. Observe that $z(t)$, being non-periodic, must be injective for $a_0 < t < b_0$. By employing a linear re-scaling $w = az + b$, $g(w) = af(z)$, it may be assumed that $z_0 = 0$ and $f(z_0) = 0$.

If $z(t) \to 0$ as $t \to b_0$ with $t \in (a_0, b_0)$, then so does $u(t) = \phi(z(t))$, where $\phi(z) = \int_0^t 1/f(s) \, ds$ near 0. But then $\dot{u} = 1$, so that $b_0 < +\infty$ and $u(t)$ extends beyond $t = b_0$, as does $z(t)$, contrary to assumption. Hence there exists an arbitrarily small positive $\sigma$ such that $z(t)$ enters and leaves the disc $D(0, \sigma) = \{z \in \mathbb{C} : |z| < \sigma\}$ infinitely often as $t \to b_0$. Because $\sigma$ is small and $f(0) = i$, there exists $\tau > 0$ such that any trajectory which meets $D(0, \sigma)$ crosses the real interval $I = (-2\sigma, 2\sigma)$ non-tangentially from below to above in increasing time, and exits $D(0, 2\sigma)$ after leaving $I$, taking at least time $\tau$ to do so: thus $b_0 = +\infty$.

It is now possible to choose a sequence $(t_n)$, with $a_0 < t_n < t_{n+1} < \infty$, such that $z(t_n)$ and $z(t_{n+1})$ both lie in $I \setminus \{0\}$ but $z(t_n) \not\in I$ for $t_n < t < t_{n+1}$. Then $t_{n+1} \geq t_n + \tau$, so $t_n \to \infty$ and $\lim \inf_{n \to \infty} |z(t_n)| = 0$. Since the trajectory is non-periodic, $z(t_n) \neq z(t_{n+1})$. Let $J_n$ be the open real interval with end-points $z(t_n)$ and $z(t_{n+1})$, let $K_n$ be the arc $\{z(t) : t_n \leq t \leq t_{n+1}\}$, and let $L_n$ be the Jordan curve formed from $J_n$ and $K_n$.

Let $P_n$ be the component of $I \setminus \{z(t_n)\}$ containing $z(t_{n+1})$, and $Q_n$ the component of $I \setminus \{z(t_{n+1})\}$ containing $z(t_n)$. Choose $u_n$ and $v_n$ with $u_n - t_{n+1}$ and $t_n - v_n$ small and positive. Then $z(u_n)$ lies in a component $\Omega_{1,n}$ of $(\mathbb{C} \cup \{\infty\}) \setminus L_n$, as do all points lying just above the open interval $P_n$. Similarly, $z(v_n)$ and all points lying just below $Q_n$ belong to the same component $\Omega_{2,n}$ of $(\mathbb{C} \cup \{\infty\}) \setminus L_n$. The fact that $J_n = P_n \cap Q_n$ gives $\Omega_{1,n} \neq \Omega_{2,n}$. All points $z(t)$ with $t > t_{n+1}$ also lie in $\Omega_{1,n}$, because $z(t)$ cannot meet $K_n$ for $t > t_{n+1}$ and cannot cross $J_n$ from above as $t$ increases. This gives $z(t_m) \not\in Q_n$ for all $m > n + 1$, because the contrary case leads to $z(v_m) \in \Omega_{2,n}$. It follows that the sequence $z(t_n)$ is monotone, with $z(t_n) \to 0$ as $n \to \infty$.

Now the integral of $1/f(z)$ along $K_n$ is real, by (5.2), but that along $J_n$ is not, and so choosing $n$ large enough makes

$$I_n = \text{Im} \left( \int_{L_n} \frac{1}{f(z)} \, dz \right)$$

arbitrarily small but non-zero. Thus the lemma is proved if $f$ has finitely many zeros in $D$, or if the trajectory $z(t)$ remains within a compact subset of $D$, because in these cases there are only finitely many zeros of $f$ which may lie inside $L_n$, and so only finitely many possible values of $I_n$, by the residue theorem.

Assume now that $f$ has infinitely many zeros, and hence infinitely many poles, in $D$. Since $f(z_0) \neq 0, \infty$ by assumption, it may be assumed further that the trajectory $z(t)$ does not remain within any compact subset of $D$ as $t \to +\infty$. Thus $\Omega_{2,n}$ must be the bounded component of $(\mathbb{C} \cup \{\infty\}) \setminus L_n$, and these bounded components satisfy $\Omega_{2,n} \subseteq \Omega_{2,n+1}$. Let $\Lambda_n$ be the domain obtained by deleting from $\Omega_{2,n+1}$ all points in the closure of $\Omega_{2,n}$, and let $n_0 \in \mathbb{N}$ be so large that for $n \geq n_0$ there are no poles of $f$ in $\Lambda_n$. Let $n \geq n_0$ and let $\Sigma_n$ be the set of $w \in \Lambda_n$ with $f(w) \neq 0$. For $w \in \Sigma_n$, follow the trajectory $\zeta_w$ through $w$ in decreasing time. The resulting path $\sigma_w$ cannot exit $\Omega_{2,n+1}$, and so remains within a compact subset of $D$. Thus by the argument of the previous paragraph, with time reversed, $\sigma_w$ must either cross $J_n$, or be periodic, or tend to a zero of $f$ in $\Lambda_n$. Here the set of $w \in \Sigma_n$ corresponding to each of these finitely many possibilities is open, by §5.4, §5.5 and §5.7, as well as Lemma 5.9.1. But there are points $w \in \Lambda_n$, close to the trajectory $z(t)$, for which $\sigma_w$ does cross $J_n$, and so by connectedness the same is true for all $w \in \Sigma_n$. A similar argument shows that for every $w \in \Sigma_n$ the trajectory $\zeta_w$ exists $\Omega_{2,n+1}$ through $J_{n+1}$ in increasing time. However, if $v$ is a zero of $f$ in $\Lambda_n$, then §5.5 and §5.7, together with Lemma 5.9.1, show that there exists $w \neq v$, close to $v$, such
that either $\zeta_w$ is periodic or $\zeta_w$ tends to $w$ in increasing or decreasing time. Hence $f$ has no zeros in $\Lambda_n$. It follows that there are only finitely many possible values for $I_n$, and this is a contradiction. □

Suppose now that $f$ is non-constant and meromorphic in $\mathbb{C}$ in (5.2), with finitely many poles. If $\gamma(t)$ is a simple trajectory for (5.2), with maximal interval of definition $(\alpha, \beta) \subseteq \mathbb{R}$, then it follows from Lemma 5.10.1, with $D = \mathbb{C}$, that the initial and final end-points $\gamma^- = \lim_{t \to \alpha^+} \gamma(t) \in \mathbb{C} \cup \{\infty\}$ and $\gamma^+ = \lim_{t \to \beta^-} \gamma(t) \in \mathbb{C} \cup \{\infty\}$ both exist, and may coincide.

If $\gamma^+ = z_0 \in \mathbb{C}$ and $f(z_0) \neq \infty$ then $z_0$ must be a sink or a multiple zero of $f$ (see §5.6), and the trajectory takes infinite time to reach $z_0$ (that is, $\beta = +\infty$). To see this, take $C > 0$ and $m \in \mathbb{N}$ with $|f(z)| \leq C|z - z_0|^m$ as $z \to z_0$. Let $n$ be large and consider any $z(t)$ such that $|z(t) - z_0| = 2^{-n}$ and $2^{-n-1} \leq |z(t) - z_0| \leq 2^{-n}$ for $t_n \leq t \leq t_{n+1}$. This yields

$$2^{-n-1} \leq |z(t_{n+1}) - z(t_n)| = \left| \int_{t_n}^{t_{n+1}} f(z(t)) \, dt \right| \leq (t_{n+1} - t_n)C2^{-nm}$$

and so $t_{n+1} - t_n \geq C^{-1}2^{(m-1)n-1} \geq 1/2C$. Similar remarks apply if $\gamma^- \in \mathbb{C}$.

### 5.11 The analytic implicit function theorem

**Lemma 5.11.1** Let the function $P(w, z)$ be $C^1$ on a neighbourhood of $(w_0, z_0) \in \mathbb{C}^2$ and satisfy the following: for each $w$ near to $w_0$, the functions $P(w, z)$ and $P_w(w, z)$ are analytic functions of $z$ on a neighbourhood of $z_0$; for each $z$ near to $z_0$, the function $q(w) = P(w, z)$ is an analytic function of $w$ on a neighbourhood of $w_0$.

Assume that $P(w_0, z_0) = 0$, and that $P_w(w_0, z_0) \neq 0$. Then there exists an analytic function $\phi(z)$ on a neighbourhood of $z_0$, with $\phi(z_0) = w_0$, such that $P(\phi(z), z) = 0$ near $z_0$.

**Proof.** It may be assumed that $w_0 = z_0 = 0$. The function $g(w) = P(w, 0)$ is analytic near 0 with $g(0) = 0$ and $g'(0) = P_w(0, 0) \neq 0$. Thus $g$ has a simple zero at 0 and, if $\varepsilon$ is small and positive,

$$\frac{1}{2\pi i} \int_{|w| = \varepsilon} \frac{g'(w)}{g(w)} \, dw = 1,$$

with all integrations once counter-clockwise. In particular, $g(w) = P(w, 0) \neq 0$ for $|w| = \varepsilon$. Hence if $|z|$ is small enough then $P(w, z) \neq 0$ for $|w| = \varepsilon$, since $P$ is $C^1$, and

$$\frac{1}{2\pi i} \int_{|w| = \varepsilon} \frac{P_w(w, z)}{P(w, z)} \, dw = 1,$$

by continuity of the integral and the argument principle applied to $q(w) = P(w, z)$. Thus, again if $|z|$ is small enough, the equation $P(w, z) = 0$ has a unique root $w = \phi(z) \in D(0, \varepsilon)$, and the residue theorem gives

$$\phi(z) = \frac{1}{2\pi i} \int_{|w| = \varepsilon} \frac{wP_w(w, z)}{P(w, z)} \, dw,$$

so that $\phi(z)$ is analytic near 0.
Chapter 6

Univalent functions and the hyperbolic metric

6.1 Basic results on univalent functions

6.1.1 The area theorem

Let \( g(z) = 1/z + \sum_{n=1}^{\infty} b_n z^n \) be analytic and univalent in \( 0 < |z| < 1 \). Then

\[
\sum_{n=1}^{\infty} |b_n|^2 \leq 1.
\]

(6.1)

Proof. We can assume that \( b_1 = a \) is real and non-negative, because if \( g \) is univalent and \( |\alpha| = 1 \) then \( \alpha g(z\alpha) = 1/z + \alpha^2 b_1 z + \ldots \) is also univalent, and this does not change \( |b_n| \). Our assumptions imply that the power series has radius of convergence at least 1. We extend \( g \) to a one-one meromorphic function on \( D(0,1) \) by setting \( g(0) = \infty \). If \( 0 < r < 1 \) then

\[
J_r(t) = g(re^{it}), \quad 0 \leq t \leq 2\pi,
\]

is a simple closed curve. For finite \( w \) not on \( J_r \), the winding number satisfies

\[
n(J_r, w) = \frac{1}{2\pi i} \int_{J_r} \frac{1}{u-w} du = \frac{1}{2\pi i} \int_{|z|=r} \frac{g'(z)}{g(z)-w} dz
\]

and by the argument principle this is zero or non-zero, depending on whether or not \( g \) takes the value \( w \) in \( 0 < |z| < r \).

For \( 0 < r < 1 \) let \( A(r) \) be the area of the set of finite complex values not taken by \( g \) in \( D(0,r) \): this is the same as the area enclosed by \( J_r \). For \( 0 < s < S < 1 \) we have

\[
A(s) - A(S) = \int_{s \leq |z| \leq S} |g'(z)|^2 r dr d\theta
\]

because the integral on the RHS (computed using polar coordinates) is the area of the image under \( g \) of \( s \leq |z| \leq S \). Differentiating gives

\[
A'(r) = -\int_{|z|=r} |g'(z)|^2 r d\theta.
\]
We write
\[ g'(z) = -z^{-2} + \sum_{n=1}^{\infty} nb_n z^{n-1}, \quad \overline{g'(z)} = -(\overline{z})^{-2} + \sum_{n=1}^{\infty} n\overline{b_n}(\overline{z})^{n-1}, \]
and use the elementary fact that, for \( j, k \in \mathbb{Z} \),
\[ \int_{|z|=r} z^j \overline{z}^k d\theta \]
is 0 unless \( j = k \), in which case the integral is \( 2\pi r^{2j} \). Thus we get
\[ -A'(r) = 2\pi \left( r^{-3} + \sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2n-1} \right), \]
and by integration there is a constant \( C \) such that
\[ A(r) = C + \pi r^{-2} - \pi \sum_{n=1}^{\infty} n |b_n|^2 r^{2n}, \quad 0 < r < 1. \]
We assert that \( C = 0 \). Suppose first that \( g(z) = 1/z + az \), still with \( a > 0 \). With this assumption,
\[ g(re^{i\theta}) = (1/r + ar) \cos \theta - i(1/r - ar) \sin \theta \]
describes an ellipse \( E_r \) enclosing an area
\[ \pi(1/r + ar)(1/r - ar) = \pi(r^{-2} - |b_1|^2 r^2) = \pi r^{-2} + O(r^2). \]
In the general case, as \( |z| = r \to 0 \) we have
\[ g(z) = 1/z + b_1 z + O(r^2) \]
and so the distance from \( J_r \) to the ellipse \( E_r \) is \( O(r^2) \). Since \( E_r \) has length \( O(1/r) \), the difference between \( A(r) \) and the area enclosed by \( E(r) \) is \( O(r) \) as \( r \to 0 \), and this gives \( A(r) = \pi r^{-2} + O(r) \) and so \( C = 0 \). Using the fact that \( A(r) \geq 0 \), and letting \( r \to 1 \), we deduce the lemma.

6.1.2 The class \( S \)
Suppose that \( h \) is analytic and univalent in \( D(0, 1) \). Then \( h'(0) \neq 0 \) and the function
\[ H(z) = \frac{h(z) - h(0)}{h'(0)}, \quad H(0) = 0, \quad H'(0) = 1, \]
is also analytic and univalent in \( D(0, 1) \). This normalization gives us the class \( S \) of functions
\[ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z + \sum_{n=2}^{\infty} a_n z^n \]
which are analytic and univalent in \( D(0, 1) \).
6.1.3 Bieberbach’s theorem

Let \( f \in S \). Then \(|a_2| \leq 2\). Further, equality holds if and only if \( f \) is a Koebe function

\[
f(z) = k_\theta(z) = \frac{z}{(1 - z e^{i\theta})^2} = z + 2z^2 e^{i\theta} + \ldots,
\]

(6.2)

for some real \( \theta \).

Proof. Take \( f \in S \), and write

\[
f(z^2) = z^2 (1 + a_2 z^2 + a_3 z^4 + \ldots) = z^2 G(z)
\]

so that, since \( G(z) \neq 0 \) in \( D(0,1) \), the function \( F \) given by

\[
F(z) = z G(z)^{1/2} = z (1 + \frac{1}{2} a_2 z^2 + \ldots)
\]

is analytic in \( D(0,1) \). We claim that \( F \) is univalent on \( D(0,1) \). To see this, suppose that \( F(u) = \pm F(v) \). Then \( f(u^2) = F(u)^2 = F(v)^2 = f(v^2) \) and so \( u^2 = v^2, u = \pm v \). But \( v = -u \neq 0 \) gives \( F(v) = -F(u) \neq 0 \), since the power series for \( F \) has only odd powers, and so \( F(u) = F(v) \) forces \( u = v \).

Now we know that \( F \) is univalent on \( D(0,1) \), we consider

\[
g(z) = \frac{1}{F(z)} = \frac{1}{z} - \frac{1}{2} a_2 z + \ldots = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,
\]

which is analytic and univalent on \( 0 < |z| < 1 \). From (6.1) we get \( \sum_{n=1}^{\infty} n |b_n|^2 \leq 1 \) and so in particular \( |b_1| = \frac{1}{2} |a_2| \leq 1 \). If \( |a_2| = 2 \) then we must have \( |b_1| = 1 \) and \( b_n = 0 \) for \( n \geq 2 \) and so, for some real \( \theta \),

\[
g(z) = \frac{1}{z} - z e^{i\theta} = \frac{1 - z^2 e^{i\theta}}{z}, \quad F(z) = \frac{z}{1 - z^2 e^{i\theta}}, \quad f(z) = F(z^{1/2})^2 = \frac{z}{(1 - z e^{i\theta})^2}.
\]

6.1.4 Koebe quarter theorem

Suppose that \( f \in S \) and that \( f \) does not take the finite value \( w \) in \( D(0,1) \). Then \(|w| \geq 1/4 \). If \(|w| = 1/4 \) then \( f \) is given by (6.2), with \( w = -\frac{1}{4} e^{-i\theta} \) for some real \( \theta \).

Proof. Assume \( f(z) \neq w \). Then

\[
\frac{w f}{w - f} = -w + \frac{w^2}{w - f} = z + (a_2 + 1/w) z^2 + \ldots
\]

is also in \( S \). This gives, by Bieberbach’s theorem,

\[
|a_2 + 1/w| \leq 2, \quad |1/w| \leq 2 + |a_2| \leq 4.
\]

Also if \(|1/w| = 4 \) then \( 1/w = -4 e^{i\theta} \) for some real \( \theta \). Since \(|a_2| \leq 2 \) and \(|a_2 + 1/w| \leq 2 \) we must have \( a_2 = 2 e^{i\theta} \) and so \( f \) is given by (6.2).

Note that

\[
k_\theta(z) = e^{-i\theta} k_0(z e^{i\theta}).
\]

Also

\[
k_0(z) = \frac{z}{(1 - z)^2} = \frac{1}{4} \left( \frac{1 + z}{1 - z} \right)^2 - \frac{1}{4},
\]

and this maps \( D(0,1) \) univalently onto the region obtained by deleting from the complex plane the half-line \( \{ w \in \mathbb{R} : w = x < -\frac{1}{3} \} \).
6.1.5 The distance to the boundary

Let \( f \in S \). By the Koebe quarter theorem we know that the distance from 0 to the boundary of \( f(D(0,1)) \) is at least \( \frac{1}{2} \). On the other hand, this distance is at most 1, for otherwise the inverse function \( F \) is defined and analytic on a disk \( D(0, R) \) with \( R > 1 \), and Schwarz’ lemma applied to \( h(z) = F(Rz) \) gives \( 1 = |1/f'(0)| = |F'(0)| \leq 1/R \).

Suppose now that \( a \) is any point in \( D(0,1) \), and that \( g \) is analytic and univalent on \( D(0,1) \). Set

\[
G(z) = g\left(\frac{z + a}{1 + \overline{a}z}\right).
\]

Then \( G'(0) = (1 - |a|^2)g'(a) \) and

\[
H(z) = \frac{G(z) - G(0)}{(1 - |a|^2)g'(a)}
\]

is in \( S \). Thus the distance from 0 to the boundary of \( H(D(0,1)) \) is at least \( \frac{1}{4} \) and at most 1. This gives

\[
\frac{1}{4}(1 - |a|^2)|g'(a)| \leq \text{dist}\{g(a), \partial(g(D(0,1)))\} \leq (1 - |a|^2)|g'(a)|. \tag{6.3}
\]

6.1.6 Koebe distortion theorem

Let \( f \in S \). Then for \( |z_0| = r < 1 \) we have

\[
\frac{1 - r}{(1 + r)^3} \leq |f'(z_0)| \leq \frac{1 + r}{(1 - r)^3}. \tag{6.4}
\]

**Proof.** Set

\[
g(z) = f\left(\frac{z + z_0}{1 + \overline{z_0}z}\right) = b_0 + b_1z + b_2z^2 + \ldots.
\]

Then \( g \) is analytic and univalent in \( D(0,1) \) and

\[
b_0 = f(z_0), \quad b_1 = g'(0) = f'(z_0)(1 - r^2), \quad 2b_2 = g''(0) = (1 - r^2)^2f''(z_0) - 2z_0(1 - r^2)f'(z_0).
\]

Applying Bieberbach’s theorem to \((g(z) - g(0))/g'(0)\) we get \(|g''(0)| \leq 4|g'(0)|\), and so

\[
f''(z_0)/f'(z_0) - 2z_0(1 - r^2)^{-1}
\]

has modulus at most \( 4(1 - r^2)^{-1} \). Multiplying through by \( z_0r^{-1} \) we get

\[
\left|\frac{z_0f''(z_0)}{rf'(z_0)} - \frac{2r}{1 - r^2}\right| \leq \frac{4}{1 - r^2}. \tag{6.5}
\]

If we write \( G = \log f'(z), \zeta = \log z, \rho = |z| \) then the Cauchy-Riemann equations give

\[
\frac{\partial \log |f'(z)|}{\partial \log \rho} = \text{Re}\left(\frac{dG}{d\zeta}\right) = \text{Re}\left(\frac{zf''(z)}{f'(z)}\right).
\]

Thus

\[
\frac{\partial \log |f'(z)|}{\partial \rho} = \rho^{-1}\text{Re}\left(\frac{zf''(z)}{f'(z)}\right)
\]

and so (6.5) tells us that

\[
\frac{2\rho - 4}{1 - \rho^2} = \frac{2\rho}{1 - \rho^2} - \frac{4}{1 - \rho^2} \leq \frac{\partial \log |f'(z)|}{\partial \rho} \leq \frac{2\rho}{1 - \rho^2} + \frac{4}{1 - \rho^2} = \frac{2\rho + 4}{1 - \rho^2}.
\]

Integrating from 0 to \( r \) with respect to \( \rho \) using partial fractions, and then taking exponentials, we get

(6.4).
6.2 The hyperbolic metric

We begin with a refinement of the standard Schwarz lemma.

6.2.1 The Schwarz-Pick lemma

Let \( f : D(0,1) \rightarrow D(0,1) \) be analytic, and let \( a \in D(0,1) \). Then we have

\[
\left| \frac{f(z) - f(a)}{1 - f(a)f(z)} \right| \leq \left| \frac{z - a}{1 - \overline{a}z} \right| \tag{6.6}
\]

and

\[
\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \tag{6.7}
\]

for all \( z \) in \( D(0,1) \). If there exists \( z \) in \( D(0,1) \) for which equality holds in (6.7), or \( z \in D(0,1) \setminus \{a\} \) for which equality holds in (6.6), then \( f \) is a conformal map of (i.e. a one-one analytic function from) the unit disc \( D(0,1) \) onto itself.

The conformal maps \( f \) of \( D(0,1) \) onto itself have the form

\[
f(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}z} \tag{6.8}
\]

for some constants \( \theta, a \) with \( \theta \) real and \( |a| < 1 \). For such \( f \), equality holds in both (6.6) and (6.7).

Proof. It is easy to check that \( f \) of the form (6.8) is a conformal map of \( D(0,1) \) onto itself: \( f \) is Möbius and so one-one, and \( f(a) = 0 \), \( f(z) \rangle = 1 \) for \( |z| = 1 \). We denote the collection of mappings of form (6.8) by \( A \). It is easy to check that \( A \) is a group under composition.

Next let \( f \) map \( D(0,1) \) analytically into itself, and let \( a \in D(0,1) \), and define \( G \) by

\[
G(z) = G_1(z)G_2(z), \quad G_1(z) = \frac{f(z) - f(a)}{1 - f(a)f(z)} , \quad G_2(z) = \frac{1 - \overline{a}z}{z - a}.
\]

Then \( G \) has a removable singularity at \( a \) and so is analytic in \( D(0,1) \). Further, we have \( |G_1(z)| \leq 1 \) on \( D(0,1) \), while \( |G_2(z)| \to 1 \) as \( |z| \to 1 \). So the maximum principle gives \( |G(z)| \leq 1 \) on \( D(0,1) \).

There are now two possibilities. The first is that \( G \) is a constant of modulus 1, so that equality holds in (6.6). Further, we can solve for \( f \), and since \( A \) is a group it follows that \( f \) is in \( A \), and is a conformal map of \( D(0,1) \) onto itself. Finally, since

\[
|f'(a)| = \lim_{z \to a} \left| \frac{f(z) - f(a)}{z - a} \right| , \tag{6.9}
\]

we get equality in (6.7).

In the converse direction, suppose that \( f \) is a conformal map of \( D(0,1) \) onto itself. Then \( |f(z)| \to 1 \) as \( |z| \to 1 \), and so \( |G(z)| \to 1 \) as \( |z| \to 1 \). Since \( f \) is one-one, \( G(z) \) is non-zero on \( D(0,1) \) (the zeros cancel out) and so \( |G(z)| = 1 \) on \( D(0,1) \), by the maximum principle applied to \( G \) and \( 1/G \). It follows that \( f \in A \) and that equality holds in (6.6) and (6.7).

Finally, suppose that \( f \) is not a conformal map of \( D(0,1) \) onto itself, and take \( a \in D(0,1) \). Then \( |G(z)| < 1 \) for all \( z \) in \( D(0,1) \), so that we have strict inequality in (6.6), for \( z \neq a \). Further, \( a \) lies in a compact set \( K_a \) on which \( |G(z)| \leq k_a < 1 \), and (6.9) gives us (6.7) for \( z = a \), with strict inequality. Since \( a \) is arbitrary the proof is complete.
6.2.2 Lemma

Let $\gamma$ be a smooth contour joining $a$ to $b$, with $|a| \leq |b|$, and let $f(z) = g(|z|) \geq 0$ be a function of $|z|$ which is upper semi-continuous on $\gamma$. Then

$$\int_{\gamma} f(z) |dz| \geq \int_{[a]}^{[b]} g(t) dt.$$  \hspace{1cm} (6.10)

**Proof.** Suppose first that $g$ is continuous. Take $\delta > 0$ and a partition $|a| = x_0 < x_1 < \ldots < x_n = |b|$ such that

$$\max \{g(t) : x_{j-1} \leq t \leq x_j\} - \delta < m_j = \min \{g(t) : x_{j-1} \leq t \leq x_j\}$$

for each $j$. Then for each $j$ there is a sub-path $\gamma_j$ of length at least $x_j - x_{j-1}$ and lying in $x_{j-1} \leq |z| \leq x_j$, on which $f(z) \geq m_j$. Thus

$$\int_{\gamma} f(z) |dz| \geq \sum_{j=1}^{n} m_j (x_j - x_{j-1}) \geq \int_{[a]}^{[b]} (g(t) - \delta) dt.$$

This proves (6.10) when $g$ is continuous. In the general case take continuous $g_n \downarrow g$ so that

$$\int_{\gamma} g_n(|z|) |dz| = \int_{A}^{B} g_n(|\gamma'(s)|)|\gamma'(s)| ds \to \int_{A}^{B} g(|\gamma'(s)|)|\gamma'(s)| ds = \int_{\gamma} g(|z|) |dz|,$$

by the monotone convergence theorem applied to $g_1 - g_n$, and

$$\int_{\gamma} g_n(|z|) |dz| \geq \int_{[a]}^{[b]} g_n(t) dt \geq \int_{[a]}^{[b]} g(t) dt.$$

6.2.3 The hyperbolic metric in the disc

Let $\gamma$ be a piecewise smooth contour in the unit disc $D(0, 1)$. The hyperbolic (non-Euclidean) length of $\gamma$ is defined to be

$$L_\gamma = \int_{\gamma} \frac{2|dz|}{1 - |z|^2},$$

in which $|dz|$ indicates that the integration is with respect to arc length (sometimes the factor 2 is omitted).

If $f$ is a conformal map of $D(0, 1)$ onto itself then the hyperbolic length of $f(\gamma)$ is

$$\int_{f(\gamma)} \frac{2|dw|}{1 - |w|^2} = \int_{\gamma} \frac{2|f'(z)||dz|}{1 - |f(z)|^2} = \int_{\gamma} \frac{2|dz|}{1 - |z|^2} = L_\gamma$$

using the fact that we have equality in (6.7). Thus the hyperbolic length is invariant under $f$.

Now suppose that $\gamma$ joins $0$ to $r \in (0, 1)$. Then Lemma 6.2.2 gives

$$L_\gamma \geq \int_{0}^{r} \frac{2dx}{1 - x^2} = \log \left( \frac{1 + r}{1 - r} \right).$$

In particular, the shortest path (in terms of hyperbolic length) from 0 to $r$ is the straight line segment.

If $z_1, z_2$ are in $D(0, 1)$ we now define the hyperbolic distance $[z_1, z_2]$ to be the infimum of $L_\gamma$ over all piecewise smooth contours $\gamma$ joining $z_1$ to $z_2$ through $D(0, 1)$. The distance is not altered if we apply
a conformal map $f$ of $D(0,1)$ onto itself. We can choose $f$ so that $f(z_1) = 0, f(z_2) = r > 0$, and the shortest path between these two points is then the straight line $S$ from 0 to $r$. Hence the shortest path from $z_1$ to $z_2$ is the arc $f^{-1}(S)$, which is either a straight line through 0 or (since $f$ is M"{o}bius) a circular arc which meets the circle $|z| = 1$ at right-angles. In particular

$$[0,r] = \log \left( \frac{1+r}{1-r} \right).$$

### 6.2.4 The hyperbolic metric on a simply connected domain

Let $D$ be a simply connected domain in the complex plane, not the whole plane. Then by the Riemann mapping theorem, there exists an analytic function $H$ mapping $D$ one-one onto $D(0,1)$. We can thus define the hyperbolic distance between $w_1, w_2$ in $D$ to be the hyperbolic distance between $H(w_1), H(w_2)$ in $D(0,1)$. This does not depend on which $H$ we choose, because if $G$ is another conformal map of $D$ onto $D(0,1)$ then $H \circ G^{-1}$ is a conformal map of $D(0,1)$ onto itself, so that $[H(w_1), H(w_2)] = [G(w_1), G(w_2)]$.

The next lemma gives a useful estimate for the hyperbolic metric on a simply connected domain. It is related in style and applicability to §14.1.6.

### 6.2.5 Lemma

Let $D$ be a simply connected domain in the finite plane, not containing the origin, and let $w_1, w_2 \in D$. For $t > 0$ let $t\theta(t)$ be the length of the longest open arc of the intersection of $D$ and the circle $|w| = t$. Then

$$[w_1, w_2]_D \geq \int_{|w_1|}^{[w_2|} \frac{dt}{t\theta(t)}, \quad (6.11)$$

**Proof.** Let $h$ map $D(0,1)$ analytically and univalently onto $D$, with $h(z_j) = w_j$. Let $\gamma$ be the hyperbolic geodesic (shortest path with respect to the hyperbolic metric) from $z_1$ to $z_2$. Then, with $\Gamma = h(\gamma)$, the estimate (6.3) gives

$$[w_1, w_2]_D = [z_1, z_2] = \int_{\gamma} \frac{2|dz|}{1-|z|^2} = \int_{\Gamma} \frac{2|dw|}{h'(z)(1-|z|^2)} \geq \int_{\Gamma} \frac{|dw|}{2\text{dist}\{w, \partial D\}} \geq \int_{\Gamma} \frac{|dw|}{|w|\theta(|w|)}$$

since $w$ can be joined to a point of $\partial D$ by a circular arc of length at most $|w|\theta(|w|)/2$. Since $D$ is open, $1/t\theta(t)$ is upper semi-continuous (see §14.1.3) and Lemma 6.2.2 now gives (6.11).
Chapter 7

Harmonic and subharmonic functions

7.1 Harmonic functions and Poisson’s formula

7.1.1 Lemma

If the real-valued function \( u(x,y) \) has continuous first and second partial derivatives on a domain \( D \) in \( \mathbb{R}^2 \) then \( u_{xy} = u_{yx} \).

To prove this, take any closed rectangle \( I = [a,b] \times [c,d] \) and use Fubini’s theorem to get

\[
\int_I u_{xy} \, dx \, dy = \int_a^b \int_c^d u_{xy} \, dy \, dx = \int_a^b u_x(x,d) - u_x(x,c) \, dx = u(b,d) - u(a,d) - u(b,c) + u(a,c).
\]

But

\[
\int_I u_{yx} \, dx \, dy = \int_c^d \int_a^b u_{yx} \, dx \, dy = \int_c^d u_y(b,y) - u_y(a,y) \, dy = u(b,d) - u(b,c) - u(a,d) + u(a,c).
\]

Since the integrals are always the same the functions must agree: if not then by continuity we have without loss of generality \( u_{xy} > u_{yx} \) on some rectangle.

7.1.2 Harmonic functions

Let \( D \) be a domain in \( \mathbb{C} \) (or \( \mathbb{R}^2 \): we shall use these interchangeably). A function \( u : D \to \mathbb{R} \) is called harmonic if \( u \) has continuous first and second partial derivatives and satisfies Laplace’s equation

\[
\Delta u = \nabla^2 u = u_{xx} + u_{yy} = 0.
\]

By the Cauchy-Riemann equations, if \( f = u + iv \) (\( u, v \) real) is analytic then \( u, v \) are harmonic.

Also, if \( u(z) = u(x,y) \) is harmonic then Lemma 7.1.1 shows that \( f = u_x - iu_y \) is analytic. If, in addition, \( D \) is simply connected then

\[
F = u(a) + \int_a^z f(w) \, dw = U + iV
\]

is analytic on \( D \), and \( f = F' = U_x + iV_x \) so \( u_x = U_x \). Also \( U_y = -V_x = u_y \). Thus \( U = u \) and \( V \) is called a harmonic conjugate of \( u \).

Note that if \( u \) is harmonic and \( h \) is analytic then the composition \( u \circ h \) is (locally) the real part of an analytic function and so harmonic.
7.2. BOUNDARY BEHAVIOUR OF HARMONIC FUNCTIONS

7.1.3 Identity theorem for harmonic functions

Suppose that \( u \) is harmonic on the domain \( D \) in \( \mathbb{C} \) and constant on a non-empty subdomain \( G \) of \( D \). Then \( u \) is constant on \( D \).

Proof. The function \( u_x - iu_y \) is analytic on \( D \) and 0 on \( G \) and so 0 on \( D \).

7.2 Boundary behaviour of harmonic functions

7.2.1 Example

The following example shows that a bounded harmonic function need not have limits at every boundary point. Let \( D = D(0, 1) \) and define \( u \) on \( D \) by \( u(z) = \arg \left( \frac{1+z}{1-z} \right) \). Then for \( |w| = 1, \text{Im}(w) > 0 \) we have \( \lim_{z \to w, z \in D} u(z) = \pi/2 \) and for \( |w| = 1, \text{Im}(w) < 0 \) we have \( \lim_{z \to w, z \in D} u(z) = -\pi/2 \). For \( w = \pm 1 \), the limit \( \lim_{z \to w, z \in D} u(z) \) does not exist, although \( u(x) = 0 \) for real \( x \).

7.2.2 Poisson’s formula

Let \( F(w) \) be a measurable function defined on \( |w| = 1 \) and taking values in \( \mathbb{R}^* = \mathbb{R} \cup \{ -\infty, \infty \} \). For \( |z| < 1 \) and \( |w| = 1 \) set

\[
K(z, w) = \text{Re} \left( \frac{w + z}{w - z} \right) = \frac{1 - |z|^2}{|w - z|^2}
\]

(the Poisson kernel) and

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) F(e^{it}) \, dt.
\]

The function \( u \) has the following properties.

(i) If \( F(e^{it}) \in L^1([0, 2\pi]) \) (i.e. \( \int_0^{2\pi} |F(e^{it})| \, dt < \infty \)) then \( u \) is harmonic in \( |z| < 1 \).

(ii) If \( |F(e^{it})| \leq M_0 < \infty \) for all \( t \) in \( [0, 2\pi] \) then \( |u(z)| \leq M_0 \) on \( |z| < 1 \).

(iii) If \( F(e^{it}) \in L^1([0, 2\pi]) \) and \( |w| = 1 \) and \( F \) is finite and continuous at \( w \) then as \( z \to w \) we have \( u(z) \to F(w) \).

(iv) If \( F \) is non-negative and \( F(e^{it}) \) is not in \( L^1([0, 2\pi]) \) then \( u(z) \equiv \infty \).

Proof. Suppose first that \( F \in L^1 \). Set

\[
Q(z, F) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} F(e^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} -F(e^{it}) \, dt + \frac{1}{2\pi} \int_0^{2\pi} \frac{2e^{it}}{e^{it} - z} F(e^{it}) \, dt = \]

\[
= \frac{1}{2\pi} \int_0^{2\pi} -F(e^{it}) \, dt + 2M(z),
\]

where

\[
M(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{w - z} F(w) \, dw.
\]
The integral $M(z)$ exists because, for fixed $z$, the term $(w-z)^{-1}$ is bounded. Now, as $h \to 0$,
\[
\frac{M(z+h) - M(z)}{h} = \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{(w-z)(w-z-h)} F(w) \, dw \to \frac{1}{2\pi i} \int_{|w|=1} \frac{1}{(w-z)^2} F(w) \, dw,
\]
by the dominated convergence theorem since, for fixed $z$ and small $h$, the term $(w-z)^{-1}(w-z-h)^{-1}$ is uniformly bounded. Thus $Q(z,F)$ is analytic on $|z| < 1$ and $u(z) = \text{Re}(Q(z,F))$ is harmonic. This proves (i). Note that if we choose $F = 1$ then $u(z) = Q(z,F) \equiv 1$ on $|z| < 1$ by the residue theorem. Since the Poisson kernel is positive, this proves (ii).

Next we prove (iii). Assume that $F(v)$ is finite, $|v| = 1$ and $F$ is continuous at $v$. Take $\varepsilon > 0$ and choose $\delta > 0$ so that $|F(e^{it}) - F(v)| < \varepsilon/2$ for all $t \in T_0 = \{s \in [0,2\pi] : |e^{is} - v| < \delta\}$. Then
\[
u(z) - F(v) = \frac{1}{2\pi} \int_0^{2\pi} K(z,e^{it})(F(e^{it}) - F(v)) \, dt.
\]
Now, since $K \geq 0$,
\[
\frac{1}{2\pi} \int_{T_0} K(z,e^{it})(F(e^{it}) - F(v)) \, dt
\]
has modulus at most
\[
(\varepsilon/2) \frac{1}{2\pi} \int_{T_0} K(z,e^{it}) \, dt \leq (\varepsilon/2) \frac{1}{2\pi} \int_{[0,2\pi]} K(z,e^{it}) \, dt = (\varepsilon/2).
\]
On the other hand
\[
\frac{1}{2\pi} \int_{[0,2\pi]\setminus T_0} K(z,e^{it})(F(e^{it}) - F(v)) \, dt
\]
has modulus at most
\[
\left(\frac{1}{2\pi} \int_{[0,2\pi]} |F(e^{it})| \, dt + |F(v)|\right) \sup\{K(z,e^{it}) : t \in [0,2\pi] \setminus T_0\} \to 0
\]
as $z \to v$, since for $t \notin T_0$ we have $|z - e^{it}| \geq \delta - |z - v|$ and so $K(z,e^{it}) \to 0$ as $z \to v$, uniformly on $[0,2\pi] \setminus T_0$.

Finally, to prove (iv) suppose that $F$ is non-negative and $\int_{[0,2\pi]} F(e^{it}) \, dt = \infty$. Since the Poisson kernel is positive and, for fixed $z$, bounded below on $[0,2\pi]$, we get $u \equiv \infty$ on $|z| < 1$.

**7.2.3 Corollary**

*Let $F$ be continuous on the circle $|z-z_0| = R > 0$. Then, with $w = z_0 + Re^{it}$,*

\[ u(z) = P(z,F,z_0,R) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z-z_0|^2}{|Re^{it} - (z-z_0)|^2} F(z_0 + Re^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z-z_0|^2}{|w-z|^2} F(w) \, dt \]

*is harmonic on $D = D(z_0,R)$ and $u(z) \to F(w)$ as $z \to w \in \partial D$.*

To prove this just put $z = z_0 + R\zeta$ and $u(z) = v(\zeta)$, where $v(\zeta)$ is the Poisson integral of $G(e^{it}) = F(w)$. 
7.3. SUBHARMONIC FUNCTIONS

7.2.4 Maximum principle: first version

Suppose that $u$ is harmonic on the bounded domain $D$ in $\mathbb{C}$ and continuous on the closure of $D$, and that $u(z) \leq M$ on $\partial D$. Then $u(z) \leq M$ on $D$.

Proof. Suppose $u(z_0) > M$ for some $z_0$ in $D$. Then, if $t > 0$ is small enough, the function $v(z) = u(z) + t(x^2 + y^2)$ is such that $v(z_0) > \max\{v(z) : z \in \partial D\}$ and so $v$ has a local maximum at some $z_1 \in D$. But at $z_1$ this gives $v_x = v_y = 0$ and $v_{xx} + v_{yy} = 4t > 0$ so that at least one of $v_{xx}, v_{yy}$ must be positive. This is a contradiction.

We will subsequently see another way to prove this, via the mean value property.

7.2.5 The mean value property

Let $D$ be a domain in $\mathbb{C}$. We say that a function $u : D \to \mathbb{C}$ has the mean value property if each $z_0$ in $D$ has $r_0 > 0$ such that

$$u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{it}) \, dt, \quad (0 < r \leq r_0).$$

Obviously these functions on $D$ form a vector space.

If $u$ is harmonic on $D$ then $u$ has the mean value property. To see this, take a disc $D(z_0, R) \subseteq D$ on which $u = \text{Re}(f)$ with $f$ analytic. Then Cauchy’s integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0} \, dz = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + re^{it}) \, dt, \quad (0 < r < R),$$

(7.1)

implies that $f$ has the mean value property and so has $u$.

7.3 Subharmonic functions

Let $D$ be a domain in $\mathbb{C}$. A function $u : D \to [-\infty, \infty)$ is subharmonic if:

(i) $u$ is upper semi-continuous (upper semi-continuous) in $D$ (see §1.5);

(ii) $u$ has the sub-mean-value-property, that to each $z_0$ in $D$ corresponds $r_0 > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{it}) \, dt, \quad (0 < r \leq r_0).$$

The integral exists because $u$ is measurable, by (i), and bounded above on the circle, again by (i).

7.3.1 Examples of subharmonic functions

Obviously harmonic functions are subharmonic.

Suppose that $f$ is analytic on the domain $D$: set $u = \log |f|$. If $a \in D$ and $f(a) \neq 0$ then we can define a branch $g$ of $\log f$ on a neighbourhood $A$ of $a$ and $u = \text{Re}(g)$ is harmonic on $A$. If $f(a) = 0$ then $u(z) \to u(a) = -\infty$ as $z \to a$. Thus $u$ is subharmonic on $D$.

It is easy to check from (7.1) that $|f(z)|$ is also subharmonic on $D$. Thus if $p > 0$ then $|f(z)|^p$ is also subharmonic (it is clearly upper semi-continuous and we need only check the sub-mean value property: this is obvious if $f(a) = 0$ and, if $f(a) \neq 0$, we write $|f(z)|^p = |f(z)|^p$ locally).
Further, if \( u, v \) are subharmonic then so are \( u + v, \max\{u, v\} \), and so subharmonic functions are more “flexible” than analytic or harmonic functions.

Thus the maximum of a finite family of subharmonic functions is subharmonic. However the sup of an infinite family need not be: for example, let \( u_n(z) = (1/n) \log |z| \). Then the sup is \(-\infty\) at 0, and is 0 for \( 0 < |z| < 1 \), and so is not upper semi-continuous.

### 7.3.2 Maximum principle: second version

Let \( D \) be a domain in \( \mathbb{C} \) and let \( u \) be subharmonic on \( D \). If \( u \) has a maximum in \( D \) then \( u \) is constant on \( D \).

#### Proof.

Assume that \( u(z) \leq u(z_0) = M \) on \( D \). If \( M = -\infty \) then the result is obvious. Assume now that \( M \in \mathbb{R} \). If \( u(z_1) = M \) then since

\[
M = u(z_1) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + re^{it}) \, dt \leq M
\]

for small positive \( r \) we must have \( u \equiv M \) on \( |z - z_1| = r \), by the fact that \( u \) is upper semi-continuous. This is because if \( u(z_1 + re^{is}) < M \) we get \( u(z_1 + re^{it}) < M' < M \) for \( t \) close to \( s \) and this makes the integral less than \( M \). So the set \( \{ z \in D : u(z) = M \} \) is non-empty and open. The set \( \{ z \in D : u(z) < M \} \) is open since \( u \) is upper semi-continuous. By connectedness, the second set must be empty.

### 7.3.3 Maximum principle: third version

Let \( D \) be a domain in \( \mathbb{C} \) and define \( \partial_{\alpha} D \) to be the collection of all boundary points of \( D \) in \( \mathbb{C}^* \), with respect to the spherical metric. Thus \( \partial_{\alpha} D \) is the finite boundary \( \partial D \) plus, if \( D \) is unbounded, the point \( \infty \). Then \( \partial_{\alpha} D \) is compact in \( \mathbb{C}^* \). If \( u \) is subharmonic in \( D \) and

\[
\limsup_{z \to \zeta, z \in D} u(z) \leq M \in [-\infty, \infty)
\]

for every \( \zeta \in \partial_{\alpha} D \), then either \( u(z) \equiv M \) on \( D \), or \( u(z) < M \) for all \( z \) in \( D \).

#### Proof.

The first assertion is obvious since \( \partial_{\alpha} D \) is closed and \( \mathbb{C}^* \) is compact. Set \( L = \sup\{u(z) : z \in D\} \) and take \( z_n \in D \) with \( u(z_n) \to L \). Assume without loss of generality that \( z_n \) converges to the point \( z^* \) in \( D \cup \partial_{\alpha} D \). Now if \( L > M \) then \( z^* \in D \) and we get \( u(z) \equiv L \) on \( D \) by Lemma 7.3.2, an obvious contradiction. So \( L \leq M \). Furthermore, either \( u < M \) on \( D \) or Lemma 7.3.2 gives \( u \equiv M \) on \( D \).

### 7.3.4 Lemma

Let \( u \) be subharmonic and bounded above on the domain \( D \) in \( \mathbb{C} \). For \( w \in \partial_{\alpha} D \), set

\[
\phi(w) = \limsup_{z \to w, z \in D} u(z).
\]

Then the function \( v(z) \) defined by \( v(z) = u(z) \) if \( z \in D \) and \( v(z) = \phi(z) \) if \( z \in \partial_{\alpha} D \) is upper semi-continuous on \( D \cup \partial_{\alpha} D \).
7.3. SUBHARMONIC FUNCTIONS

Proof. We only need consider $w$ on the boundary. Suppose $\phi(w) < s < t$. Then there is some spherical disc $D_q(w, r) = \{z \in \mathbb{C}^* : q(z, w) < r\}$ such that

$$u(z) < s, \quad (z \in D \cap D_q(w, r)).$$

But then if $x \in \partial D \cap D_q(w, r)$ we have $\phi(x) \leq s < t$. So $v(x) < t$ for all $x \in D \cup \partial \infty D$ which are sufficiently close to $w$.

7.3.5 Theorem (comparison with a Poisson integral)

Let $u$ be subharmonic on the disc $D(z_0, R)$. Let $v(w)$ be upper semi-continuous on $|w - z_0| = R$, taking values in $[-\infty, \infty]$, with

$$\limsup_{z \to w, z \in D(z_0, R)} u(z) \leq v(w), \quad (|w - z_0| = R).$$

Then for $z \in D(z_0, R)$ we have

$$u(z) \leq P(z, v) = P(z, v, z_0, R) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z - z_0|^2}{|Re^{it} - (z - z_0)|^2}v(z_0 + Re^{it})\,dt. \quad (7.2)$$

If $|w - z_0| = R$ then

$$\limsup_{z \to w, z \in D(z_0, R)} P(z, v) \leq v(w), \quad (7.3)$$

and if $u \neq -\infty$ on $D(z_0, R)$ then $P(z, v)$ is harmonic there.

If $u$ is harmonic in $D(z_0, R)$ and continuous on $|z - z_0| \leq R$ then setting $v = u$ gives equality in (7.2), so that $u$ is the Poisson integral of its boundary values.

Note that if $u$ is subharmonic in a domain containing the set \{ $z \in \mathbb{C} : |z - z_0| \leq R$ \} then we may take $v = u$, since $u$ is upper semi-continuous. Further, since the circle $|z - z_0| = R$ is compact, $v$ is bounded above there.

Proof. To prove the theorem take a sequence of continuous functions $f_n$ on $|z - z_0| = R$, decreasing pointwise to $v$. Such a sequence exists by Theorem 1.5.1. Let $u_n(z) = P(z, f_n) = P(z, f_n, z_0, R)$ be as defined by (7.2). Then $u_n$ is harmonic on $D(z_0, R)$ and $u_n(z) \to f_n(w)$ as $z \to w \in \partial D(z_0, R)$ with $z \in D(z_0, R)$.

Hence $u - u_n$ is subharmonic in $D(z_0, R)$ and since $v \leq f_n$ we have

$$\limsup_{z \to w, z \in D(z_0, R)} (u - u_n)(z) \leq 0, \quad (w \in \partial D(z_0, R)).$$

Thus the maximum principle 7.3.3 gives, for $z \in D(z_0, R),$

$$u(z) \leq u_n(z) = P(z, f_n).$$

Let $M = \max\{f_1(w) : |w - z_0| = R\}$. Then $M - f_{n+1} \geq M - f_n \geq 0$ and the monotone convergence theorem gives

$$P(z, M - f_n) \to P(z, M - v), \quad P(z, f_n) \to P(z, v)$$

for every $z \in D(z_0, R)$. Thus $u(z) \leq P(z, v)$, which is (7.2).

Further, if $|w - z_0| = R$ and $z \to w$ with $z \in D(z_0, R)$ then

$$\limsup P(z, v) \leq \limsup P(z, f_n) = \limsup u_n(z) = f_n(w) \to v(w),$$
which gives (7.3).

To prove that $P(z, v)$ is harmonic if $u \neq -\infty$ we can assume without loss of generality that $v \leq 0$ (since $v$ is bounded above on the compact set $|u - z_0| = R$). We then apply (after rescaling) the Poisson formula 7.2.2: if $\int_{0}^{2\pi} v(z_0 + Re^{it}) dt = -\infty$ then $P(z, v) \equiv -\infty$ and $u(z) \equiv -\infty$ on $D(z_0, R)$. On the other hand if $v(z_0 + Re^{it}) \in L^1([0, 2\pi])$ then $P(z, v)$ is harmonic.

Finally, if $u$ is harmonic on $D(z_0, R)$ and continuous on the closure we set $v = u$ and apply the above to $u$ and $-u$ to get $u = P(z, u)$ on $D(z_0, R)$.

7.3.6 Corollary

Suppose that $u : D \to \mathbb{R}$ is continuous and has the mean value property 7.2.5 on the domain $D$ in $\mathbb{C}$. Then $u$ is harmonic on $D$.

Note that the hypotheses are equivalent to $u$ and $-u$ both being subharmonic on $D$.

Proof. Take any disc $D(z_0, R)$ whose closure lies in $D$. Form the Poisson integral $\tilde{u}$ on $D(z_0, R)$ with boundary values $u(z_0 + Re^{it})$. Then $\tilde{u}$ is harmonic on $D(z_0, R)$. Since $u$ and $-u$ are subharmonic, Theorem 7.3.5 gives

$$u(z) \leq \tilde{u}(z), \quad -u(z) \leq -\tilde{u}(z), \quad (z \in D(z_0, R))$$

and so $u = \tilde{u}$ on $D(z_0, R)$.

The following result addresses the issue of on how large a set a non-constant subharmonic function can be $-\infty$.

7.3.7 Theorem

Let $u$ be subharmonic on a domain $D$ in $\mathbb{C}$ and let $0 < s < r$ and $D(z_0, r) \subseteq D$. Suppose that $u(z) \equiv -\infty$ on a subset of the circle $S(z_0, s)$ of positive angular measure. Then $u(z) \equiv -\infty$ on $D$.

In particular if $u \equiv -\infty$ on $D(z_0, r)$ then $u \equiv -\infty$ on $D$.

Proof. Since $u$ is bounded above on $S(z_0, s)$ we get $\int_{0}^{2\pi} u(z_0 + se^{it}) d\theta = -\infty$. Thus the Poisson integral of $u$ is identically $-\infty$ on $D(z_0, r)$ and Theorem 7.3.5 gives $u(z) \equiv -\infty$ on $D(z_0, s)$.

Now let $F$ be the set of $w \in D$ such that $u \equiv -\infty$ on a neighbourhood of $w$. Obviously $F$ is open, and we will show that $F$ is also closed (in $D$) so that the result follows by connectivity. Let $w_n \in F$ and $w_n \to w \in D$. Then for arbitrarily small $t$ we have $u(z) \equiv -\infty$ on a subset of $S(w, t)$ of positive measure. Hence $u(z) \equiv -\infty$ on $D(w, t)$ and $w \in F$.

7.3.8 Lemma (Poisson modification of a subharmonic function)

Let $u$ be subharmonic on the domain $D$ in $\mathbb{C}$ and let the closure of $D(z_0, R)$ be contained in $D$. Define $U(z) = u(z)$ on $D \setminus D(z_0, R)$, and on $D(z_0, R)$ let $U$ be the Poisson integral of $u(z_0 + Re^{it})$ i.e.

$$U(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{R^2 - |z - z_0|^2}{|Re^{it} - (z - z_0)|^2} u(z_0 + Re^{it}) \, dt.$$ 

Then $U$ is subharmonic with $U \geq u$ on $D$. If $u \not\equiv -\infty$ on $D$ then $U$ is harmonic on $D(z_0, R)$.

Note that in particular this gives

$$u(z_0) \leq \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + Re^{it}) \, dt$$
for every $R > 0$ such that the closure of $D(z_0, R)$ is contained in $D$, and not just for $0 < r \leq r_0$ as in the definition 7.3 of a subharmonic function.

Proof. We already know that $u \leq U$, by Theorem 7.3.5. Thus we only need check that $U$ is upper semi-continuous and has the sub-mean value property at all $z_1$ with $|z_1 - z_0| = R$. First, for small $r$,

$$U(z_1) = u(z_1) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} U(z_0 + re^{it}) \, dt.$$

Next,

$$\limsup_{z \to z_1, z \in D(z_0, R)} U(z) \leq u(z_1)$$

by Theorem 7.3.5 (see (7.3), while

$$\limsup_{z \to z_1, z \notin D(z_0, R)} U(z) = \limsup_{z \to z_1, z \notin D(z_0, R)} u(z) \leq u(z_1)$$

since $u$ is upper semi-continuous.

Hence $U$ is subharmonic on $D$. Finally if $U$ is not harmonic on $D(z_0, R)$ then $U \equiv -\infty$ there and the same is true of $u$ on $D(z_0, R)$ and hence on $D$.

7.3.9 Harnack’s inequality

Let $u$ be harmonic and non-negative on $|z - z_0| \leq R$. If $|z - z_0| = r < R$ then

$$\left( \frac{R - r}{R + r} \right) u(z_0) \leq u(z) \leq \left( \frac{R + r}{R - r} \right) u(z_0).$$

This follows at once from Poisson’s formula.

7.3.10 Harnack’s theorem

Let $D$ be a domain in $\mathbb{C}$. Let $u_n$ be harmonic functions on $D$ with $u_1 \leq u_2 \leq u_3 \leq \ldots$. Let $v(z) = \lim_{n \to \infty} u_n(z)$. Then either $v \equiv \infty$ on $D$, or $v$ is harmonic on $D$, in which case $u_n \to v$ locally uniformly on $D$.

Proof. Suppose first that $v(w) < \infty$. Take a disc $D(w, 4R) \subseteq D$. Then we assert that $u_n \to v$ uniformly on $D(w, 2R)$. Take $\delta > 0$. Then there exists $N$ such that for all $n \geq m \geq N$ we have

$$0 \leq u_n(w) - u_m(w) < \delta$$

and so Harnack’s inequality applied on $|z - w| \leq 3R$ gives

$$|u_n(z) - u_m(z)| = u_n(z) - u_m(z) < 5\delta$$

for all $z$ in $D(w, 2R)$. Letting $n \to \infty$ we see that $v(z)$ is finite and $|u_m(z) - v(z)| \leq 5\delta$ on $D(w, 2R)$. Hence $u_m \to v$ uniformly, and so $v$ is continuous, on $D(w, 2R)$. Now on $D(w, R)$ we have, denoting the Poisson integral by $P(z, u)$,

$$v(z) = \lim u_n(z) = \lim P(z, u_n) = P(z, v).$$
by uniform convergence. Thus $v$ is harmonic on $D(w, R)$.

Now suppose that $v(w) = \infty$. Fix $m$. For $M \in (0, \infty)$ we have $u_n(w) - u_m(w) > M$ for large $n$. This time Harnack’s inequality gives $u_n(z) - u_m(z) > M/5$ on $D(w, 2R)$, and we have thus shown that $u_n \to \infty$ uniformly on $D(w, R)$.

The sets $\{w : v(w) < \infty\}$ and $\{w : v(w) = \infty\}$ are thus open, and by connectedness one of them is empty.
Chapter 8

Perron’s method

8.1 The Dirichlet problem

Let $D$ be a domain in $\mathbb{C}$ and let $f$ be a bounded real-valued function on $X = \partial_\infty D$ (the boundary with respect to the extended plane). The Dirichlet problem is to find, if possible, a harmonic function $h = h_f$ on $D$ such that
\[ \lim_{z \to w, z \in D} h(z) = f(w) \quad \text{for every } w \text{ in } X. \quad (8.1) \]

When $D$ is a disc, and $f$ is continuous, this is achieved by means of the Poisson integral (see 7.2.2 and 7.2.3).

8.1.1 The Perron family and Perron function

Let $D$ be a domain in $\mathbb{C}$ and let $f$ be a bounded real function on $X = \partial_\infty D$. The Perron family $U(f)$ is the collection of all subharmonic functions $u$ on $D$ such that for every $w \in X = \partial_\infty D$ we have
\[ \limsup_{z \to w, z \in D} u(z) \leq f(w). \]

The Perron function $v_f$ is then defined by
\[ v(z) = v_f(z) = \sup \{ u(z) : u \in U(f) \}. \]

Obviously if $f \leq g$ then $v_f \leq v_g$.

If a function $h$ satisfying (8.1) exists then $h = v_f$: to see this, note first that $h \in U(f)$, so that $h \leq v_f$. Further, for every $u \in U(f)$, we have $\limsup_{z \to w} (u(z) - h(z)) \leq 0$ for every $w \in X$, and so $u \leq h$ on $D$, by the maximum principle. Thus $v_f \leq h$. So if the Dirichlet problem is solvable, then the solution $h_f$ equals the Perron function $v_f$. Most of this section will be concerned with the converse direction: that is, proving that if $f : X \to \mathbb{R}$ is continuous and $X$ is sufficiently regular, then $h = v_f$ does indeed satisfy (8.1). However, we first look at an example.

8.1.2 Example

This example shows that the Dirichlet problem is not always solvable. Let $D_1 = D(0,1) \setminus \{0\}$ and let $f(x) = 0$ for $|x| = 1$, with $f(0) = 1$. Now let $v \in U(f)$ and set $u = \max\{v, 0\}$. Then $u \in U(f)$ and $0 \leq u(z) \leq 1$ on $D_1$ by the maximum principle. Thus for $0 < t < 1$ we get
\[ u(z) \leq w(z) = \frac{\log 1/|z|}{\log 1/t}, \quad (t < |z| < 1). \]
This is because $u - w$ is subharmonic on $t < |z| < 1$ and at most 0 on the boundary. Fixing $z$ and
letting $t \to 0$ we see that $u(z) \equiv 0$. This implies that $v_f \equiv 0$. Hence the Dirichlet problem for $f$ and
$D_1$ cannot have a solution $h$, because if it did we would have $h = v_f \equiv 0$ by 8.1.1.

8.1.3 Lemma

Let $f$ be a bounded real-valued function on $X$ with $|f| \leq M$ on $X$, and let $v_f$ be its Perron function.
Then the following are true:

(i) we have $|v_f| \leq M$;

(ii) the function $v = v_f$ is harmonic on $D$.

Proof. First, $v \geq -M$ since $-M \in U(f)$. Further, each $u$ in $U(f)$ has $\lim \sup_{z \to w, z \in D} u(z) \leq M$ and
so $u \leq M$ by the maximum principle.

To prove that $v$ is harmonic, we take a disc $D_1 = D(z_0, R)$ whose closure lies in $D$, and we make
the following observations. First, the maximum of finitely many elements of $U(f)$ is subharmonic on
$D$ and is an element of $U(f)$. Second, if $u_0 \in U(f)$ then there exists an element $U_0$ of $U(f)$ which
is harmonic on $D_1$ and satisfies $u_0 \leq U_0$ on $D$. To see this, let $U_0(z) = u_0(z)$ on $D \setminus D_1$, but for $z$ in $D_1$
let $U_0(z)$ equal the Poisson integral $P(z, u_0)$, where

$$P(z, g) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z - z_0|^2}{|Re^{it} - (z - z_0)|^2} g(z_0 + Re^{it}) \, dt.$$ 

Then by Lemma 7.3.8, $U_0$ is subharmonic with $U_0 \geq u_0$ on $D$. Also $U_0$ is equal to $u_0$ outside $D_1$, so
that $U_0 \in U(f)$.

So we start by taking $v_n \in U(f)$ such that $v_n(z_0) \to v(z_0)$ and setting

$$u_n(z) = \max \{v_1(z), \ldots, v_n(z)\}.$$ 

This gives a sequence $(u_n)$ in $U(f)$ such that $u_n \leq u_{n+1}$ on $D$ and $u_n(z_0) \to v(z_0)$. Next, let $U_n$
be $u_n$ but with its values in $D_1$ replaced by the Poisson integral $P(z, u_n)$, so that $U_n \in U(f)$. Since
$u_n(z_0) \to v(z_0)$ and $u_n \leq U_n \leq v$, we have $U_n(z_0) \to v(z_0)$. We also claim that that $U_n \leq U_{n+1}$ on
$D$: this is clear on $D \setminus D_1$ and in $D_1$ we just compare the Poisson integrals. By doing this we have found
a non-decreasing sequence $(u_n)$ in $U(f)$ such that $U_n$ is harmonic on $D_1$ and $U_n(z_0) \to v(z_0)$. Since
$U_n \leq M$ on $D_1$, Harnack’s theorem 7.3.10 gives us a harmonic function $u$ on $D_1$ such that $U_n \to u$,
and clearly $u(z_0) = v(z_0)$. The idea now is to show that $u = v$ on all of $D_1$, so that $v$ is harmonic on
$D_1$ and hence on $D$.

To do this take any other point $z_1 \in D_1$. The same construction gives $W_n \in U(f)$ such that
$W_n \leq W_{n+1}$ on $D$ and $W_n(z_1) \to v(z_1)$ and $W_n$ is harmonic on $D_1$. We then combine $U_n$ and $W_n$:
let $h_n(z) = \max \{U_n(z), W_n(z)\}$ and define $H_n$ to be $P(z, h_n)$ on $D_1$ and $h_n(z)$ outside $D_1$, so that
$H_n \geq h_n$. Again we have $H_n \leq H_{n+1}$ on $D$, because we clearly have $h_n \leq h_{n+1}$ and in $D_1$ we compare
Poisson integrals again. Also, the function $H_n$ is in $U(f)$ and is harmonic on $D_1$. Thus, on $D_1$,

$$U_n(z) \leq H_n(z) \leq H_n(z) \leq v(z) \leq M, \quad W_n(z) \leq H_n(z) \leq v(z) \leq M,$$

and so $H_n(z_0) \to v(z_0)$ and $H_n(z_1) \to v(z_1)$.

By Harnack’s theorem 7.3.10 there is a harmonic function $h$ on $D_1$ such that $H_n \to h$. We also have $u \leq h$ on $D_1$, since $U_n \leq H_n$. But $u(z_0) = h(z_0) = v(z_0)$ and so $u = h$ on $D_1$, by the maximum
principle, since $(u - h)(z) \leq (u - h)(z_0)$ on $D_1$. This gives

$$v(z_1) = h(z_1) = u(z_1)$$

and, since $z_1$ is arbitrary, $v = u = h$ on $D_1$. 

CHAPTER 8. PERRON’S METHOD
8.1.4 The barrier

Following Ransford [59], let $D$ be a domain in $\mathbb{C}$ and let $x_0 \in X = \partial_\infty D$. A barrier (for $D$) at $x_0$ is a subharmonic function $b$ defined on $D \cap N$, where $N$ is an open neighbourhood of $x_0$, such that

$$b(z) < 0, \quad (z \in D \cap N), \quad \lim_{z \to x_0, z \in D \cap N} b(z) = 0.$$

If the barrier exists then $x_0$ is called a regular boundary point, and $D$ is called regular if all its boundary points are regular.

Note that if $G$ is a subdomain of $D$ and $x$ is a boundary point of both $D$ and $G$, and is regular for $D$, then $x$ is regular for $G$.

Note also that simply connected proper subdomains $D$ of $\mathbb{C}$ are regular, as we can write $b(z) = \log |F(z)|$, where $F : D \to D(0,1)$ is the analytic bijection between $D$ and $D(0,1)$ arising from the Riemann mapping theorem.

The following is an example of a non-regular boundary point. Let $D = D(0,1) \setminus \{0\}$, and let $x_0 = 0$. If a barrier $b$ exists at $x_0$ then because $b(z) < 0$ on $D \cap N$ we get $b(z) \leq t < 0$ on a circle $|z| = s$ with $s$ small and positive, by Lemma 1.5.2. We also have $b(z) \to 0$ as $z \to 0$. By taking $b(z)/|t|$ we can assume that $t = -1$. But then the function $w(z) = 1 + b(sz)$ belongs to the family $U(f)$ from Example 8.1.2 (since $w(z) \leq 0$ for $|z| = 1$ and $\lim_{z \to 0} w(z) \leq 1$), which forces $w \leq v_f = 0$ and hence $b(z) \to -1$ as $z \to 0$, a contradiction.

The next lemma is also from [59] and will be used to prove the boundary properties of the Perron function.

8.1.5 Bouligand’s lemma

Let $x_0$ be a regular boundary point of $D$ and let $N_0$ be a spherical disc centred at $x_0$. Let $\delta > 0$. Then there exists a function $w$ subharmonic on $D$ such that

$$w(z) < 0 \quad (z \in D), \quad w(z) = -1 \quad (z \in D \setminus N_0), \quad \liminf_{z \to x_0, z \in D} w(z) \geq -\delta.$$

Thus $w$ is negative on $D$ and $-1$ away from $x_0$, but not too negative near $x_0$.

Proof. The idea of the proof is to modify a barrier function by subtracting a Poisson integral. Assume without loss of generality that $0 < \delta < 1$. Choose a neighbourhood $N$ and a barrier function $b$ as in the definition of barrier. Choose an open disc $G$ centred at $x_0$ with closure satisfying $\text{Cl}(G) \subseteq N \cap N_0$. (If $x_0$ is finite then $G$ is a Euclidean disc, while if $x_0 = \infty$ then $G$ is a set $\{z \in \mathbb{C}^* : |z| > R\}$.) Let $E = \partial G \cap D$. Then $E$ is a relatively open subset of $\partial G$. Choose a compact subset $K$ of $E$ so that $L = E \setminus K$ has angular measure $2\pi \sigma < \delta$. Again, $L$ is relatively open.

We can use the Poisson integral formula to make a harmonic function $u$ on $G \cap \mathbb{C}$, which satisfies $0 \leq u \leq 1$ and is such that $u(z) \to 1$ as $z \to \eta \in L$ and $u(z) \to \sigma$ as $z \to x_0$. If $x_0$ is finite we just use the Poisson integral formula on $G$ with boundary values $1$ on $L$ and $0$ on $\partial G \setminus L$, while if $x_0 = \infty$ we have to first use a map $z \to 1/z$.

Now let $\sup \{b(z) : z \in K\} = -m$. Then $-m < 0$ by Lemma 1.5.2, because $K$ is a compact subset of $D \cap N$ and $b$ is negative and subharmonic, and so upper semi-continuous, on $D \cap N$. We may assume that $m = 1$. For $\eta \in K \subseteq E = \partial G \cap D$ we have

$$\limsup_{z \to \eta, z \in D \cap G} (b(z) - u(z)) \leq b(\eta) \leq -1$$

since $b$ is upper semi-continuous and $u \geq 0$. On the other hand if $\eta \in L = E \setminus K$ we have

$$\limsup_{z \to \eta, z \in D \cap G} (b(z) - u(z)) \leq -1$$
since \( b < 0 \) and \( u(z) \to 1 \) as \( z \) approaches \( \eta \). This implies that, for every \( \eta \in E = \partial G \cap D \),
\[
\limsup_{z \to \eta, z \in D \cap G} (b(z) - u(z)) \leq -1.
\]

So we define
\[
w(z) = \max \{-1, b(z) - u(z)\} \quad (z \in D \cap G), \quad w(z) = -1 \quad (z \in D \setminus G).
\]

Then \( w \) is subharmonic in \( D \). Since \( b < 0 \) and \( u \geq 0 \) on \( D \cap G \) we have \( w < 0 \), and for \( w \in D \setminus N_0 \) we have \( w = -1 \). Also as \( z \to x_0 \) with \( z \in D \) then \( z \in G \) and \( b(z) \to 0 \) and \( u(z) \to \sigma < \delta \) so \( w(z) \to -\sigma > -\delta \).

### 8.1.6 Lemma

Let \( f \) and \( g \) be bounded real functions on \( X \). Then \( v_f + v_g \leq v_{f+g} \) on \( D \). In particular, \( v_f(z) \leq -v_{-f}(z) \) on \( D \).

**Proof.** Let \( u_f \in U(f) \) and \( u_g \in U(g) \). Then \( u_f + u_g \in U(f + g) \) and so
\[
u_f(z) + u_g(z) \leq v_{f+g}(z)
\]
on \( D \). Now take the suprema over \( U(f) \) and \( U(g) \).

### 8.1.7 Theorem

Let \( x_0 \) be a regular boundary point of \( D \) and let \( f \) be bounded on \( X \). Then
\[
M_0 = \liminf_{x \to x_0} f(x) \leq \liminf_{z \to x_0, z \in D} v_f(z) \leq \limsup_{z \to x_0, z \in D} v_f(z) \leq M_1 = \limsup_{x \to x_0} f(x).
\]

In particular, if \( f \) is continuous at \( x_0 \) then \( v_f(z) \to f(x_0) \) as \( z \to x_0 \) with \( z \in D \). Hence if \( f \) is continuous on \( X \) and \( D \) is regular then \( v_f \) solves the Dirichlet problem for \( f \) on \( D \).

**Proof.** Let \( M = \{\sup |f(x)| : x \in X\} \). Then \( M + M_0 \geq 0 \). Let \( \delta > 0 \) and take a neighbourhood \( N_0 \) of \( x_0 \) such that \( f(x) > M_0 - \delta \) on \( X \cap N_0 \). Take a spherical disc \( N \) centred at \( x_0 \), whose closure lies in \( N_0 \). Define \( \omega \) as in Bouligand’s lemma, using \( \delta \) and the disk \( N \).

Set
\[
u(z) = M_0 - \delta + (M + M_0)w(z).
\]

Then \( u \) is subharmonic on \( D \). Let \( x \in X \). If \( x \in N_0 \) then as \( z \to x \) with \( z \in D \) we have, since \( w < 0 \),
\[
\limsup_{z \to x_0} u(z) \leq M_0 - \delta \leq f(x).
\]

On the other hand if \( x \notin N_0 \) then as \( z \to x_0 \) with \( z \in D \) we have \( \z \notin N \) and so \( w(z) \leq -1 \) and
\[
\limsup_{z \to x_0} u(z) \leq -\delta - M < f(x).
\]

Hence \( u(z) \leq v_f(z) \). But then
\[
\liminf_{z \to x_0, z \in D} v_f(z) \geq \liminf_{z \to x_0, z \in D} u(z) \geq M_0 - \delta - (M + M_0).
\]

Since \( \delta \) is arbitrary we get
\[
\liminf_{z \to x_0, z \in D} v_f(z) \geq M_0.
\]
to Applying the same argument to \(-f\) gives
\[
\lim_{z \to x_0, z \in D} \inf_{v \to f(z)} \geq -M_1
\]
and so, using Lemma 8.1.6,
\[
\lim_{z \to x_0, z \in D} \sup_{v} v_f(z) \leq \lim_{z \to x_0, z \in D} \sup_{v} -v_f(z) \leq M_1.
\]

8.1.8 A sufficient condition for existence of the barrier

Let \(x_0 \in X\). Let \(E\) be the component of \(X\) which contains \(x_0\) (i.e. the union of all connected subsets of \(X\) which contain \(x_0\)) and suppose that \(E \neq \{x_0\}\). Then \(x_0\) is regular. In particular if there exists a path in \(X\) joining \(x_0\) to \(x \neq x_0\) then \(x_0\) is regular.

**Proof.** Suppose first that \(x_0 = \infty\) and choose \(x_1 \in E \setminus \{\infty\}\). If \(\gamma\) is a closed PSC in \(D\) then the winding number \(n(\gamma, z)\) is integer-valued and continuous on \((C \cup \{\infty\}) \setminus \gamma\); here we set \(n(\gamma, \infty) = 0\). Thus \(n(\gamma, x_1) = 0\) (because otherwise \(E\) would be partitioned into relatively open sets \(\{z \in E : n(\gamma, z) = 0\}\) and \(\{z \in E : n(\gamma, z) \neq 0\}\), the second non-empty by assumption and the first non-empty since \(\infty\) belongs to it). Assume without loss of generality that \(x_1 = 1\).

Thus we can define an analytic branch of \(\log z = u(z) + iv(z)\) on \(D\) and
\[
b(z) = -\Re \left( \frac{1}{\log z} \right) = \frac{-u}{u^2 + v^2}
\]
is harmonic on \(D\) and has \(b(z) < 0\) on \(D \cap \{z : |z| > 1\}\) and \(|b(z)| \leq 1/|u(z)| \to 0\) as \(z \to \infty\).

If \(x_0\) is finite then without loss of generality \(x_0 = 0\) and we first apply the transformation \(z \to 1/z\).

8.2 Convexity and subharmonic functions

8.2.1 Theorem

Let \(u\) be subharmonic in \(a < |z| < b\). For \(a < r < b\) set
\[
I(r, u) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\tau})d\tau.
\]

Then \(I(r, u)\) is a convex function of \(\log r\) on \((a, b)\) i.e.
\[
I(s, u) \leq \frac{\log t/s}{\log t/r} I(r, u) + \frac{\log r/s}{\log r/t} I(t, u)
\]
for \(a < r < s < t < b\).

If \(a = 0\) and \(I(r, u)\) is bounded above as \(r \to 0^+\), then \(I(r, u)\) is non-decreasing on \((0, b)\).

**Proof.** Let \(a < r < s < t < b\). Take continuous \(f_n\) such that \(f_{n+1} \leq f_n\) and \(f_n \to u\) pointwise on the union of the circles \(|z| = r, |z| = t\) (on which \(u\) is upper semi-continuous: we can do this by 1.5.1). Let \(D\) be the annulus \(r < |z| < t\). If \(\zeta \in \partial D\) then since \(u\) is upper semi-continuous we get \(\limsup_{z \to \zeta, z \in D} u(z) \leq f_n(\zeta)\). Thus \(u \in U(f_n)\) in the terminology of Perron’s method. Solving the Dirichlet problem for \(f_n\) gives functions \(u_n\) harmonic on \(r < |z| < t\) and continuous on \(r \leq |z| \leq t\) and equal to \(f_n\) on the boundary circles. Further, \(u \leq u_n\) for \(r \leq |z| \leq t\).
Now, on \((r, t)\), we have, with \(v = u_n\),
\[
\frac{d^2 I(s, v)}{d \log s^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 v}{\partial \tau^2} d\tau = -\frac{1}{2\pi} \int_0^{2\pi} v_{\tau \tau} d\tau = 0, \quad \sigma = \log s,
\]
using the fact that \(v\) is locally the real part of an analytic function and so the real part of an analytic function of \(\log z = \sigma + i\tau\). Thus on \([r, t]\) we have
\[
I(s, u_n) = p_n \log s + q_n
\]
for some constants \(p_n, q_n\).

We now have, for \(r < s < t\),
\[
I(s, u) \leq I(s, u_n) = \frac{\log t/s}{\log t/r} I(r, u_n) + \frac{\log r/s}{\log r/t} I(t, u_n) \to \frac{\log t/s}{\log t/r} I(r, u) + \frac{\log r/s}{\log r/t} I(t, u),
\]
by the monotone convergence theorem (use the fact that \(0 \leq u_1 - u_n \uparrow u_1 - u\)). This proves (8.3).

Now assume that \(a = 0\) and \(I(r, u)\) is bounded above as \(r \to 0^+\). Letting \(r \to 0^+\), we note that the \(\frac{\log t/s}{\log t/r}\) term is positive but tends to 0. Since \(\frac{\log r/s}{\log r/t} \to 1\), we get \(I(s, u) \leq I(t, u)\).

### 8.2.2 Theorem

Let \(u\) be subharmonic and bounded above in \(0 < |z| < R\). Then setting \(u(0) = K = \lim_{r \to 0^+} I(r, u)\) makes \(u\) subharmonic in \(D(0, R)\).

**Proof.** Take \(M > 0\) such that \(u \leq M\) on \(0 < |z| < R\). Since \(I(r, u)\) is a non-decreasing function of \(r\), and tends to \(K\) as \(r \to 0^+\), we automatically get \(u(0) \leq I(r, u)\) and the sub-mean value property. Thus we only need to show that \(u\) is upper semi-continuous at 0. Let \(0 < s < R\).

**Claim:** We have \(\limsup_{z \to 0} u(z) \leq I(s, u)\).

Let \(f_n\) be continuous on \(|z| = s\), with \(f_{n+1} \leq f_n\) and \(f_n \to u\) pointwise (again these exist by 1.5.1). Using Poisson’s formula let \(u_n\) be harmonic on \(|z| < s\), continuous on \(|z| \leq s\) and equal to \(f_n\) on \(|z| = s\). Then \(u_n(0) = I(s, f_n)\) by Poisson’s formula.

For a given \(n\) take \(N > 0\) such that \(u_n(0) + N > 0\) and \(r_n > 0\) such that \(u_n(z) + N > 0\) for \(|z| \leq r_n\). Let \(0 < r \leq r_n\) and \(D = \{z : r < |z| < s\}\) and set
\[
v_n(z) = (M + N) \frac{\log s/|z|}{\log s/r} + u_n(z).
\]
Then \(v_n(z) = u_n(z)\) on \(|z| = s\), while \(v_n(z) > M\) on \(|z| = r\). Hence \(u \leq v_n\) on \(D\), since \(\limsup_{z \to \zeta, \zeta \in D} (u(z) - v_n(z)) \leq 0\) for every \(\zeta \in \partial D\), using the fact that \(u\) is upper semi-continuous with \(u_n\).

Keeping \(z\) fixed and letting \(r \to 0^+\) we get \(u(z) \leq u_n(z)\) for \(0 < |z| < s\) and so
\[
\limsup_{z \to 0} u(z) \leq \limsup_{z \to 0} u_n(z) = u_n(0) = I(s, f_n) \to I(s, u)
\]
as \(n \to \infty\).

This proves the Claim. Now letting \(s \to 0^+\) we get \(\limsup_{z \to 0} u(z) \leq K\).
8.2.3 Example

The following construction gives a subharmonic function on $\mathbb{C}$ which is not continuous at 0. Let

$$u(z) = \sum_{n=1}^{\infty} \frac{1}{n^2} \log \left| z - \frac{1}{n} \right|.$$ 

Then $u$ is subharmonic on $\mathbb{C}\setminus\{0\}$; to see this, note that if $0 \neq z_0 \in \mathbb{C}$ then there exists a neighbourhood $U_0$ of $z_0$ containing at most one of the singularities $1/n$, say $1/n_0$, which implies that

$$u(z) = \frac{1}{n_0^2} \log \left| z - \frac{1}{n_0} \right| + \operatorname{Re} \left( \sum_{n \neq n_0} \frac{1}{n^2} \log \left( z - \frac{1}{n} \right) \right)$$

on $U_0$. Since $u(z) \leq \sum_{n=1}^{\infty} \frac{\log 2}{n^2}$ for $0 < |z| < 1$, Theorem 8.2.2 shows that $u$ extends to be subharmonic on $\mathbb{C}$. For $x < 0$ and $m \in \mathbb{N}$ we have $|x - 1/n| \geq 1/n$ and so

$$u(x) \geq -\sum_{n=1}^{\infty} \frac{\log n}{n^2} > -\infty = u(1/m).$$

Thus the extension of $u$ to $\mathbb{C}$ is not continuous at 0.

8.2.4 Theorem

Let $D, G$ be domains in $\mathbb{C}$ and let $u$ be subharmonic on $G$, and $f : D \to G$ analytic. Then $v = u \circ f$ is subharmonic on $D$.

**Proof.** We assume that $f$ is non-constant and that $u \neq -\infty$, since otherwise the result is obvious. We show first that $v$ is upper semi-continuous: if $v(z_0) < L$ then $u(f(z_0)) < L$ so $u(w) < L$ near $f(z_0)$ and so $v(z) < L$ near $z_0$.

Assume that $r$ is small and positive and that $f$ is one-one near $z_0$. Take continuous functions $v_n$, decreasing pointwise to $v$ on $S(z_0, r)$, and form the Poisson integrals $V_n$. Then $V_n$ is harmonic on $D(z_0, r)$ and $V_n(z) \to v_n(u)$ as $z \to u \in S(z_0, r)$ from inside the circle. Define $h_n$ on the closure of $W = f(D(z_0, r))$ by $h_n(f(z)) = V_n(z)$. Then $h_n$ is harmonic on $W$ and continuous on the closure of $W$. As $w_m \to w \in \partial W, w_m \in W$, we have, since $u$ is upper semi-continuous,

$$\limsup u(w_m) \leq u(w) = v(f^{-1}(w)) \leq v_n(f^{-1}(w)) = h_n(w)$$

and we get $u \leq h_n$ on $W$. Hence $v \leq v_n$ on $D(z_0, r)$ and

$$v(z_0) \leq v_n(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} v_n(z_0 + re^{i\theta})d\theta \to \frac{1}{2\pi} \int_{0}^{2\pi} v(z_0 + re^{i\theta})d\theta,$$

by the monotone convergence theorem applied to $v_1 - v_n$. Hence $v$ has the sub-mean value property at $z_0$.

Finally, the multiple points $z^*$ of $f$ are isolated, since they are zeros of $f'$. By Theorem 8.2.2 these $z^*$ are removable singularities of $v$, since $v(z) = u(f(z))$ is bounded above as $z \to z^*$. 

8.2.5 Theorem

Let \( u \) be subharmonic and bounded above in \( \mathbb{C} \). Then \( u \) is constant.

Proof. Assume \( u \) non-constant. Then without loss of generality \( u(0) = 0 \). Let \( v = \max\{u,0\} \) so that \( v \) is subharmonic. Let \( 0 < r < s < t < \infty \). Then convexity gives

\[
I(s) = I(s, v) \leq \frac{\log t/s}{\log t/r} I(r) + \frac{\log r/s}{\log r/t} I(t).
\]

Let \( t \to \infty \). Since \( \frac{\log r/s}{\log r/t} \to 0 \) and since \( 0 \leq I(t) < M \) for some fixed \( M \) we get \( I(s) \leq I(r) \). Since \( I \) is non-decreasing we have \( I \) constant on \((0,\infty)\). Since \( v \) is upper semi-continuous and \( v(0) = 0 \) we get \( I(s) \equiv 0 \). By Theorem 7.3.5 we have, for \( |z| \leq s \),

\[
0 \leq v(z) \leq 3I(2s, v) = 0
\]

and so \( v \equiv 0 \). But then \( u \) has a maximum at 0 and so is constant.

We give another proof of this result (Beardon). Assume that \( u \) is non-constant. We can also assume WLOG that \( \sup\{u(z) : z \in \mathbb{C}\} = 0 \). Let \( m = \max\{u(z) : |z| = 1\} \), which exists because \( u \) is upper semi-continuous. Then \( m < 0 \), since otherwise \( u \) has a maximum in \( \mathbb{C} \) and so is constant. Now fix \( z_0 \) with \( |z_0| > 1 \). Let \( \varepsilon > 0 \) and let \( R > 1 \) be large. The function

\[
v(z) = u(z) - \varepsilon \log |z|
\]

is subharmonic in \( 0 < |z| < +\infty \), and we have \( v(z) \leq m \) for \( |z| = 1 \) and for \( |z| = R \), since \( R \) is large. Hence we get

\[
u(z_0) \leq v(z_0) + \varepsilon \log |z_0| \leq m + \varepsilon \log |z_0|,
\]

and so \( u(z_0) \leq m < 0 \) since \( \varepsilon \) may be chosen arbitrarily small. Thus \( u(z) \leq m < 0 \) for \( |z| > 1 \) and so on \( \mathbb{C} \) by the maximum principle, contradicting the assumption that the supremum of \( u \) is 0.

8.2.6 Exercises

(a) Prove Iversen’s theorem: if \( f \) is a non-constant entire function then there exists a path \( \gamma \) tending to infinity such that \( f(z) \to \infty \) as \( z \) tends to infinity on \( \gamma \). (Hint: consider a component \( C_n \) of the set \( E_n = \{z : |f(z)| > n\} \). Prove that \( f \) is unbounded on \( C_n \) and take a component of \( E_{n+1} \)).

(b) Let \( u \) be subharmonic in \( \mathbb{C} \) such that \( u = 0 \) on the imaginary axis but \( u(z) > 0 \) for at least one \( z \) in the right half-plane. Let \( 0 < s < 1/2 \). Prove that there exists a path tending to infinity in the right half-plane on which \( u(z) > |z|^s \). (Hint: take \( s < t < 1/2 \) and \( \delta \) small and positive and consider the function \( u(z) - \delta \Re (z^t) \)).

8.2.7 Lemma

Let \( u \) be subharmonic and bounded above on the domain \( D \) in \( \mathbb{C} \). Suppose that

\[
\limsup_{z \to \zeta, z \in D} u(z) \leq 0
\]

for at least one, and for all but finitely many, \( \zeta \in X = \partial_D \). Then \( u \leq 0 \) on \( D \).
Proof. Let \( \zeta_1, \ldots, \zeta_n \) be the points in \( X \) at which (8.4) fails. Let \( G = \mathbb{C} \setminus \{\zeta_1, \ldots, \zeta_n\} \) and define \( v \) on \( G \) as follows. On \( G \setminus D \) we set \( v = 0 \), while on \( D \) we set \( v = \max\{u, 0\} \).

Then \( v \) is subharmonic and bounded above in \( G \) (it clearly has the sub-mean value property and the fact that \( v \) is upper semi-continuous follows from (8.4)), and hence subharmonic and bounded above in \( \mathbb{C} \). So \( v \) is constant.

Now let \( \zeta^* \in X \) be such that (8.4) does hold. Then there exists a sequence \( z_n \in D \) such that \( z_n \to \zeta^* \) and \( u(z_n) \to 0 \), so that \( v(z_n) \to 0 \). Since \( v \) is constant this gives \( v \equiv 0 \) and \( u \leq 0 \).

8.2.8 Example

This example shows that in the last lemma we cannot delete the hypothesis that \( u \) is bounded above. Let

\[
u(z) = \Re \left( \frac{1 + z}{1 - z} \right), \quad |z| < 1.
\]

Then for \( |w| = 1, w \neq 1 \) we have \( u(z) \to 0 \) as \( z \to w \), but \( u \) is unbounded in \( D(0, 1) \).
Chapter 9

Harmonic measure

9.1 Definition of the harmonic measure

For suitable domains $D$ and subsets $E$ of $X = \partial_{\infty} D$ the harmonic measure $\omega(z, E, D)$ will be defined for $z \in D$. It will then turn out that the harmonic measure is for fixed $E$ a harmonic function of $z$, while for fixed $z$ it is a measure on a suitable collection of subsets of $X$.

One of the main applications of harmonic measure is the two-constants theorem 9.2.10 which gives a powerful improvement of the maximum principle for subharmonic functions.

9.1.1 Semi-regular domains

Let $D$ be a domain in $\mathbb{C}$. We say that $D$ is semi-regular if $X = \partial_{\infty} D$ is infinite and all but finitely many $x \in X = \partial_{\infty} D$ are regular. Here we use the spherical metric on $\mathbb{C}^*$, which makes $\mathbb{C}^*$ compact.

Note that if $U$ is an open subset of $X$ then $U = V \cap X$ for some open subset $V$ of $\mathbb{C}^*$, by definition of the relative topology. Also any closed subset of $X$ is compact, as is $X$, because a closed subset of a compact set is compact.

For a set $Y$ a collection $S$ of subsets of $Y$ is called a $\sigma$-algebra if it is non-empty and has the following two properties: (i) $A \in S$ implies that $Y \setminus A$ is in $S$; (ii) if $A_1, A_2, \ldots$ are countably many elements of $S$ then their union is in $S$. Obviously the power set of $Y$ is a $\sigma$-algebra. It is easy to prove that if $S_t$ is a $\sigma$-algebra of subsets of $Y$ for every $t \in T$ then $\bigcap_{t \in T} S_t$ is also a $\sigma$-algebra of subsets of $Y$. So for any collection $U$ of subsets of $Y$, taking the intersection of all $\sigma$-algebras $S$ of subsets of $Y$ with $U \subseteq S$ gives a $\sigma$-algebra, which is said to be generated by $U$.

If $Y$ is also a topological space then we can form the $\sigma$-algebra generated by the open subsets of $Y$, which is the smallest $\sigma$-algebra of subsets of $Y$ containing all the open subsets of $Y$. Its elements are called Borel sets.

We now identify the Borel sets of $X = \partial_{\infty} D$. We claim that the Borel subsets of $X$ are precisely the sets $B \cap X$ where $B$ is a Borel subset of $\mathbb{C}^*$. To see this let $B_1$ be the collection of Borel subsets of $X$ and let $F$ be the collection of Borel subsets of $\mathbb{C}^*$. Then $B_2 = \{ B \cap X : B \in F \}$ is a $\sigma$-algebra and every open subset of $X$ is an element of $B_2$ since it is $U \cap X$ for some open $U \in F$. So $B_1 \subseteq B_2$.

But $B_3 = \{ W \subseteq \mathbb{C}^* : W \cap X \in B_1 \}$ is a $\sigma$-algebra and it contains all open subsets of $\mathbb{C}^*$, so $F \subseteq B_3$. Hence $V \in B_2$ gives $V = B \cap X$ with $B \in F$ and hence $B \in B_3$, so that $V = B \cap X \in B_1$. Thus $B_1 = B_2$. 

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9.1.2 An example of a semi-regular domain

For \( n \in \mathbb{N} \) let \( C_n \) be the circle \(|z - n| = \frac{1}{4}\). Let \( D \) be the unbounded component of \( \mathbb{C} \setminus \bigcup_{n=1}^{\infty} C_n \) i.e.

\[
D = \left\{ w \in \mathbb{C} : |w - n| > \frac{1}{4} \quad \text{for all} \quad n \in \mathbb{N} \right\}.
\]

For \( x \in C_n \) the component of \( \partial_\infty D \) containing \( x \) is not \( \{x\} \), since \( C_n \) is itself connected, and so \( x \) is a regular point of \( X \) by 8.1.8. On the other hand if \( \infty \in E \subseteq X \) and \( E \) is connected then \( E = \{\infty\} \), because if \( y \neq \infty \) is in \( E \) we can take a large \( n \in \mathbb{N} \) and partition \( E \) as

\[
\left\{ x \in E : |x| < n + \frac{1}{2} \right\} \cup \left\{ x \in E : |x| > n + \frac{1}{2} \right\}
\]

with both sets relatively open and non-empty. Thus the component of \( \partial_\infty D \) containing \( \infty \) is just \( \{\infty\} \). So \( \infty \) fails to satisfy the sufficient condition 8.1.8 for a barrier, but our definition of harmonic measure will still make sense for \( D \).

9.1.3 A linear functional

Let \( D \) be a semi-regular domain in \( \mathbb{C} \), with boundary \( X = \partial_\infty D \). Let \( Y \) be the vector space of functions \( f : X \to \mathbb{R} \) which are bounded on \( X \) and continuous at all but finitely many points of \( X \). Applying Perron’s method gives a harmonic Perron function \( v_f \) on \( D \), and Theorem 8.1.7 shows that

\[
\lim_{z \to x, z \in D} v_f(z) = f(x)
\]

for all but finitely many \( x \in X \).

If \( f, g \in Y \) and \( f = g \) except on a finite set then \( v_f - v_g \) is harmonic and bounded and has boundary values \( 0 \) except on a finite set, so that \( v_f = v_g \) by Lemma 8.2.7. Similarly if \( f, g \in Y \) then \( v_{f+g} - v_f - v_g \) is harmonic and bounded and again has boundary values \( 0 \) except on a finite set, so we get \( v_{f+g} = v_f + v_g \). Also if \( f \in Y \) with \( f \leq 0 \) on \( X \) then \( v_f \leq 0 \), again by Lemma 8.2.7. Finally if \( f \) is a constant (say \( M \)) on \( X \) then \( v_f \) has boundary values \( M \) except on a finite set and so is \( M \) by Lemma 8.2.7.

Fix \( z \) in \( D \). Then

\[
f \mapsto L(z, f) = v_f(z)
\]

is a non-negative linear functional on \( Y \) (this just means that \( f \geq 0 \) on \( X \) implies that \( v_f(z) \geq 0 \) on \( D \)).

The rest of this section will be devoted to proving the following theorem from first principles: it can, however, be deduced rather quickly from the Riesz representation theorem (2.14 of W. Rudin, Real and Complex Analysis).

9.1.4 Theorem

Let \( D \) be as above and fix \( z_0 \) in \( D \). Then there exist a \( \sigma \)-algebra \( \Pi \) of subsets of \( X \) and a probability measure \( \mu \) on \( \Pi \) (this means a measure \( \mu : \Pi \to [0, 1] \) with \( \mu(X) = 1 \)) such that:

(i) every Borel subset \( E \) of \( X \) is in \( \Pi \);

(ii) for \( E \) in \( \Pi \), the measure \( \mu(E) \) is the infimum of \( \mu(V) \) over all open \( V \) containing \( E \);
(iii) \( \mu(E) \) is the supremum of \( \mu(K) \) over all compact \( K \subseteq E \);

(iv) if \( A \subseteq B \) and \( \mu(B) = 0 \) then \( \mu(A) \) exists and is 0;

(v) if \( V \) is open then \( \mu(V) \) is the supremum of \( L(z_0, g) = v_g(z_0) \) taken over all continuous functions \( g : X \to [0, 1] \) such that \( g \leq \chi_V \) on \( X \).

In order to be a measure, \( \mu \) must satisfy \( \mu(\bigcup E_j) = \sum \mu(E_j) \) whenever the \( E_j \) are countably many pairwise disjoint elements of \( \Pi \). Now \( V \) is an open subset of \( X \) if and only if \( V = U \cap X \) where \( U \) is open in \( \mathbb{C}^* \), and so if and only if \( K = X \setminus V \) is of form \( K = F \cap X \) where \( F \) is closed in \( \mathbb{C}^* \). But then \( K \) is compact, since \( X \) is compact, and \( \mu(V) = 1 - \mu(K) \) and \( \mu(E) = 1 - \mu(X \setminus E) \). Hence properties (ii) and (iii) are equivalent provided \( \mu \) is a measure.

Note that (ii) and (v) imply that this measure \( \mu \) is unique (because there is a unique definition for open \( V \) and hence for every \( E \in \Pi \)).

9.1.5 Example

Let \( D = D(0, 1) \). For a Borel subset \( A \) of \( X = \partial D \), let \( \chi_A(t) \) be 1 on \( A \) and 0 elsewhere, and set

\[
\mu(A) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) \chi_A(e^{it}) \, dt,
\]

in which \( K(z, e^{it}) \) is the Poisson kernel. Note that if we keep \( A \) fixed then what we get is a harmonic function of \( z \) on \( D \). Also (i) is satisfied because this integral exists for every Borel set \( A \).

To check (ii) let \( r = |z| < 1 \). Obviously if \( A \subseteq V \) then \( \mu(A) \leq \mu(V) \). But given \( A \subseteq X \) and \( \delta > 0 \) we can look at \( B = \{ t \in [0, 2\pi] : e^{it} \in A \} \) and the Lebesgue measure \( \lambda(B) \) of \( B \) is the infimum of the Lebesgue measure of \( U \) over all open sets \( U \) with \( B \subseteq U \subseteq \mathbb{R} \). So we can find an open \( U \subseteq \mathbb{R} \) such that \( B \subseteq U \) and \( U \setminus B \) has Lebesgue measure less than \( \delta \). Now let \( V = \{ e^{it} : t \in U \} \). Then \( V \) is an open subset of \( X \) with \( A \subseteq V \) and \( \chi_V(e^{it}) = 1 \) implies that \( t \in U \). Hence \( \chi_V(e^{it}) = \chi_A(e^{it}) \) for all \( t \in [0, 2\pi] \) apart from a set of Lebesgue measure at most \( \delta \). This gives, since

\[
0 \leq K(z, e^{it}) \leq \frac{1 + r}{1 - r},
\]

the inequality

\[
\mu(A) \leq \mu(V) \leq \mu(A) + \delta \left( \frac{1 + r}{1 - r} \right),
\]

which proves (ii).

To check (iv) let \( B \) be a Borel subset of \( X \) with \( \mu(B) = 0 \) and let \( A \) be any subset of \( B \). Let \( |z| = r < 1 \). Since

\[
K(z, t) \geq \frac{1 - r}{1 + r}
\]

we have

\[
\int_0^{2\pi} \chi_B(e^{it}) \, dt = 0.
\]

So the set \( C = \{ t \in [0, 2\pi] : e^{it} \in B \} \) has Lebesgue measure 0. Then every subset of \( C \) is Lebesgue measurable with Lebesgue measure 0, and \( \chi_A(e^{it}) = 0 \) for every \( t \in [0, 2\pi] \) apart from a set of Lebesgue measure 0. Hence (9.1) implies that

\[
\frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) \chi_A(e^{it}) \, dt
\]
exists and is 0.

Now we check (v). Let \( A \subseteq X \) be open. First if \( g : X \to [0, 1] \) is continuous and \( g \leq \chi_A \) then

\[
L(z, g) = \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) g(e^{it}) \, dt \leq \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) \chi_A(e^{it}) \, dt = \mu(A).
\]

Next, \( 1 - \chi_A \) is upper semi-continuous on \( X \) and by Theorem 1.5.1 we can take continuous \( f_n \uparrow \chi_A \) on \( X \). Thus

\[
\mu(A) \geq \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) f_n(e^{it}) \, dt = L(z, f_n) \uparrow \frac{1}{2\pi} \int_0^{2\pi} K(z, e^{it}) \chi_A(e^{it}) \, dt = \mu(A)
\]

by the monotone convergence theorem. Thus \( \mu \) satisfies conditions (i) to (v).

The fact that \( \mu \) is a measure in this example is easily seen from the fact that if \( E_1, E_2, \ldots \) are pairwise disjoint subsets of \( X \) with union \( E \) then \( \chi_E = \sum \chi_{E_j} \) and so the integrals add up. Also \( \mu(X) = 1 \) because the Poisson extension to \( D \) of a constant function on \( X \) is constant.

For general semi-regular domains \( D \) the situation is more complicated and the first step in the proof of Theorem 9.1.4 involves looking at upper semi-continuous functions. Throughout this section \( D \) will be a semi-regular domain and \( X \) will be \( \partial_\infty D \).

9.1.6 Theorem

Let \( f : X \to \mathbb{R} \) be upper semi-continuous and let \( H_f \) be the family of all continuous real-valued \( g \) with \( g \geq f \) on \( X \). Choose continuous functions \( f_n \) such that \( f_n \downarrow f \) on \( X \), and let \( u_n(z) = L(z, f_n) = v_{f_n}(z) \). Then \( u = \lim u_n \) is either identically \( -\infty \) on \( D \) or harmonic in \( D \), and \( u(z) = \inf \{ L(z, g) : g \in H_f \} \).

In particular, \( u \) is independent of the particular choice of the sequence \( (f_n) \).

**Proof.** The functions \( f_n \) exist by Theorem 1.5.1, and the first assertion follows from Harnack’s theorem 7.3.10, since \( u_n \leq u_{n-1} \) on \( D \). Also the set \( H_f \) makes sense, since \( f \) is bounded above on \( X \).

To prove that \( u(z) = \inf \{ L(z, g) : g \in H_f \} \), take any \( g \in H_f \), and \( \varepsilon > 0 \), and any \( w_0 \) in \( X \). Set \( G(x) = g(x) + \varepsilon \). Then for some large \( N \) we have \( G(w_0) > f_N(w_0) \) and so \( G(w) > f_N(w) \) for all \( w \) in a relatively open neighbourhood \( V \) of \( w_0 \). Hence \( G(w) > f_M(w) \) for all \( w \) in \( V \) and all \( n \geq N \). Now the compact set \( X \) can be covered by finitely many such \( V \), and so there exists \( M \) such that \( G(w) > f_M(w) \) for all \( w \) in \( X \). Thus \( u(z) \leq u_M(z) \leq L(z, G) = L(z, g) + \varepsilon \) for all \( z \) in \( D \). This gives \( u(z) \leq L(z, g) \) on \( D \).

Next, take any \( z \) in \( D \), and \( K > u(z) \), and \( n \in \mathbb{N} \) with \( u_n(z) < K \). Now \( f_n \geq f \) on \( X \) and \( L(z, f_n) = u_n(z) < K \), so that \( K \) is not a lower bound for \( \{ L(z, g) : g \in H_f \} \). Thus \( u(z) \) is the greatest lower bound as asserted.

9.1.7 Definition

For an upper semi-continuous function \( f \) on \( X \) we define \( u(z) \) on \( D \) as follows. For each \( z \) in \( D \), we set \( u(z) = u_f(z) \) to be the infimum of \( L(z, g) = v_g(z) \) over all continuous \( g \) with \( g \geq f \) on \( X \). We have just seen that \( u \) is harmonic or identically \( -\infty \) on \( D \), and we call \( u \) the harmonic extension of \( f \) to \( D \). Note that if \( f \) is itself continuous, then \( u(z) = v_f(z) = L(z, f) \); in particular this is true if \( f \) is constant. It is clear that if \( f_1 \) and \( f_2 \) are upper semi-continuous on \( X \) with \( f_1 \leq f_2 \) then the harmonic extension of \( f_1 \) is bounded above by that of \( f_2 \). Hence if \( A, B \) are real numbers and \( A \leq f \leq B \) on \( X \) then \( A \leq u(z) \leq B \) on \( D \).
For a closed subset $E$ of $X$, the characteristic function $\chi_E$ is upper semi-continuous, and we write

$$\omega(z, E, D) = u_{\chi_E}(z).$$

This is a harmonic function on $D$, bounded above by 1 and below by 0. We will refer to $\omega(z, E, D)$ as the harmonic measure of $E$ with respect to $D$. Note that if $g$ is continuous on $X$ with $g \geq \chi_E$ then $g \geq 0$ and $h(x) = \min\{g(x), 1\}$ is also continuous with $h \geq \chi_E$. So in fact

$$\omega(z, E, D) = \inf\{L(z, h) : h : X \to [0, 1], h \geq \chi_E, h \text{ continuous}\}.$$

To see this, observe that the set of $h$ as above is a subset of the set of $g$, so any lower bound for the $L(z, g)$ is a lower bound for the $L(z, h)$, and hence the infimum of the $L(z, h)$ is not less than that of the $L(z, g)$; on the other hand, given $g$ there exists an $h$ with $\chi_E \leq h \leq g$ and so any lower bound for the $L(z, h)$ is a lower bound for the $L(z, g)$ also.

Obviously,

$$\omega(z, X, D) = 1, \quad \omega(z, \emptyset, D) = 0,$$

since in both cases the characteristic function is constant.

Note that if $A$ and $B$ are closed subsets of $X$ and $A \subseteq B$ then any $g : X \to [0, 1]$ which satisfies $g \geq \chi_B$ also satisfies $g \geq \chi_A$, so $\omega(z, B, X) \geq \omega(z, A, X)$.

### 9.1.8 Urysohn’s lemma

Let $Y$ be a compact metric space and let $K \subseteq V \subseteq Y$, with $K$ compact and $V$ open (in both cases relative to $Y$). Then there exists a continuous function $g : Y \to [0, 1]$, with $g = 1$ on $K$ and $g = 0$ off $V$.

We just define $g$ by

$$1 - g(y) = \frac{d(y, K)}{d(y, K) + d(y, V^c)}$$

in which $V^c = Y \setminus V$ and $d$ denotes the metric. Here the distance $d(y, A)$ is defined for any closed (and hence compact) $A \subseteq Y$ and is the minimum of the continuous function $d(y, a)$ over $a \in A$. This distance is continuous for a given closed $A$, and we cannot have $d(y, K) + d(y, V^c) = 0$, because if $d(y, K) = 0$ then $y \in K \subseteq V$ and so $d(y, V^c) > 0$.

### 9.1.9 Boundary behaviour of the harmonic measure of a closed set

Let $E$ be a closed subset of $X$.

(a) If $x \in X \setminus E$ and $x$ is a regular boundary point of $D$ then $\omega(z, E, D) \to 0$ as $z \to x$ in $D$.

To see this, just take $K = E$ and $V = X \setminus \{x\}$ in Urysohn’s lemma. This gives a continuous $g : X \to [0, 1]$ with $g \geq \chi_E$ on $X$ and $g(x) = 0$. On $D$ we have

$$0 \leq \omega(z, E, D) \leq v_g(z) \to g(x) = 0, \quad z \to x.$$

In particular this is always the case if $D$ is simply connected, by §8.1.4.

(b) If $x$ is a regular boundary point of $D$, and an interior point of $E$ (with respect to $X$), then $\omega(z, E, D) \to 1$ as $z \to x$ in $D$. 
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To see this, take \( r > 0 \) with \( V = D(x,r) \cap X \subseteq E \). Then apply Urysohn’s lemma to get a continuous \( g : X \to [0,1] \) with \( g(x) = 1 \) and \( g = 0 \) off \( V \) (and so \( g = 0 \) off \( E \)). If \( h : X \to [0,1] \) is continuous with \( h \geq \chi_E \) on \( X \), then we have \( h \geq g \) on \( X \) and so \( L(z,h) \geq L(z,g) \) on \( D \). Thus, on \( D \),

\[
1 \geq \omega(z,E,D) \geq L(z,g) \to g(x) = 1, \quad z \to x.
\]

(c) If \( D \) is regular then, with \( f = \chi_E \), the harmonic measure \( \omega(z,E,D) \) agrees with the Perron function \( v_f(z) \) defined in §8.1.1.

To prove (c), first let \( g : X \to [0,1] \) be continuous with \( g \geq f \), and let \( y \) be a member of the Perron family \( U(f) \) as defined in §8.1.1. Then for every \( \zeta \in X \) we have

\[
\limsup_{z \to \zeta, z \in D} y(z) \leq f(x) \leq g(x) = \lim_{z \to \zeta, z \in D} v_g(z), \quad \limsup_{z \to \zeta, z \in D} (y(z) - v_g(z)) \leq 0
\]

and so the maximum principle shows that \( y(z) \leq v_g(z) = L(z,g) \) on \( D \). Thus \( y(z) \leq \omega(z,E,D) \) on \( D \), and taking the supremum over these \( y \) we get \( v_f(z) \leq \omega(z,E,D) \) on \( D \). But we also know that if \( x \in X \setminus E \) then \( \omega(z,E,D) \to 0 \) as \( z \to x, z \in D \). Since \( \omega(z,E,D) \leq 1 \) we get \( \omega(z,E,D) \in U(f) \) and so \( \omega(z,E,D) \leq v_f(z) \).

9.1.10 Example

In §9.1.9(a), we cannot delete the hypothesis that \( x \) is a regular boundary point. Let \( D_1 \) be the domain \( 0 < |z| < 1 \) as in Example 8.1.2, let \( A \) be the circle \( |x| = 1 \), and let \( g = \chi_A \). Then \( g \) is continuous on \( X \) and \( g = 1 \) except at one point, and so \( v_g = v_1 = 1 \) by the argument in §9.1.3. Since \( L(z,h) \geq L(z,g) \) for continuous \( h \) with \( h \geq g \) on \( X \), this now shows that \( \omega(z,A,D_1) = L(z,g) \equiv 1 \) on \( D_1 \), and so \( \omega(z,A,D_1) \) fails to tend to 0 as \( z \to 0 \).

We return to this theme in §14.1.8.

9.1.11 The harmonic measure of an open set

For a closed subset \( E \) of \( X \), we have defined \( \omega(z,E,D) \) to be the infimum of \( L(z,g) = v_g(z) \) over all continuous \( g : X \to [0,1] \) with \( g \geq \chi_E \) on \( X \).

If \( U \) is an open subset of \( X \), we define \( \omega(z,U,D) \) to be the supremum of \( L(z,g) \) over all continuous \( g : X \to [0,1] \) with \( g \leq \chi_U \) on \( X \).

Note that if \( A \) and \( B \) are relatively open subsets of \( X \) and \( A \subseteq B \) then any \( g : X \to [0,1] \) which satisfies \( g \leq \chi_A \) also satisfies \( g \leq \chi_B \), so \( \omega(z,B,X) \geq \omega(z,A,X) \).

Note also that if \( E \) is a clopen (closed and open) subset of \( X \) then \( h = \chi_E \) is continuous, and so the two definitions both give \( \omega(z,E,D) = v_h(z) = L(z,h) \) and in particular they agree.

The next lemma shows that this definition gives the “expected” result that

\[
\omega(z,U,D) = 1 - \omega(z,X \setminus U,D).
\]

9.1.12 Lemma

Let \( U \) be an open subset of \( X \) and let \( E = X \setminus U \). Then for every \( z \in D \) we have \( \omega(z,U,D) = 1 - \omega(z,E,D) \).
Proof. Obviously if \( g : X \to [0,1] \) then \( g \leq \chi_U \) if and only if \( h = 1 - g \geq 1 - \chi_U = \chi_E \) and \( g \) is continuous if and only if \( h \) is. Thus

\[
\omega(z, U, D) = \sup \{ L(z, g) : g : X \to [0,1], g \leq \chi_U, g \text{ continuous} \}
\]

\[
= \sup \{ L(z, g) : g : X \to [0,1], h = 1 - g \geq \chi_E, g \text{ continuous} \}
\]

\[
= \sup \{ 1 - L(z, h) : h : X \to [0,1], h \geq \chi_E, h \text{ continuous} \}
\]

\[
= 1 - \inf \{ L(z, h) : h : X \to [0,1], h \geq \chi_E, h \text{ continuous} \}
\]

\[
= 1 - \omega(z, E, D).
\]

9.1.13 \( \mu \)-measurable sets

Fix \( z_0 \) in \( D \). For any subset \( C \) of \( X \) we define:

\( \mu^+(C) \) to be the infimum of \( \omega(z_0, U, D) \) over all open \( U \) with \( C \subseteq U \subseteq X \);

\( \mu^-(C) \) to be the supremum of \( \omega(z_0, E, D) \) over all closed \( E \) with \( E \subseteq C \subseteq X \).

Obviously \( \mu^+(C) \) and \( \mu^-(C) \) both exist, and they are in \([0,1]\).

We say that \( C \) is \( \mu \)-measurable if \( \mu^+(C) = \mu^-(C) \), in which case we denote the common value by \( \mu(C) \).

9.1.14 Lemma

Let \( A \subseteq X \) and let \( B = X \setminus A \). Then:

(a) \( \mu^-(A) \leq \mu^+(A) \);

(b) \( \mu^+(A) = 1 - \mu^-(B) \);

(c) if \( A \) is \( \mu \)-measurable then so is \( B \), and \( \mu(A) = 1 - \mu(B) \).

Proof. (a) Take closed \( E \) and open \( U \) with \( E \subseteq A \subseteq U \), and using Urysohn’s lemma let \( g : X \to [0,1] \) be continuous, with \( g = 1 \) on \( E \) and \( g = 0 \) off \( U \). Then \( \chi_E \leq g \leq \chi_U \) and so

\[
\omega(z_0, E, D) \leq L(z_0, g) \leq \omega(z_0, U, D).
\]

This proves (a), and also establishes (iv) of Theorem 9.1.4, because if \( A \subseteq C \) and \( \mu(C) = 0 \) then

\[
0 \leq \mu^-(A) \leq \mu^+(A) \leq \mu^+(C) = 0.
\]

(b) Here

\[
\mu^+(A) = \inf \{ \omega(z_0, U, D) : A \subseteq U \ (U \text{ open}) \}
\]

\[
= \inf \{ 1 - \omega(z_0, E, D) : E \subseteq B \ (E \text{ closed}) \}
\]

\[
= 1 - \sup \{ \omega(z_0, E, D) : E \subseteq B \ (E \text{ closed}) \}
\]

\[
= 1 - \mu^-(B).
\]

Similarly we get \( \mu^+(B) = 1 - \mu^-(A) \) and (c) follows.

9.1.15 Lemma

Let \( U \) be an open subset of \( X \). Then \( U \) is \( \mu \)-measurable and \( \mu(U) = \omega(z_0, U, D) \).
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Proof. Obviously \( \mu^+(U) = \omega(z_0, U, D) \). Let \( \delta > 0 \). Then by the definition of \( \omega \) for open \( U \) there exists a continuous \( g : X \to [0, 1] \) with \( g \leq \chi_U \) on \( X \) and

\[
L(z_0, g) > \omega(z_0, U, D) - \delta.
\]

Let \( E \) be the closed set \( E = \{ x \in X : g(x) \geq \delta \} \). Then \( E \subseteq U \), since \( g \leq \chi_U \). This time using the definition of the harmonic measure for closed sets, choose a continuous \( h : X \to [0, 1] \), with \( h \geq \chi_E \) on \( X \), and with

\[
L(z_0, h) < \omega(z_0, E, D) + \delta.
\]

Now, on \( E \) we have \( g \leq 1 \leq \chi_E \leq h \) and on \( X \setminus E \) we have \( g < \delta \) and \( h \geq 0 \). Thus

\[
g \leq h + \delta
\]
on \( X \). So

\[
\omega(z_0, E, D) > L(z_0, h) - \delta \geq L(z_0, g) - 2\delta > \omega(z_0, U, D) - 3\delta.
\]

Since \( E \) is closed and is a subset of \( U \), we have

\[
\mu^-(U) \geq \omega(z_0, E, D) \geq \omega(z_0, U, D) - 3\delta = \mu^+(U) - 3\delta,
\]

and the lemma follows since \( \delta \) is arbitrary.

In particular, for open \( U \) we have \( \mu(U) = \mu^+(U) = \omega(z_0, U, D) \) and this is the supremum of \( L(z, g) \) over all continuous \( g \) with \( g \leq \chi_U \) on \( X \), which immediately establishes assertion (v) of Theorem 9.1.4.

It follows from the last three lemmas that closed sets \( E \) are also \( \mu \)-measurable with \( \mu(E) = \omega(z_0, E, D) \). Using the definitions of \( \mu^- \) and \( \mu^+ \), we now have (ii) and (iii) of Theorem 9.1.4 for \( \mu \)-measurable subsets of \( X \).

The next few results deal with the effect of taking unions. For general sets this is rather involved, so it is convenient first to look at disjoint unions of open (and then closed) sets.

9.1.16 Lemma

Let \( U_1, U_2 \) be disjoint open subsets of \( X \). Then

\[
\mu(U_1) + \mu(U_2) = \mu(W), \quad W = U_1 \cup U_2.
\]

Once we have this result for two disjoint open subsets it extends by induction to finitely many pairwise disjoint open subsets.

Proof. Let \( E \subseteq W \) be closed. Then so are \( E_1 = E \cap U_1 = E \setminus U_2 \) and \( E_2 = E \cap U_2 \). Using Urysohn’s lemma form continuous \( g_1, g_2 : X \to [0, 1] \) with \( g_j = 1 \) on \( E_j \) and \( g_j = 0 \) off \( U_j \). Then \( g = g_1 + g_2 : X \to [0, 1] \) is continuous and \( \chi_E \leq g \) on \( X \). Since \( g_j \leq \chi_U \) this gives

\[
\omega(z_0, E, D) \leq L(z_0, g) = L(z_0, g_1) + L(z_0, g_2)
\]

\[
\leq \omega(z_0, U_1, D) + \omega(z_0, U_2, D) = \mu(U_1) + \mu(U_2).
\]

Taking the sup over closed \( E \subseteq W \) we get

\[
\mu(W) = \mu^-(W) \leq \mu(U_1) + \mu(U_2).
\]
Now let \( h_j : [0, 1] \to X \) be continuous with \( h_j \leq \chi_{U_j} \). Then \( h = h_1 + h_2 : X \to [0, 1] \) satisfies \( h \leq \chi_W \).

So
\[
L(z_0, h_1) + L(z_0, h_2) = L(z_0, h) \leq \omega(z_0, W, D) = \mu(W).
\]

Taking the sup over \( h_1 \) and \( h_2 \) gives
\[
\mu(U_1) + \mu(U_2) = \omega(z_0, U_1, D) + \omega(z_0, U_2, D) \leq \mu(W).
\]

9.1.17 Lemma

Let \( E_1, E_2 \) be disjoint closed subsets of \( X \), with union \( E \). Then
\[
\mu(E_1) + \mu(E_2) = \mu(E).
\]

Again this result extends to finitely many pairwise disjoint closed sets.

Proof. Take an open \( U \) with \( E \subseteq U \subseteq X \). Since the distance between \( E_1 \) and \( E_2 \) is positive, as is the distance from \( E \) to \( X \setminus U \), we can form disjoint open sets \( U_j \) with \( E_j \subseteq U_j \subseteq U \). To do this we can take a small positive \( \rho \) and the union of the open discs of centre \( x \in E_j \) and radius \( \rho \). Now
\[
\mu(E_1) + \mu(E_2) = \mu^+(E_1) + \mu^+(E_2)
\]
\[
\leq \omega(z_0, U_1, D) + \omega(z_0, U_2, D)
\]
\[
= \mu(U_1) + \mu(U_2) = \mu(U_1 \cup U_2) \leq \mu(U)
\]

by the result for disjoint open sets. Now taking the infimum over \( U \supseteq E \) gives \( \mu(E_1) + \mu(E_2) \leq \mu(E) \).

Next, take \( \delta > 0 \) and open sets \( V_j \) with \( E_j \subseteq V_j \) and \( \mu(V_j) = \omega(z_0, V_j, D) < \mu(E_j) + \delta \). By intersecting with open sets \( U_j \) as in the previous part it may be assumed that \( V_1, V_2 \) are disjoint. Now
\[
\mu(E) = \mu^+(E) \leq \mu(V_1 \cup V_2) = \mu(V_1) + \mu(V_2) < \mu(E_1) + \mu(E_2) + 2\delta,
\]
and letting \( \delta \to 0 \) completes the proof.

9.1.18 Theorem

Let \( V_j \) be open subsets of \( X \) and let \( W = \bigcup_{j=1}^{\infty} V_j \). Then \( \mu(W) \leq \sum_{j=1}^{\infty} \mu(V_j) \) (Thus \( \mu \) is countably sub-additive on open sets).

Proof. We first take the case where \( W = V_1 \cup V_2 \). Let \( H \) be a closed (and so compact) subset of \( W \). We claim that there exists \( \rho > 0 \) such that for each \( x \in H \) the (spherical) disc \( D(x, \rho) \) is contained in one of the \( V_k \). To see this note that each \( x \in H \) has \( \rho(x) > 0 \) such that \( X \cap D(x, 2\rho(x)) \) is contained in one of the \( V_j \). So by compactness there exist \( x_1, \ldots, x_N \) with
\[
H \subseteq \bigcup_{j=1}^{N} D(x_j, \rho(x_j)).
\]

Let \( \rho \) be the minimum of the \( \rho(x_j) \), for \( j = 1, \ldots, N \). Then each \( x \in H \) lies in one of the \( D(x_j, \rho(x_j)) \), and \( D(x, \rho) \) is a subset of \( D(x_j, 2\rho(x_j)) \), which in turns lies in one of the \( V_k \).

Let
\[
H_j = \{ x \in V_j : \text{dist} (x, X \setminus V_j) \geq \rho \} = \{ x \in X : D(x, \rho) \subseteq V_j \}.
\]
Then each $H_j$ is closed by continuity and so compact, and $H \subseteq H_1 \cup H_2$. Use Urysohn’s lemma to define continuous $g_j : X \rightarrow [0,1]$ such that $g_j = 1$ on $H_j$ and $g_j = 0$ off $V_j$. Then

$$
\chi_H(x) \leq g(x) = \min \left\{ 1, \sum_{j=1}^{2} g_j(x) \right\}, \quad g_j(x) \leq \chi_{V_j}(x).
$$

So, since $H$ is closed and $P_j$ is open,

$$
\omega(z_0, H, D) \leq L(z_0, g) \leq \sum_{j=1}^{2} L(z_0, g_j) \leq \sum_{j=1}^{2} \omega(z_0, V_j, D) = \sum_{j=1}^{2} \mu(V_j).
$$

Taking the infimum over closed $H \subseteq \mathcal{W}$ is arbitrary completes the proof in this case.

This result now extends by induction to the union of finitely many $V_j$. To handle the general case just note that if $H \subseteq \mathcal{W}$ is closed then $H$ is compact and lies in the union of finitely many $V_j$, so that

$$
\omega(z_0, H, D) = \mu^+(H) \leq \mu \left( \bigcup_{j=1}^{N} V_j \right) \leq \sum_{j=1}^{N} \mu(V_j).
$$

9.1.19 Lemma

$\mu^+$ is countably sub-additive on subsets of $X$.

Proof. Let $A_j$ be subsets of $X$, $j = 1, 2, \ldots$, let $B = \bigcup_{j \in \mathbb{N}} A_j$ and $\varepsilon > 0$ and choose open $U_j$ such that

$$
A_j \subseteq U_j, \quad \omega(z_0, U_j, D) < \mu^+(A_j) + \varepsilon/2^j.
$$

Then $B$ is a subset of the union of the $U_j$ and so, by Lemma 9.1.18,

$$
\mu^+(B) \leq \mu^+ \left( \bigcup U_j \right) = \mu \left( \bigcup U_j \right) \leq \mu(U) = \omega(z_0, U_j, D) \leq \varepsilon + \sum \mu^+(A_j).
$$

9.1.20 Lemma

Let $A$ be a $\mu$-measurable subset of $X$ and let $\delta > 0$. Then there exist closed $E$ and open $U$ with $E \subseteq A \subseteq U$ and $\mu(U \setminus E) < \delta$.

Proof. Let $\rho > 0$ and choose closed $E$ and open $U$ with $E \subseteq A \subseteq U$ and $\mu(A) = \mu^-(A) < \mu(E) + \rho$, $\mu(A) = \mu^+(A) > \mu(U) - \rho$.

Take a closed subset $G$ of the open set $U \setminus E$ such that

$$
\mu(U \setminus E) < \mu(G) + \rho.
$$

Then

$$
\mu(U \setminus E) + \mu(E) < \mu(G) + \mu(E) + \rho = \mu(G \cup E) + \rho \leq \mu(U) + \rho < \mu(E) + 3\rho.
$$
9.1.21 Theorem
Let $A_1, A_2$ be $\mu$-measurable subsets of $X$. Then $H = A_1 \cup A_2$ is $\mu$-measurable.

Proof. Take $\delta > 0$ and closed $E_j$ and open $U_j$ with $E_j \subseteq A_j \subseteq U_j$ and $\mu(U_j \setminus E_j) < \delta$. Let $K = E_1 \cup E_2$. Then $H \setminus K \subseteq (U_1 \setminus E_1) \cup (U_2 \setminus E_2)$ and

$$\mu^+(H) \leq \mu^+(K) + \mu^+(H \setminus K) \leq \mu^+(K) + \mu^+(U_1 \setminus E_1) + \mu^+(U_2 \setminus E_2) < \mu^+(K) + 2\delta = \mu^-(K) + 2\delta \leq \mu^-(H) + 2\delta.$$ 

It follows at once that the union of finitely many $\mu$-measurable sets is $\mu$-measurable.

9.1.22 Lemma
Let $A_1, A_2$ be pairwise disjoint $\mu$-measurable subsets of $X$. Then

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2).$$

Proof. First, we have

$$\mu(A_1 \cup A_2) = \mu^+(A_1 \cup A_2) \leq \mu^+(A_1) + \mu^+(A_2) = \mu(A_1) + \mu(A_2)$$

by Lemma 9.1.19. Next, choose closed sets $B_j$ with $B_j \subseteq A_j$. Then $B_1 \cap B_2 = \emptyset$ and so

$$\mu(B_1) + \mu(B_2) = \mu(B_1 \cup B_2) \leq \mu^-(A_1 \cup A_2) = \mu(A_1 \cup A_2).$$

Since $B_1$ and $B_2$ are arbitrary we get

$$\mu(A_1) + \mu(A_2) = \mu^-(A_1) + \mu^-(A_2) \leq \mu(A_1 \cup A_2).$$

This lemma obviously extends to the union of finitely many disjoint $\mu$-measurable sets.

9.1.23 Theorem
Let $A_j, j \in \mathbb{N}$, be pairwise disjoint $\mu$-measurable sets, with union $B$. Then $B$ is $\mu$-measurable and

$$\mu(B) = \sum \mu(A_j).$$

Proof. We know that

$$\mu^+(B) \leq \sum \mu^+(A_j) = \sum \mu(A_j).$$

But, if $N$ is finite,

$$\sum_{j=1}^N \mu(A_j) = \mu^+ \left( \bigcup_{j=1}^N A_j \right) = \mu^- \left( \bigcup_{j=1}^N A_j \right) \leq \mu^-(B)$$

and the result follows on letting $N \to \infty$. 

9.1.24 Theorem

The \( \mu \)-measurable subsets of \( X \) form a \( \sigma \)-algebra. All Borel subsets of \( X \) are \( \mu \)-measurable.

Proof. We already know that if \( A \) is \( \mu \)-measurable then so is \( X \setminus A \), and that finite unions of \( \mu \)-measurable sets are \( \mu \)-measurable. If \( A_j \) are \( \mu \)-measurable for \( j \in \mathbb{N} \) we just set

\[
E_1 = A_1, \quad E_{n+1} = A_{n+1} \setminus \bigcup_{j=1}^{n} A_j.
\]

Now \( X \setminus A_{n+1} \) is \( \mu \)-measurable by Lemma 9.1.14, and so is \( \bigcup_{j=1}^{n} A_j \), by Lemma 9.1.21, and hence so is \( (X \setminus A_{n+1}) \cup \bigcup_{j=1}^{n} A_j \), the complement of which is \( E_{n+1} \).

Then the union \( B \) of the \( A_j \) is the union of the pairwise disjoint sets \( E_j \) and so is \( \mu \)-measurable, with

\[
\mu(B) = \sum \mu(E_j) \leq \sum \mu(A_j).
\]

This gives our \( \sigma \)-algebra of subsets of \( X \), and each open set belongs to this \( \sigma \)-algebra by Lemma 9.1.15.

9.2 Properties of the harmonic measure

We have now established Theorem 9.1.4. The measure \( \mu \) has been constructed, for a fixed \( z_0 \), and the next step is to investigate what happens as \( z_0 \) varies.

For each \( z \) in \( D \), we construct the measure \( \mu = \mu_z \). For every Borel subset \( A \) of \( X \) we define

\[
\omega(z,A,D) = \mu_z(A),
\]

and we know that \( \omega(z,A,D) \) is the infimum of \( \omega(z,U,D) \), and the supremum of \( \omega(z,E,D) \), over all open \( U \) and closed \( E \) with \( E \subseteq A \subseteq U \).

If \( A \) is closed then we have already seen that \( \omega(z,A,D) \) is a harmonic function of \( z \) on \( D \). The same is true if \( A \) is open, because

\[
\omega(z,A,D) = 1 - \omega(z,X \setminus A,D).
\]

The next theorem shows that \( \omega(z,A,D) \) is harmonic for every Borel subset \( A \) of \( X \), thus justifying the term harmonic measure.

9.2.1 Theorem

Let \( A \) be a Borel subset of \( X \). Then \( \omega(z,A,D) \) is harmonic on \( D \).

Proof. Take \( z_0 \) in \( D \) and \( \delta > 0 \). Choose closed \( E \), open \( U \), with

\[
E \subseteq A \subseteq U, \quad \omega(z_0,E,D) > \omega(z_0,A,D) - \delta, \quad \omega(z_0,U,D) < \omega(z_0,A,D) + \delta.
\]

If \( z \) is sufficiently close to \( z_0 \) then, since \( \omega(z,E,D) \) and \( \omega(z,U,D) \) are harmonic and so continuous, we have

\[
\omega(z,A,D) \leq \omega(z,U,D) \leq \omega(z_0,A,D) + 2\delta, \quad \omega(z,A,D) \geq \omega(z,E,D) \geq \omega(z_0,A,D) - 2\delta,
\]

and this proves that \( \omega(z,A,D) \) is continuous on \( D \).
Next, if $r$ is small and positive,

$$\omega(z_0, A, D) = \frac{1}{2\pi} \int_0^{2\pi} \omega(z_0 + re^{it}, U, D)dt - \delta \geq \frac{1}{2\pi} \int_0^{2\pi} \omega(z_0 + re^{it}, A, D)dt - \delta$$

and

$$\omega(z_0, A, D) < \omega(z_0, E, D) + \delta = \frac{1}{2\pi} \int_0^{2\pi} \omega(z_0 + re^{it}, E, D)dt + \delta \leq \frac{1}{2\pi} \int_0^{2\pi} \omega(z_0 + re^{it}, A, D)dt + \delta.$$

Since $\delta$ is arbitrary, we see that $\omega(z, A, D)$ has the mean value property on $D$ and so is harmonic there.

The same proof obviously works for any subset $A$ of $X$ such that $A$ is $\mu_z$-measurable for every $z$ in $D$.

9.2.2 Corollary

Let $E$ be a Borel subset of $X$. If $\omega(z, E, D) = 0$ for some $z \in D$ then $\omega(z, E, D) = 0$ for all $z \in D$.

This follows at once since $1 - \omega(z, E, D)$ has a maximum in $D$. We then say that $E$ has zero harmonic measure.

9.2.3 Lemma

Let $x_0 \in X$. Then $\{x_0\}$ has zero harmonic measure.

This is by Lemma 8.2.7, since $\omega(z, \{x_0\}, D)$ has boundary values 0 except on a finite set. It follows using sub-additivity that every countable subset of $X$ has zero harmonic measure.

9.2.4 Theorem

Let $D$ be regular, let $E$ be a Borel subset of $X$ and let $f = \chi_E$. Then $\omega(z, E, D) = v_f(z)$ on $D$, where $v_f$ is the Perron function defined in §8.1.

Thus the conclusion of 9.1.9(c) extends to all Borel subsets of $X$. For a more general result see Theorem 9.3.3.

Proof. First let $E \subseteq V \subseteq X$, with $V$ open (i.e. a relatively open subset of $X$). Let $u \in U(f)$, where $U(f)$ is the Perron family of $f$ as in §8.1. Then, since $D$ is semi-regular and $V$ is open we have, for all $w \in V$,

$$\limsup_{z \to w, z \in D} (u(z) - \omega(z, V, D)) \leq 0$$

by 9.1.9(a), and the same is true for all $w \in X \setminus V$, since $u \in U(f)$. Thus Lemma 8.2.7 gives, for every $z \in D$,

$$u(z) \leq \omega(z, V, D).$$

Applying Theorem 9.1.4(ii) we obtain $u(z) \leq \omega(z, E, D)$, and taking the supremum over $u \in U(f)$ gives $v_f(z) \leq \omega(z, E, D)$.

Now let $K \subseteq E$ be compact. Then $\omega(z, K, D)$ belongs to $U(f)$, and so

$$\omega(z, K, D) \leq v_f(z).$$

Taking the supremum over $K$ we get $\omega(z, E, D) \leq v_f(z)$ by Theorem 9.1.4(iii).
9.2.5 Definition
For a Borel measurable $f : X \to \mathbb{R}^*$ (this means that the set $\{x \in X : f(x) < y\}$ is a Borel set for every $y \in \mathbb{R}$), define

$$M(z, f) = \int_X f(x) d\mu_z(x).$$

Here $\mu_z$ is the measure constructed for $z$. Clearly if $f(x)$ is a constant $c$ then $M(z, f) = c\mu_z(X) = c$.

9.2.6 Theorem
Let $f : X \to [0, \infty]$ be Borel measurable. Then $M(z, f)$ is either harmonic or identically $\infty$ on $D$.

Proof. If $s$ is a real-valued simple Borel function on $X$ then $M(z, s)$ is a linear combination of harmonic measures of Borel sets and so Theorem 9.2.1 shows that $M(z, s)$ is harmonic. Now just take non-negative simple Borel functions $s_n$ such that $0 \leq s_1 \leq s_2 \leq \ldots$ and $s_n \to f$ pointwise on $X$. Then

$$M(z, s_n) = \int_X s_n(x) d\mu_z(x) \to \int_X f(x) d\mu_z(x) = M(z, f)$$

by the monotone convergence theorem. The result now follows from Harnack’s theorem. It follows at once that if $f$ is a bounded Borel measurable function on $X$ then $M(z, f)$ is harmonic on $D$.

9.2.7 Lemma
Let $f$ be a bounded Borel measurable function on $X$, and let $f$ be continuous at all but finitely many $x \in X$. Then

$$\lim_{z \to x_0, z \in D} M(z, f) = f(x_0).$$

Proof. Assume without loss of generality that $f(x_0) = 0$, and take $\delta > 0$. Assume $|f| \leq M < \infty$ on $X$. Take an open subset $U$ of $X$ with $x_0 \in U$ and $|f(x)| < \delta$ on $U$. Then

$$\left| \int_U f(x) d\mu_z(x) \right| \leq \delta \mu_z(U) \leq \delta$$

while

$$\left| \int_{X \setminus U} f(x) d\mu_z(x) \right| \leq M \omega(z, X \setminus U, D) \to 0$$

as $z \to x_0$.

9.2.8 Corollary
Let $g$ be a bounded function on $X$ which is continuous at all but finitely many $x \in X$. Then

$$v_g(z) = L(z, g) = M(z, g).$$

Next, let $f$ be upper semi-continuous on $X$. Then $M(z, f)$ agrees with the harmonic extension $u(z)$ of $f$ defined in Theorem 9.1.6.

Proof. We have $L(z, g) = v_g(z)$ by definition, and

$$\lim_{z \to x_0, z \in D} (v_g(z) - M(z, g)) = 0$$
for all but finitely many \( x_0 \in X \). Now apply Lemma 8.2.7.

Next, take continuous \( f_n \downarrow f \) on \( X \) and let \( u_n(z) = L(z, f_n) \). Then Theorem 9.1.6 gives

\[
    u(z) = \lim u_n(z) = \lim M(z, f_n) = M(z, f),
\]

the last step using the monotone convergence theorem.

9.2.9 Theorem: the principle of harmonic measure

Let \( u \) be subharmonic and bounded above on \( D \). Let \( f \) be a Borel function, bounded above on \( X \), such that

\[
    \phi(x) = \limsup_{z \to x, z \in D} u(z) \leq f(x)
\]

for all \( x \in X \setminus E \), where \( E \) has harmonic measure 0. Then

\[
    u(z) \leq \int_X f(x) d\mu_z(x)
\]

on \( D \).

Proof. We know from Lemma 7.3.4 that \( \phi \) is upper semi-continuous on \( X \). Take continuous \( f_n \) decreasing pointwise to \( \phi \) on \( X \), and write \( u_n(z) = v_{f_n}(z) = L(z, f_n) = M(z, f_n) \). For all but finitely many \( x \in X \) we have

\[
    \limsup_{z \to x, z \in D} (u(z) - u_n(z)) \leq 0,
\]

and so, since \( u - u_n \) is bounded above, we get \( u(z) \leq u_n(z) \) on \( D \). Letting \( n \to \infty \) we have

\[
    \int_X (f_1 - f_n)(x) d\mu_z(x) \to \int_X (f_1 - \phi)(x) d\mu_z(x)
\]

by the monotone convergence theorem and so

\[
    u(z) \leq \lim u_n(z) = \lim \int_X f_n(x) d\mu_z(x) = \int_X \phi(x) d\mu_z(x) \leq \int_X f(x) d\mu_z(x).
\]

The following example shows that the principle may fail for unbounded functions. For \( D = D(0, 1) \) and \( u(z) = \text{Re} \left( \frac{1+z}{1-z} \right) \) we may take \( f = 0 \), but \( u \) is positive on \( D \).

9.2.10 “Two-constants” theorem

Let \( E_j \) be finitely many pairwise disjoint Borel subsets of \( X \), with union \( X \). Let \( u \) be a function subharmonic and bounded above on \( D \), and let \( M_j \in \mathbb{R} \) be such that

\[
    \limsup_{z \to x, z \in D} u(z) \leq M_j \quad \text{for} \quad x \in E_j.
\]

Then

\[
    u(z) \leq \sum_j M_j \omega(z, E_j, D), \quad z \in D.
\]

Proof. Assume first that all \( M_j \) are positive. Take open \( U_j \) with \( E_j \subseteq U_j \) and set

\[
    v(z) = \sum_j M_j \omega(z, U_j, D).
\]
Then for all but finitely many \( x \in U_j \) we have
\[
\lim_{z \to x, z \in D} \omega(z, U_j, D) = 1, \quad \limsup_{z \to x, z \in D} (u(z) - v(z)) \leq 0.
\]
So \( u(z) \leq v(z) \) on \( D \). Using (ii) of Theorem 9.1.4 we get
\[
u(z) \leq \sum_j M_j \omega(z, E_j, D).
\]
If any \( M_j \) is negative, take a large positive \( M \). Then we get
\[
u(z) + M \leq \sum_j (M_j + M) \omega(z, E_j, D) = M + \sum_j M_j \omega(z, E_j, D).
\]

### 9.2.11 Comparison principle

Let \( D_1, D_2 \) be semi-regular domains in \( \mathbb{C} \), with \( D_1 \subseteq D_2 \), and let \( X_j = \partial_\infty D_j \). Let \( E \subseteq X_1 \cap X_2 \) be a Borel subset of \( X_1 \) and of \( X_2 \). Then
\[
\omega(z, E, D_1) \leq \omega(z, E, D_2).
\]

**Proof.** Let \( F \) be a closed subset of \( X_1 \) and \( U \) an open subset of \( X_2 \) such that \( F \subseteq E \subseteq U \). Let
\[
u(z) = \omega(z, F, D_1) - \omega(z, U, D_2).
\]
Let \( z \to x \in X_1 \) with \( z \in D_1 \). If \( x \notin F \) and \( x \) is regular for \( D_1 \) then \( \omega(z, F, D_1) \to 0 \) and so \( \limsup u(z) \leq 0 \). If \( x \in F \) then \( x \in U \) and so if \( x \) is regular for \( D_2 \) we get \( \omega(z, U, D_2) \to 1 \) and again \( \limsup u(z) \leq 0 \). This means that \( \limsup_{z \to x, x \in D_1} u(z) \leq 0 \) for all but finitely many \( x \in X_1 \) and so \( u \leq 0 \) on \( D_1 \). Now take the supremum over \( F \) and infimum over \( U \).

### 9.2.12 Conformal invariance

Let \( D_1, D_2 \) be semi-regular domains in \( \mathbb{C} \), and let \( X_j = \partial_\infty D_j \). Let \( B_j \) be a Borel subset of \( X_j \) and let \( f : D_1 \cup B_1 \to D_2 \cup B_2 \) be continuous, such that \( f : D_1 \to D_2 \) is analytic and \( f \) maps \( B_1 \) into \( B_2 \). Suppose further that at most finitely many \( x \) in \( B_1 \) are such that \( f(x) \) is an irregular boundary point of \( D_2 \). Then
\[
\omega(z, B_1, D_1) \leq \omega(f(z), B_2, D_2).
\]

**Proof.** Take a compact subset \( F \) of \( X_2 \setminus B_2 \) and an open subset \( V \) of \( X_1 \) containing \( X_1 \setminus B_1 \). Define\[
u(z) = \omega(f(z), F, D_2), \quad z \in D_1.
\]
Then \( u \) is harmonic on \( D_1 \). We assert that
\[
\limsup_{z \to x, z \in D_1} (u(z) - \omega(z, V, D_1)) \leq 0 \tag{9.2}
\]
for all but finitely many \( x \in X_1 \). Let \( x \in X_1 \) be regular. If \( x \) is in \( V \), then \( \omega(z, V, D_1) \to 1 \) as \( z \to x \) and so (9.2) holds. If \( x \) is not in \( V \) then \( x \) is in \( B_1 \) and so \( f(x) \in B_2 \). In particular, for all but finitely
many $x \in X_1 \setminus V$ we have $f(z) \to f(x) \notin F$ and $u(z) \to 0$ as $z \to x$. This proves (9.2). Thus $u(z) \leq \omega(z, V, D_1)$ on $D_1$. Now (ii) and (iii) of Theorem 9.1.4 give
\[ \omega(f(z), F, D_2) \leq \omega(z, X_1 \setminus B_1, D_1) \]
and
\[ \omega(f(z), X_2 \setminus B_2, D_2) \leq \omega(z, X_1 \setminus B_1, D_1). \]

9.3 Comparing the harmonic measure and the Perron function

Let $D$ be a semi-regular domain, let $E$ be a Borel subset of $X$ and let $f = \chi_E$. It is natural to ask whether the harmonic measure $\omega(z, E, D)$ agrees on $D$ with the Perron function $v_f(z)$ defined in §8.1, and the following leads to a more general version of 9.1.9(c) and Theorem 9.2.4.

9.3.1 Lemma

Let $E$ be a finite subset of $X$, and let $g : X \to \mathbb{R}$ with $g = 0$ on $X \setminus E$. Then $v_g = 0$ on $D$.

Proof. The function $g$ is continuous on $X \setminus E$, and $v_g$ is bounded and harmonic on $D$. By Theorem 8.1.7, we have $v(z) \to 0$ as $z \to w$ from within $D$, for all but finitely many $w \in X$. Since a finite set has zero harmonic measure, we obtain $v_g \leq 0$ on $D$ from the two-constants theorem, and $-v_g \leq 0$ in the same way.

9.3.2 Lemma

Let $E$ be a finite subset of $X$, and let $f$ and $g$ be bounded functions on $X$ with $f \leq g$ on $X$ and $f = g$ off $E$. Then $v_f = v_g$ on $D$.

Proof. Write $f = g + h$, where $h \leq 0$ on $X$ and $h = 0$ off $E$. If $u_1 \in U(g)$ and $u_2 \in U(h)$ then $u_1 + u_2 \in U(f)$ and so (as in Lemma 8.1.6)
\[ v_g + v_h \leq v_f \leq v_g. \]
Since $v_h = 0$ by the previous lemma the result follows.

9.3.3 Theorem

Let $D$ be a semi-regular domain, let $E$ be a Borel subset of $X$ and let $f = \chi_E$. Then $\omega(z, E, D) = v_f(z)$ on $D$, where $v_f$ is the Perron function defined in §8.1.

Thus the conclusion of 9.1.9(c) extends to all Borel subsets of $X$, even for semi-regular domains.

Proof. First let $E \subseteq V \subseteq X$, with $V$ open (i.e. a relatively open subset of $X$). Let $u \in U(f)$, where $U(f)$ is the Perron family of $f$ as in §8.1. Then, since $D$ is semi-regular and $V$ is open we have, for all but finitely many $w \in V$,
\[ \limsup_{z \to w, z \in D} (u(z) - \omega(z, V, D)) \leq 0 \]
by 9.1.9(a), and the same is true for all $w \in X \setminus V$, since $u \in U(f)$. Thus Lemma 8.2.7 gives, for every $z \in D$,
\[ u(z) \leq \omega(z, V, D). \]
Applying Theorem 9.1.4(ii) we obtain \( u(z) \leq \omega(z, E, D) \), and taking the supremum over \( u \in U(f) \) gives \( v_f(z) \leq \omega(z, E, D) \).

Now let \( K \subseteq E \) be compact, and let \( Z \) be the (finite) set of non-regular boundary points. Set \( g = f + \chi_Z \). Since \( \omega(z, K, D) \) tends to 0 as \( z \) tends in \( D \) to any regular point in \( X \setminus K \), we see that \( \omega(z, K, D) \) belongs to \( U(g) \), and so

\[
\omega(z, K, D) \leq v_g(z) = v_f(z),
\]

using the previous lemma. Taking the supremum over \( K \) we get \( \omega(z, E, D) \leq v_f(z) \) by Theorem 9.1.4(iii).
Chapter 10

Jordan domains and boundary behaviour

10.1 Introduction

The first part of this chapter describes a fairly simple analytic proof of the Jordan curve theorem, which roughly-speaking states that a simple closed curve in $\mathbb{C}$ divides its complement in $\mathbb{C}^*$ into two components, each a domain without holes. The proof is taken from

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A Jordan arc is a continuous one-one function $g : [a, b] \to \mathbb{C}$.

A Jordan curve is a continuous one-one function $g : T \to \mathbb{C}$, in which $T = \{e^{it} : 0 \leq t \leq 2\pi\}$.

In either case, the image $H$ is compact and, since $\mathbb{C}$ is Hausdorff, the inverse function is continuous and $g$ is a homeomorphism.

Falconer, Geometry of Fractal Sets, p.115 gives a continuous $f : [0, 1] \to \mathbb{R}$ whose graph (obviously a Jordan arc) has Hausdorff dimension in $(1, 2)$.

10.1.1 Preliminaries

Let $U$ be an open subset of $\mathbb{C}$. For $x \in U$ the component $C_x$ of $U$ containing $x$ is the union of all open subsets of $U$ each containing $x$. If $y \in C_x$ then $C_y = C_x$. Also each $C_x$ is open, and its boundary is a subset of $\partial U$. The number of components is countable.

From now on in this chapter, $X$ will always be a compact subset of $\mathbb{C}$. The complement of $X$ in $\mathbb{C}$ consists of countably many components, one of which is unbounded.

As usual $C(X)$ will denote the set of all continuous $f : X \to \mathbb{C}$, with $L_\infty$ distance $\rho(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$. 

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10.1.2 Some groups

For a compact subset $X$ of $\mathbb{C}$ let $C^*(X)$ be the set of all continuous (non-vanishing) $f : X \to \mathbb{C} \setminus \{0\}$. Obviously $C^*(X)$ is a multiplicative group.

Next, 

$$e^{C(X)} = \{e^f : f \in C(X)\}$$

is clearly a subgroup of $C^*(X)$. It is the collection of functions in $C^*(X)$ which have a continuous logarithm.

We can then form the quotient group 

$$H_X = C^*(X)/e^{C(X)}.$$ 

This is the same as defining an equivalence relation on $C^*(X)$ by $f \sim g$ iff $f/g \in e^{C(X)}$, and then a multiplication on the classes given by $[f][g] = [fg]$. The collection of equivalence classes is $H_X$.

Note that if $\phi$ is a homeomorphism from $X$ onto a compact $Y \subseteq \mathbb{C}$, then $H_X$ and $H_Y$ are isomorphic via 

$$[g]_{H_Y} \to [g(\phi)]_{H_X}.$$ 

To see this, obviously 

$$[g_1][g_2] = [g_1g_2] \to [g_1(\phi)g_2(\phi)] = [g_1(\phi)][g_2(\phi)]$$

while if $[g]$ is sent to the identity of $H_X$ then $g(\phi) = e^u$ on $X$ and so $g = e^{u(\phi^{-1})}$ on $Y$.

10.1.3 Lemma

If $f, g \in C(X)$ with $|g| < |f|$ on $X$ then $f \sim f + g$.

We remark that this result is similar in statement and proof to Rouché’s theorem.

Proof. We have $\text{Re}(1 + g/f) > 0$ on $X$. Hence the principal logarithm of $1 + g/f$ is continuous on $X$.

10.1.4 Lemma

Let $f \in C^*(X)$ . Then there exists $\delta > 0$ such that $g \in C^*(X)$ and $g \sim f$ for all $g \in C(X)$ with $\rho(f, g) < \delta$.

Proof. The function $|f|$ has a positive minimum $\delta$ on $X$. Thus if $\rho(f, g) < \delta$ we have $|g - f| < \delta$ on $X$ and so $g \neq 0$, while writing $g = f + (g - f)$ shows that $g \sim f$.

10.1.5 Lemma

Let $f \in C^*(X)$. Then $[f] = \{g \in C^*(X) : g \sim f\}$ is an open and closed subset of $C^*(X)$.

Proof. We’ve just seen that $[f]$ is open. Now suppose that $g \not{\sim} f$. Then there exists $\delta > 0$ such that $\rho(g, h) < \delta$ implies $h \sim g$ and so $h \not{\sim} f$. 
10.1.6 Theorem

Let \( f, g \in C^*(X) \). Then \( f \sim g \) if and only if there exists a continuous \( F : X \times [0, 1] \to \mathbb{C} \setminus \{0\} \) such that

\[
F(z, 0) = f(z), \quad F(z, 1) = g(z)
\]

for all \( z \) in \( X \).

Proof. If \( f \sim g \) we can write \( f = ge^h \) with \( h \in C(X) \). We then just set \( F(z, t) = f(z)e^{-th(z)} \).

Now suppose that such a \( F \) exists. Define \( f_t \) by \( f_t(z) = F(z, t) \). Now \( J = \{ t \in [0, 1] : f_t \sim f_0 \} \) is open and closed in \([0, 1] \), by Lemma 10.1.5, and is non-empty. So \( J = [0, 1] \).

10.1.7 Corollary

Let \( X \) have the property that \( z \in X \) implies that \( tz \in X \) for all \( t \in [0, 1] \) (starlike about 0). Then \( C^*(X) = e^{C(X)} \).

Obviously this applies if \( X \) is the closed unit disc, or is \([0, 1] \) (which is henceforth always denoted \( I \)).

Proof. Just define \( F(z, t) = f(tz) \). This shows that \( f \sim 1 \) for every \( f \in C^*(X) \).

10.1.8 Lemma

\( e^{C(X)} \) is the maximal connected subset of \( C^*(X) \) containing 1.

Proof. Suppose \( A \subseteq C^*(X) \), and \( A \) properly contains \( e^{C(X)} \). Since \( e^{C(X)} = [1] \) is an open and closed subset of \( C^*(X) \), we may partition \( A \) as \( e^{C(X)}, A \setminus e^{C(X)} \) and both are relatively open. So \( A \) is not connected.

It remains only to show that \( e^{C(X)} \) is connected. But if \( f, g \in C(X) \) then \( t \to e^{tg+(1-t)f} \) is a continuous function from \( I \) to \( e^{C(X)} \), sending 0 to \( e^f \) and 1 to \( e^g \).

10.2 Janiszewski’s theorem

10.2.1 Lemma

Let \( n \in \mathbb{Z} \) and (henceforth) let \( T = \{ e^{i\theta} : 0 \leq \theta \leq 2\pi \} \). If \( z^n \in e^{C(T)} \) then \( n = 0 \).

Proof. Suppose that \( n \neq 0 \) and \( z^n = e^{i\theta} \) on \( T \) for some \( h \in C(T) \). The principal argument \( \alpha(z) = \text{Arg} z \) is continuous on \( T_1 = T \setminus \{-1\} \), taking values in \((-\pi, \pi) \), and \( e^{i\text{Arg}(z)} = z^n \) on \( T_1 \). Thus \( (h(z) - in(z))/2\pi i \) is continuous and integer-valued and so constant on \( T_1 \). This is a contradiction since \( h \) is continuous on all of \( T \) but \( \alpha(z) \) is not.

Obviously the same proof shows that if \( X \) is the circle \(|z - a| = R > 0 \) and \( n \) is a non-zero integer then \( (z - a)^n \not\in e^{C(X)} \).
10.2.2 Corollary

If \( a \in \mathbb{C}, R > 0 \) and \( n \) is a non-zero integer then there is no \( f \in C^*(\overline{B}(a, R)) \) which equals \( (z - a)^n \) on \( |z - a| = R \).

Here \( \overline{B}(a, R) \) denotes the closed unit disc.

**Proof.** Assume without loss of generality that \( a = 0, R = 1 \). Suppose we had such an \( f \). By Corollary 10.1.7 we have \( f = e^{g} \) for some \( g \) continuous on the closed unit disc. So \( z^n = e^{g(z)} \) on \( T \) and this contradicts Lemma 10.2.1.

10.2.3 Theorem

Let \( U \) be a bounded open subset of \( \mathbb{C} \) and let \( a \in U \). Let \( H = \partial U \) and let \( n \) be a non-zero integer.
Then there is no \( f \in C^*(U) \) which equals \( (z - a)^n \) on \( H \).

**Proof.** We assume that \( U \subseteq D(a, R) \). If \( f \) is a continuous non-zero function on the closure of \( U \), which equals \( (z - a)^n \) on \( H \), then we extend \( f \) to the closed disc by setting \( f(z) = (z - a)^n \) for \( z \) not in the closure of \( U \). This extended function is continuous, and this contradicts Corollary 10.2.2.

10.2.4 Tietze’s extension theorem

Let \( A \) be a closed subset of \( \mathbb{C} \) and let \( f : A \to \mathbb{R} \) be continuous. Then there is a continuous function \( g : \mathbb{C} \to \mathbb{R} \) such that \( g = f \) on \( A \).

**Proof.** Urysohn’s lemma gives us the following: if \( B, C \) are disjoint closed subsets of \( \mathbb{C} \) then there exists a continuous \( k : \mathbb{C} \to [0, 1] \) with \( k = 0 \) on \( B \) and \( k = 1 \) on \( C \). In fact, \( k \) is

\[
k(z) = \frac{d(z, B)}{d(z, B) + d(z, C)}
\]

with \( d(z, B) \) the distance from \( z \) to \( B \). Clearly there is then a continuous \( K_0 : \mathbb{C} \to [-1/3, 1/3] \) with \( K_0 = -1/3 \) on \( B \) and \( K_0 = 1/3 \) on \( C \).

Now let \( f : A \to (-1, 1) \) be continuous. Let \( B = \{x \in A : f(x) \leq -1/3\} \) and let \( C = \{x \in A : f(x) \geq 1/3\} \). Then \( B, C \) are closed and there is a continuous \( h_1 : \mathbb{C} \to [-1/3, 1/3] \) such that \( h_1 = -1/3 \) on \( B \) and \( h_1 = 1/3 \) on \( C \). Thus \( |h_1 - f| \leq 2/3 \) on \( A \).

We claim that there exist continuous \( h_n : \mathbb{C} \to \mathbb{R} \) such that \( |h_n(x)| \leq 2^{n-3} - 3^{n-3} \) on \( \mathbb{C} \) and

\[
|f(x) - \sum_{j=1}^{n} h_j(x)| \leq 2^n 3^{n-3}, \quad x \in A.
\]

Assuming that \( h_1, \ldots, h_n \) exist, we apply the first part to the function

\[
(3/2)^n (f(x) - \sum_{j=1}^{n} h_j(x)) = K(x).
\]

This gives \( H(x) \) with \( |H| \leq 1/3 \) on \( \mathbb{C} \) and \( |H - K| \leq 2/3 \) on \( A \), and we just set \( h_{n+1} = (2/3)^n H \).
The function $h(x) = \sum_{j=1}^{\infty} h_j(x)$ is then continuous on $\mathbb{C}$ and equal to $f$ on $A$.

For a general $f$, we apply the above proof to $F = f/(1 + |f|), f = F/(1 - |F|)$.

10.2.5 Lemma

Let $f \in e^{C(X)}$. Then $f$ can be extended to a continuous non-zero function on $\mathbb{C}$.

Proof. Just write $f = e^{g}$ on $X$ and extend $g$ to a continuous function on $\mathbb{C}$ using Tietze’s extension theorem.

10.2.6 Notation

We continue to use $X$ to denote a compact subset of $\mathbb{C}$. Let the distinct components be $U_0, U_1, \ldots$, with $U_0$ unbounded. Fix $a_k \in U_k$.

10.2.7 Lemma

Let $a, b \in U_k$. Then $z - a \sim z - b$.

If $a \in U_0$ then $z - a \sim 1$.

Proof. Take a path $\gamma : [0,1] \to U_k$ with $\gamma(0) = a, \gamma(1) = b$. Let $F(z, t) = z - \gamma(t)$. Then $F$ is continuous and non-zero on $X \times I$ and $F(z, 0) = z - a, F(z, 1) = z - b$. Apply Theorem 10.1.6.

Next, choose positive $R$ so large that $\text{Re}(z + R) > 0$ on $X$ and $R \in U_0$. Then $\log(z + R)$ (the principal log) is continuous on $X$, so $z + R \sim 1$. Hence $z - a \sim 1$ for all $a$ in $U_0$.

10.2.8 Lemma

Let $q \in \mathbb{N}$ and let

$$f(z) = \prod_{k=0}^{q} (z - a_k)^{n_k} \quad (10.1)$$

with each $n_k \in \mathbb{Z}$. If $f \in e^{C(X)}$ then $n_k = 0$ for all $k \geq 1$.

Proof. N.B. since $a_j \in U_j$ the hypotheses assume that $\mathbb{C} \setminus X$ has at least two components. Assume that $f = e^h$ on $X$. By Tietze’s extension theorem we can assume that $h$ is continuous on $\mathbb{C}$. Thus there is a continuous non-zero function $F$ on $\mathbb{C}$ (Lemma 10.2.5), which equals $f$ on $X$. Now set

$$g(z) = (z - a_0)^{n_0} \prod_{k=2}^{q} (z - a_k)^{n_k}$$

(with $g(z) = (z - a_0)^{n_0}$ if $q = 1$). Then $g$ is continuous and non-zero on $X \cup U_1$. So there is a continuous non-zero function $F/g$ on $X \cup U_1$, which equals $(z - a_1)^{n_1} = f/g$ on $X$ and so on $\partial U_1$. This forces $n_1 = 0$, by Theorem 10.2.3.

Remark: this is one of the two key steps of the method. The other will be to show that each $F \in C(X)$ has $F \sim f$ for some $f$ of the form (10.1).
10.3. CONVOLUTIONS AND RUNGE’S THEOREM

10.3.1 Convolutions

We describe the following ideas for \( \mathbb{C}, \mathbb{R}^2 \), but they work equally well in \( \mathbb{R}^n \). A function \( g : \mathbb{C} \to \mathbb{R} \) is said to have compact support if there exists a positive real \( R \) with \( g(z) = 0 \) for \( |z| > R \). We say \( g \) is \( C^n \) if \( g \) has continuous \( n \)’th order partial derivatives on all of \( \mathbb{C} \). Note that if \( g \) is \( C^1 \) then writing

\[
 g(x, y) - g(a, b) = g(x, y) - g(x, b) + g(x, b) - g(a, b)
\]

and using the MVT shows that \( g \) is continuous. \( C^\infty \) is the intersection of the \( C^n \) and we say \( g \) is \( C^m_0 \) if \( g \) is \( C^n \) with compact support.

10.2.9 Corollary

If \( C^*(X) = e^{C(X)} \) then \( \mathbb{C} \setminus X \) is connected.

For otherwise there is a bounded component \( U_1 \), and \( z - a_1 \not\sim 1 \).

10.2.10 Corollary

If \( X \) is a Jordan arc (i.e. \( X \) is homeomorphic to \( I = [0, 1] \)) then \( \mathbb{C} \setminus X \) is connected.

Proof. We have \( C^*(I) = e^{C(I)} \) by Corollary 10.1.7. If \( f \in C^*(X) \) we have \( f(\phi) = e^g \in e^{C(T)} \), where \( \phi : T \to X \) is the homeomorphism. Thus \( f = e^{g(\phi^{-1})} \) and \( C^*(X) = e^{C(X)} \).

Note that \( \partial U_0 = X \) in this case. Otherwise some \( x \in X \) is not a limit point of \( U_0 \) and so some \( D(x, r) \) is a subset of \( X \). But then \( D(x, r) \) is the homeomorphic image of a connected relatively open subset of \( I \) and so of an interval. This is a contradiction, as deleting a point disconnects an interval but not \( D(x, r) \).

10.2.11 Corollary

If \( a, b \in \mathbb{C} \setminus X \) and \( z - a \sim z - b \) then \( a \) and \( b \) lie in the same \( U_j \).

Proof. Otherwise we have, with \( j \neq k \),

\[
 \frac{z - a_j}{z - a_k} \sim \frac{z - a}{z - b} \sim 1
\]

on \( X \). This contradicts Lemma 10.2.8.

10.2.12 Janiszewski’s theorem

Let \( X, Y \) be compact subsets of \( \mathbb{C} \) and let \( a, b \in \mathbb{C} \) have the property that \( a \) and \( b \) both lie in the same component of \( \mathbb{C} \setminus X \), and \( a \) and \( b \) both lie in the same component of \( \mathbb{C} \setminus Y \). If \( X \cap Y \) is connected then \( a \) and \( b \) both lie in the same component of \( \mathbb{C} \setminus (X \cup Y) \).

Proof. We can write \( (z - a)/(z - b) = e^{g(z)} \) on \( X \), and \( (z - a)/(z - b) = e^{h(z)} \) on \( Y \). Thus \( e^{g - h} = 1 \) on \( X \cap Y \). But then \( g - h \) is constant on \( X \cap Y \) and, adding an integer multiple of \( 2\pi i \) to \( h \) if necessary, we get \( (z - a)/(z - b) \sim 1 \) on \( X \cup Y \).

10.3 Convolutions and Runge’s theorem

10.3.1 Convolutions

We describe the following ideas for \( \mathbb{C}, \mathbb{R}^2 \), but they work equally well in \( \mathbb{R}^n \). A function \( g : \mathbb{C} \to \mathbb{R} \) is said to have compact support if there exists a positive real \( R \) with \( g(z) = 0 \) for \( |z| > R \). We say \( g \) is \( C^n \) if \( g \) has continuous \( n \)’th order partial derivatives on all of \( \mathbb{C} \). Note that if \( g \) is \( C^1 \) then writing

\[
 g(x, y) - g(a, b) = g(x, y) - g(x, b) + g(x, b) - g(a, b)
\]

and using the MVT shows that \( g \) is continuous. \( C^\infty \) is the intersection of the \( C^n \) and we say \( g \) is \( C^m_0 \) if \( g \) is \( C^n \) with compact support.
Let \( g \) be a real-valued bounded measurable function, zero off the compact set \( Y \), and let \( h \) be measurable and locally integrable i.e. the Lebesgue integral
\[
\int_{B(0,R)} |h(z)| dx dy
\]
is finite for every \( R > 0 \). Define \( H \) by
\[
H(w) = (h * g)(w) = \int_C h(z)g(w-z)dx dy = \int_C h(w-z)g(z)dx dy = \int_Y h(w-z)g(z)dx dy.
\]
**Fact 1:** if \( h \) has compact support so has \( H \).

This is because if \(|w| \) is large enough then \( h(w-z) = 0 \) for all \( z \) in \( Y \).

**Fact 2:** if \( g \) is continuous then so is \( H \).

Fix \( w \) and take \( \varepsilon_1 > 0 \). Take \( S > 0 \) such that \( g(u-z) = 0 \) for \(|u-w| < 1 \) and \(|z| > S \). Since \( g \) is uniformly continuous we can take \( \delta \in (0,1) \) such that \(|g(u-z) - g(w-z)| < \varepsilon_1 \) for \(|u-w| < \delta \) and for all \( z \). Hence for these \( u \) we have
\[
|H(u) - H(w)| \leq \varepsilon_1 \int_{B(0,S)} |h(z)| dx dy.
\]

**Fact 3:** if \( g \) is \( C^1 \) then so is \( H \), and \( \partial_j H = h * \partial_j g \).

To see this, fix \( w \) and \( S \) as before, let \( t \) be real, small and non-zero, and write
\[
\frac{H(w+t) - H(w)}{t} = \int_{B(0,S)} h(z) \frac{g(w+t-z) - g(w-z)}{t} dx dy.
\]
But the MVT gives
\[
\frac{g(w+t-z) - g(w-z)}{t} = \partial_1 g(c),
\]
and this is uniformly bounded, since \( \partial_1 g \) is continuous with compact support. Taking any sequence \( t_n \to 0 \) and using the dominated convergence theorem we get the result.

**10.3.2 Lemma**

Let \( Y \subseteq V \subseteq \mathbb{C} \), with \( Y \) compact and \( V \) open. Then there exists a \( C^\infty_0 \) function \( F : \mathbb{C} \to [0,1] \) with \( F = 1 \) on \( Y \) and \( F = 0 \) off \( V \).

**Proof.** Take a small positive \( t \) and a non-negative \( C^\infty \) function \( \phi \), vanishing off \( D(0,t) \), and with
\[
\int_C \phi(z) dx dy = \int_{D(0,t)} \phi(z) dx dy = 1.
\]
For example, we may take
\[
\phi(x+iy) = \lambda \exp(-1/(t^2 - x^2 - y^2)), \quad x^2 + y^2 < t^2,
\]
with \( \phi(z) = 0 \) otherwise, and with some suitable positive \( \lambda \).

Now let \( h(z) = 1 \) for \( z \) in a \( t \)-neighbourhood of \( Y \) (i.e. the distance from \( z \) to \( Y \) is less than \( t \)), with \( h = 0 \) otherwise. Set
\[
F(w) = \int_C h(z) \phi(w - z) dxdy = \int_C h(w - z) \phi(z) dxdy = \int_{D(0,t)} h(w - z) \phi(z) dxdy.
\]
Obviously
\[
0 \leq F(w) \leq \int_{D(0,t)} \phi(z) dxdy = 1.
\]
Also \( F \) is \( C^\infty \), since \( \phi \) is. Next, if \( w \not\in V \) then \( h(w - z) = 0 \) for \( |z| < t \), provided \( t \) was chosen small enough. Thus \( F(w) = 0 \). Finally, if \( w \in Y \) then \( h(w - z) = 1 \) for \( |z| < t \) and so \( F(w) = 1 \).

10.3.3 Lemma

Let \( Y \) be a compact subset of \( \mathbb{C} \) and let \( F: Y \to \mathbb{C} \) be continuous. Let \( \delta > 0 \). Then there exists a \( C^\infty_0 \) function \( G \) such that \( |G(w) - F(w)| \leq \delta \) for all \( w \) in \( Y \).

**Proof.** By Tietze’s extension theorem we can assume that \( F \) is continuous on all of \( \mathbb{C} \). By multiplying by a \( C_0 \) function which is 1 on \( Y \) we can assume \( F \) has compact support. Thus \( F \) is uniformly continuous on \( \mathbb{C} \). Now take a small positive \( t \), so small that \( |F(w) - F(w - z)| < \delta \) for all \( w \in Y \) and all \( z \in D(0,t) \). Take \( \phi \) as in the previous lemma and set
\[
G(w) = \int_C F(z) \phi(w - z) dxdy = \int_{D(0,t)} F(w - z) \phi(z) dxdy.
\]
This convolution is \( C^\infty_0 \) since \( \phi \) is (and \( F \) has compact support) and for \( w \in Y \) we have
\[
F(w) - G(w) = \int_{D(0,t)} (F(w) - F(w - z)) \phi(z) dxdy
\]
with modulus at most
\[
\delta \int_{D(0,t)} \phi(z) dxdy = \delta.
\]

10.3.4 The \( \overline{\partial} \) operator

We define
\[
\overline{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]
If \( h \) is complex-valued and has continuous first partials on a domain \( D \) in \( \mathbb{C} \) and \( \overline{\partial} h \equiv 0 \) then \( h \) is analytic, by Cauchy-Riemann.

Also Green’s theorem
\[
\int_{\partial A} P dx + Q dy = \int_A Q_x - P_y dxdy
\]
(the integral around \( \partial A \) once in the positive sense) can be written in the form
\[
\int_{\partial A} g(z) dz = \int_{\partial A} g(z) dx + ig(z) dy = \int_A ig_x - g_y dxdy = 2i \int_A \overline{\partial} g dxdy.
\]
10.3.5 Lemma

Let $g$ be a complex-valued $C^1$ function on $\mathbb{C}$, with compact support. Define $Tg$ by

$$Tg(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(z)}{w - z} \, dx \, dy = \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(w - z)}{z} \, dx \, dy. \quad (10.2)$$

Then $Tg$ is $C^1$ on $\mathbb{C}$, and we have

$$\overline{\partial}(Tg) = T(\partial g) = g.$$

Proof. The fact that $Tg$ is $C^1$ and $\overline{\partial}(Tg) = T(\partial g)$ follows from §10.3.1 (note that $1/z$ is locally integrable).

Fix $w$, and choose $R$ so large that $g(z) = 0$ for $|w - z| > R/2$, and take a small positive $\delta$. Let $A = \{z \in \mathbb{C} : \delta < |w - z| < R\}$, and let $B$ be the boundary of $A$, described once in the positive sense (keeping the interior to the left).

Now

$$(T(\partial g))(w) = \lim_{\delta \to 0} \frac{1}{\pi} \int_{A} \frac{(\partial g)(z)}{w - z} \, dx \, dy.$$ But

$$\frac{(\partial g)(z)}{w - z} = \overline{\partial} \left( \frac{g(z)}{w - z} \right)$$

and so Green’s theorem gives

$$(T(\partial g))(w) = \lim_{\delta \to 0} \frac{1}{\pi} \int_{A} \overline{\partial} \left( \frac{g(z)}{w - z} \right) \, dx \, dy = \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{B} \frac{g(z)}{w - z} \, dz = g(w).$$

10.3.6 Lemma

Let $K \subseteq U \subseteq \mathbb{C}$ with $K$ compact and $U$ open. Let $g$ be analytic on $U$, and let $\delta > 0$. Then there exists a rational function $R$, with no poles in $K$, such that $|g(w) - R(w)| < \delta$ for all $w \in K$.

Proof. Replacing $g$ by $\phi g$, where $\phi$ is a $C_0^\infty$ function as in Lemma 10.3.2 which is 1 on some $\{z : \text{dist}(z, K) < t\}$, with $t > 0$, we can assume that $g$ is in fact $C^1_0$. The function

$$h = \overline{\partial} g$$

is then continuous with compact support, and there is a compact set $H$ not meeting $K$ such that $h = 0$ off $H$. Lemma 10.3.5 then gives, for $w \in K$,

$$g(w) = (Th)(w) = \frac{1}{\pi} \int_{H} \frac{h(z)}{w - z} \, dx \, dy.$$

To form $R(w)$ we then just approximate the integral by a Riemann sum.

We outline the details. We can assume $H$ is a union of closed rectangles. Let $d$ be the (positive) distance from $K$ to $H$, and let $|h| \leq M$ on $H$. Take $\rho > 0$ and partition $H$ into closed rectangles $H_k$. 
disjoint apart from boundary, and so small that $|z - z_k| < \rho \text{ and } |h(z) - h(z_k)| < \rho$ for all $z \in H_k$, with $z_k$ the centre of $H_k$. Let $m(H_k)$ be the area of $H_k$, and let

$$R(w) = \frac{1}{\pi} \sum \frac{h(z_k)}{w - z_k} m(H_k).$$

Then, for $w \in K$,

$$|g(w) - R(w)| \leq I_1 + I_2,$$

in which

$$I_1 = \left| \sum \int_{H_k} \frac{h(z) - h(z_k)}{w - z} \, dxdy \right| \leq \rho d^{-1} m(H)$$

and

$$I_2 = \left| \sum \int_{H_k} h(z_k) \left( \frac{1}{w - z} - \frac{1}{w - z_k} \right) \, dxdy \right| = \left| \sum \int_{H_k} h(z_k) \frac{z - z_k}{(w - z)(w - z_k)} \, dxdy \right| \leq M m(H) \rho d^{-2}.$$

**Remark:** this is a weak version of Runge’s theorem, which states that $R$ can be chosen so that all its poles lie in the set $\{a_k\}$, in which as before each $a_k \in U_k$ and $U_0, \ldots$ are the components of $C \setminus K$. To see this, suppose that $R$ has a pole at $b \in U_k$. Join $b$ to $a_k$ by a path $\sigma$ in $U_k$, and so by a finite sequence of points $z_j$ such that

$$|z_j - z_{j-1}| < \frac{1}{4} \text{dist}(K, \sigma).$$

Now just use the fact that if $|A - B| < \frac{1}{4} \text{dist}(B, K)$ then for $z \in K$ and $m \in \mathbb{N}$ we can write

$$(z - A)^{-m} = (z - B + B - A)^{-m} = (z - B)^{-m} (1 + (B - A)/(z - B))^{-m}$$

and expand out in negative powers of $z - B$, using the fact that $|z - B| > 2|A - B|$ for all $z$ in $X$.

### 10.4 Proof of the Jordan curve theorem

#### 10.4.1 Theorem

Let $f(z) \in C^* (X)$. Let $N$, with $0 \leq N \leq \infty$, be the number of bounded components of $C \setminus X$. Then there exist integers $n_k$, all but finitely many of them 0, such that

$$f(z) \sim \prod_{k=1}^{N} (z - a_k)^{n_k}.$$

The $n_k$ are uniquely determined by $f$. If $N = 0$ then $f \sim 1$.

**Proof.** Suppose first that $f$ is a rational function of $z$ with no zeros or poles in $X$. Write $f = P/Q$ and factorize $P$ and $Q$. If $f$ has a zero of multiplicity $m$ at $\alpha \in U_k$, Lemma 10.2.7 gives $(z - \alpha)^m \sim (z - a_k)^m$, and $(z - \alpha)^m \sim 1$ if $k = 0$.

Now let $f$ be any function in $C^* (X)$. It suffices to show that there exists a rational function $R$, with no zeros or poles in $X$, such that $f \sim R$. By Lemmas 10.1.4 and 10.3.3 there exists a function $p \in C_0^\infty$ such that $f \sim p$ i.e. $f/p \in e^{C(X)}$. Let $t > 0$, and let

$$V_j = \{ z : \text{dist}(z, X) < jt \}, \quad j = 1, 2, 3.$$
Since \( p \neq 0 \) on \( X \), we have \( p \neq 0 \) on \( V_3 \), provided \( t \) is small enough. We can then use Lemma 10.3.2 to form a function \( \phi \) which is \( C^\infty \), is 1 on \( V_1 \), and 0 off \( V_2 \). Define \( g \) by

\[
g = \phi \frac{\partial p}{p}
\]

on \( V_3 \), with \( g = 0 \) off \( V_3 \). Then \( g \) is \( C^\infty \), and Lemma 10.3.5 gives us a \( C^\infty \) function \( h = Tg \) with \( \partial h = g \). Let \( F = pe^{-h} \). Then \( f \sim F \). Also, \( F \) is \( C^\infty \) and on \( V_1 \) we have

\[
\partial F = e^{-h} \partial p - (\partial h) pe^{-h} = e^{-h} \partial p - gpe^{-h} = 0.
\]

So \( F \) is analytic on \( V_1 \). Let \( s = \min \{|F(z)| : z \in X\} > 0 \) and choose a rational \( R \), with \( |R(w) - F(w)| < s \) on \( X \). Then Lemma 10.1.4 gives \( R \sim F \) and so \( R \sim F \sim f \).

Finally, the uniqueness follows from Lemma 10.2.8.

10.4.2 Theorem

Let \( X, Y \) be compact subsets of \( \mathbb{C} \) and let \( \phi : X \to Y \) be a homeomorphism. Then \( \mathbb{C} \setminus X, \mathbb{C} \setminus Y \) have the same number of components.

**Proof.** Obviously it suffices to prove that the number of components of \( \mathbb{C} \setminus Y \) is at most that of \( \mathbb{C} \setminus X \). If \( \mathbb{C} \setminus X \) is connected then \( C^*(X) = e^{C(X)} \) by Theorem 10.4.1 and so for any \( f \in C^*(Y) \) we have \( g = f(\phi) \in C^*(X) \) and \( g = e^h, h \in C(X) \), which gives \( f = e^{h(\phi^{-1})} \in e^{C(Y)} \).

Now suppose that the bounded components of \( \mathbb{C} \setminus X \) are \( U_1, \ldots, U_n \), and that \( \mathbb{C} \setminus Y \) has distinct bounded components \( V_1, \ldots, V_{n+1} \). Choose \( a_k \in U_k, b_j \in V_j \), and set

\[
f_k(z) = z - a_k, \quad F_k = f_k(\phi^{-1}), \quad g_j(z) = z - b_j.
\]

For each \( j \), the function \( g_j(\phi) \) is in \( C^*(X) \). So we can find integers \( q_{j,k}, 1 \leq k \leq n \), such that

\[
g_j(\phi) \sim \prod_{k=1}^n f_k^{q_{j,k}},
\]

by which we mean that

\[
g_j(\phi) \prod_{k=1}^n f_k^{-q_{j,k}} \in e^{C(X)},
\]

and so

\[
g_j \sim \prod_{k=1}^n F_k^{q_{j,k}}.
\]

The matrix with entries \( q_{j,k} \) has rank at most \( n \) over \( \mathbb{Q} \), and so we can find integers \( m_1, \ldots, m_{n+1} \), not all zero, such that

\[
m_1 q_{1,k} + m_2 q_{2,k} + \ldots + m_{n+1} q_{n+1,k} = 0
\]

for all \( k \). But this gives (in \( C^*(Y) \))

\[
\prod_{j=1}^{n+1} g_j^{m_j} \sim 1,
\]

which is a contradiction.
10.4.3 The Jordan curve theorem

Let \( X \) be a Jordan curve i.e. homeomorphic to \( T \). Then \( \mathbb{C} \setminus X \) has two components \( U_0, U_1 \), each with boundary \( X \).

We only need show that each \( U_j \) has boundary \( X \). Obviously \( \partial U_j \subseteq X \). If \( \partial U_j \neq X \) for some \( j \) then \( \partial U_j \) is a subset of a Jordan arc \( Y \subseteq X \). So we can join \( a_0 \) to \( a_1 \) by a path \( \gamma \) in \( \mathbb{C} \setminus Y \) which does not meet \( \partial U_j \). This says that both \( a_m \) are in \( U_j \) and this is obviously a contradiction.

10.5 Boundary extension for Jordan domains

10.5.1 Theorem

Let \( D \) be a bounded domain in \( \mathbb{C} \) and let \( f \) map \( \Delta = D(0,1) \) conformally (i.e. one-one analytically) onto \( D \). Then the following are equivalent:

(i) \( f \) has a continuous extension to the closed unit disc \( \Delta \cup T \);

(ii) \( \partial D \) is a closed curve;

(iii) \( J = \partial D \) has the following property: to each \( \epsilon > 0 \) corresponds \( \delta > 0 \) such that if \( w_1, w_2 \in J \) with \( |w_1 - w_2| < \delta \) then \( w_1, w_2 \) lie in a compact connected subset \( B \) of \( J \) with the diameter of \( B \) less than \( \epsilon \).

Proof. This is adapted from Pommerenke’s book Boundary behaviour of conformal maps.

(i) implies (ii) is easy. If \( f \) extends continuously to the closed disc then \( f(T) \) is a closed curve, and standard results show that \( f(T) \subseteq \partial D \). Further, \( f(T) = \partial D \), since \( f(\Delta \cup T) \) is closed.

(ii) implies (iii). Let \( \lambda : [0,1] \to \mathbb{C} \) be any curve. Let \( \epsilon > 0 \). Since \( \lambda \) is uniformly continuous, we may partition \( [0,1] \) into closed subintervals \( I_k \) such that \( J_k = \lambda(I_k) \) has diameter less than \( \epsilon/2 \). Let \( \delta > 0 \) be such that if \( J_k \cap J_j = \emptyset \) then the distance from \( J_k \) to \( J_j \) is at least \( 2\delta \). Then if \( w_1, w_2 \) lie on \( \lambda \) and \( |w_1 - w_2| < \delta \) we may write \( w_1 \in J_1, w_2 \in J_k, J_1 \cap J_2 \neq \emptyset \), and we may take \( B = J_j \cup J_k \).

(iii) implies (i). We set out to prove that \( f \) is uniformly continuous on \( \Delta \). The extension to \( T \) then follows easily. We may assume that \( f(0) = 0 \) and hence that \( D(0,s) \subseteq D \) for some \( s > 0 \). Let \( \epsilon \) be positive, small compared to \( s \), and choose \( \delta \in (0,\epsilon) \) as in (iii).

Let \( z_0 \in \Delta \) and let \( \rho \) be small and positive. For \( \rho \leq r \leq \rho^{1/2} \) let \( \gamma_r = S(z_0, r) \cap \Delta \) (recall that \( S(z_0, r) \) is the circle of centre \( z_0 \), radius \( r \), and let \( L(r) \) be the length (possibly infinite) of \( f(\gamma_r) \). Parametrizing \( \gamma_r \) by \( z = z_0 + re^{i\theta} \) the Cauchy-Schwarz inequality gives

\[
L(r)^2 = \left( \int_{\gamma_r} |f'(z)|rd\theta \right)^2 \leq \left( \int_{\gamma_r} rd\theta \right) \left( \int_{\gamma_r} |f'(z)|^2rd\theta \right) \leq 2\pi r \left( \int_{\gamma_r} |f'(z)|^2rd\theta \right).
\]

Dividing by \( r \) and integrating from \( \rho \) to \( \rho^{1/2} \) we thus have, with \( A \) the (finite) area of \( D \),

\[
\int_{\rho}^{\rho^{1/2}} \frac{L(r)^2}{r}dr \leq 2\pi \int_{\rho}^{\rho^{1/2}} \int_{\gamma_r} |f'(z)|^2rd\theta dr \leq 2\pi A.
\]
Hence there exists $r$ with $\rho \leq r \leq \rho^{1/2}$ such that

$$L(r)^2 \leq \frac{8\pi A}{\log 1/\rho}.$$ 

Assume that $\rho$ is chosen so small that $L(r) \leq \delta/2$, and let $C = \gamma_r$. Then $f(C)$ is a curve in $D$ of length at most $\delta/2$.

Suppose first that $C = S(z_0, r)$. Thus $f(C)$ is a closed curve in $D$. Choose $z^* \in C$. Then $|f(z) - f(z^*)| \leq \delta/2 \leq \varepsilon/2$ for all $z$ on $C$, and the same holds for $z \in D(z_0, r)$, by the maximum principle. Thus $|f(z) - f(z_0)| \leq \varepsilon$ for $z \in D(z_0, r)$.

We assume henceforth that $C \neq S(z_0, r)$. Thus $C$ is a circular arc whose closure, when described counter-clockwise, joins end-points $z_1, z_2 \in T$. Since $f(C)$ has finite length, the limits

$$w_j = \lim_{z \to z_j, z \in C} f(z) \in J$$

exist, and $|w_1 - w_2| \leq \delta/2$. Thus there exists a compact connected subset $K$ of $J$ of diameter at most $\varepsilon$, with $w_1, w_2 \in K$. Let $M = K \cup f(C)$. Then $M$ has diameter at most $2\varepsilon$.

Let $z', z'' \in D(z_0, r) \cap \Delta$. We assert that the distance between $f(z')$, $f(z'')$ is at most $16\varepsilon$. If this is not the case then at least one of these points, without loss of generality $w' = f(z')$, lies at distance at least $4\varepsilon$ from $M$. Hence there is a path joining 0 to $w'$ and not meeting $M$. There is also a path joining 0 to $w'$ in $D$, and so not meeting $J$. Since $J \cap M = K$ is connected, Janiszewski’s theorem gives a path $\sigma$ from 0 to $w'$ and not meeting $M \cup J$. Thus $\sigma$ is a path in $D$, not meeting $f(C)$, and so $f^{-1}(\sigma)$ is a path in $\Delta$ from 0 to $z'$, not meeting $C$. By the definition of $C$, this is impossible.

We have thus shown, in both cases, that $|f(z) - f(z_0)| \leq 16\varepsilon$ for $|z - z_0| < \rho$, with $\rho$ independent of $z_0$. Thus $f$ is uniformly continuous on $\Delta$, as asserted.

### 10.5.2 Remark

If $D_1$ is a simply connected proper subdomain in $\mathbb{C}$ then the following method may be used to map $D_1$ conformally onto a bounded simply connected domain. Take $a \in D_1, b \in \mathbb{C} \setminus D_1$. The function $u(z) = (z-b)^{1/2}$ is analytic and one-one on $D_1$. Further, if $z_0 \in D_1$ then $u(z)$ does not take the value $-u(z_0)$ on $D_1$. So there is some $r > 0$ such that $D(u(a), r) \subseteq u(D_1)$ and $|u(z) + u(a)| \geq r$ on $D_1$, so that $v(z) = (u(z) + u(a))^{-1}$ is bounded and conformal on $D_1$.

### 10.5.3 Theorem

Let $D$ be a Jordan domain in $\mathbb{C}$ i.e. a bounded simply connected domain in $\mathbb{C}$ such that $J = \partial D$ is a Jordan curve. Let $f$ map $\Delta$ conformally (i.e. analytically and one-one) onto $D$. Then $f$ has a continuous extension mapping $\Delta \cup T$ one-one onto $D \cup J$.

**Proof.** Since $J$ is a curve $f$ has a continuous extension mapping $\Delta \cup T$ onto $D \cup J$, by Theorem 10.5.1, and it remains only to show that the extended function is homeomorphic. Certainly $f$ maps $T$ onto $J$. Let $a \in J$. The set

$$E = \{z \in T : f(z) \neq a\}$$

is a relatively open subset of $T$. Further, $F = T \setminus E$ has measure zero, as may be seen by applying the two-constants theorem to the function $u(z) = \log |f(z) - a|$, which is subharmonic and bounded above.
on $\Delta$.

Assume that $|z_1| = |z_2| = 1, z_1 \neq z_2, f(z_1) = f(z_2) = a$. The set $T \setminus \{z_1, z_2\}$ consists of two open arcs of $T$, denoted $A_1, A_2$, and we choose $u_j \in A_j$ such that $v_j = f(u_j) \neq a$. Join $z_1$ to $z_2$ by a straight line segment $L$.

Now $\Gamma = f(L) \cup \{a\}$ is a Jordan curve in $D \cup J$, with $f(L) \subseteq D$. So $C \setminus \Gamma$ has two components, $U_0, U_1$, with $U_0$ unbounded.

Claim 1: Each $v_j$ lies in $U_0$.

By rotating $D$ if necessary we may assume that there exists $b \in J$ with $\text{Re}(b) \leq \text{Re}(w)$ for all $w$ in $J$ and $\text{Re}(b) < \text{Re}(a)$. Thus the line $N$ given by $z = b - t, t \geq 0$, does not meet $D \cup \{a\}$. Hence $v_j$ can be joined to points of arbitrarily large modulus by a path not meeting $D \cup \{a\}$ (follow an arc of $J$ to $b$ and then follow $N$), and so not meeting $\Gamma$. This proves Claim 1.

Since $f(L) \subseteq \Gamma = \partial U_1$, there are points arbitrarily close to $L$ whose images under $f$ lie in $U_1$. Thus we can choose a simple curve $M$ from $u_1$ to $u_2$, consisting of two straight line segments, such that $M$ lies in $\Delta$ (apart from its end-points), $f(M)$ meets $U_1$, and $M$ intersects $L$ at precisely one point, $v$. If $z^* \in M \setminus \{v\}$ then the line segment $M^*$ from $z^*$ to one of the $u_j$ is a path not meeting $L \cup f^{-1}(\{a\})$ and so $f(M^*)$ is a path from $v_j$ to $f(z^*)$ not meeting $\Gamma$. Thus $f(z^*) \in U_0$, for all $z^* \in M \setminus \{v\}$, contradicting the assumption that $f(M)$ meets $U_1$.

### 10.5.4 Extending the conformal mapping to the plane

Let $X$ be a Jordan curve in $\mathbb{C}$. Assume that $0 \in U_1$. The Riemann mapping theorem gives an analytic homeomorphism $f : D(0, 1) \to U_1$ with $f(0) = 0$, and we have seen that $f$ extends to a homeomorphism of $|z| \leq 1$ onto $U_1 \cup X$. By first using the map $z \to 1/z$, and applying the Riemann mapping theorem again, we obtain a homeomorphism $g$ of $\{z : 1 \leq |z| \leq \infty\}$ onto $X \cup U_0 \cup \{\infty\}$ with $g(\infty) = \infty$. Thus $\phi(z) = g^{-1}(f(z))$ is a homeomorphism of $T$ onto itself, and so $f(z) = g(\phi(z))$ for $|z| = 1$. If we set

$$G(z) = f(z), \quad |z| < 1,$$

with $G(\infty) = \infty$ and

$$G(z) = g(|z|\phi(z)/|z|)), \quad 1 < |z| < \infty,$$

then $f$ has been extended to a homeomorphism $G$ of the extended plane onto itself, fixing $\infty$.

### 10.6 Totally disconnected sets

A non-empty set $H$ is called totally disconnected if its only connected subsets are singleton sets $\{x\}, x \in H$. Compact totally disconnected sets arise for example as the Julia set of $z^2 + c$, for any $c$ not in the Mandelbrot set (in particular for $|c| > 2$).

#### 10.6.1 Lemma

Let $E$ be a connected subset of $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, with $\infty \in E$. Then all components $U_j$ of $\mathbb{C}^* \setminus E$ are simply connected domains in $\mathbb{C}$.
10.6.2 Lemma

Let $U$ be a domain in $\mathbb{C}$. Then there exists a simply connected domain $V$ such that $U \subseteq V \subseteq \mathbb{C}$ and $\partial V \subseteq \partial U$.

(All boundaries here are with respect to $\mathbb{C}^*$).

Proof. Let $E$ be the component of $\mathbb{C}^* \setminus U$ containing $\infty$. Let $V$ be the component of $F = \mathbb{C}^* \setminus E$ containing $U$. Then $V$ is a simply connected domain, and $\partial V \subseteq \partial F = \partial E \subseteq \partial U$.

10.6.3 Theorem

Let $X$ be a totally disconnected compact subset of $\mathbb{C}$. Then $\mathbb{C} \setminus X$ is connected.

Proof. Assume that $\mathbb{C} \setminus X$ has a bounded component $U$. Then there is a simply connected domain $V$ with $U \subseteq V \subseteq \mathbb{C}$, and $W = \partial V \subseteq \partial U \subseteq X$, so that $V$ is bounded. Since $V$ is simply connected, $W$ is a connected subset of $X$, and so at most a singleton, which is plainly impossible.

Note that Browder’s paper suggests an alternative approach to this, based on partitioning $X$ into a disjoint union of compact sets of small diameter.

10.7 Boundary behaviour of analytic functions

10.7.1 Schwarz reflection principle

Let $r > 0$ and $D^+ = \{z : |z| < r, \text{Im}(z) > 0\}$ and $D^- = \{z : |z| < r, \text{Im}(z) < 0\}$ and let $u$ be harmonic on $D^+ = \{z : |z| < r, \text{Im}(z) > 0\}$ with $\lim_{z \to x} u(z) = 0$ for every $x \in (-r, r)$. Then $u$ extends to a harmonic function on $D(0, r)$ satisfying $u(z) = -u(\bar{z})$.

Proof. Let $0 < s < r$ and let $f(t) = u(se^{it})$ on $(0, \pi)$. Extend $f$ to an odd continuous function on $[-\pi, \pi]$. Let $U$ be the Poisson integral of $f(t)$ in $D(0, s)$. Then $U = 0$ on $(-s, s)$ since the Poisson kernel is even when $z$ is real, while $f$ is odd. Thus $U - u = 0$ on $\{z : |z| < s, \text{Im}(z) > 0\}$ since $U - u \to 0$ as $z$ tends to any point on the boundary.

The reflection principle has a very powerful consequence for the boundary behaviour of analytic functions. Suppose that $D$ is a domain in $\mathbb{C}$ and that $D^+ \subseteq D$ and $D^- \cap D = \emptyset$, so that $I = (-r, r) \subseteq \partial D$. Let $f$ be analytic on $D$ such that $u(z) = \text{Re}(f(z)) \to 0$ as $z \to x \in I$. Thus $u$ extends across $I$ to a harmonic function on $D(0, r)$, which is in turn the real part of an analytic function $g$ on $D(0, r)$. Thus $g$ is an analytic extension of $f$ to $D(0, r)$.

We can also handle “corners” as follows. Suppose that $r > 0$ and $0 < \alpha < 2\pi$ and that $D$ is a domain such that

$$D_1 = \{z : 0 < |z| < r, 0 < \arg z < \alpha\} \subseteq D, \quad \{z : 0 < |z| < r, \alpha < \arg z < 2\pi\} \cap D = \emptyset.$$
Thus the corner

\[ J = \{ 0 \} \cup \{ z : 0 < |z| < r, \arg z = 0, \alpha \} \]

forms part of the boundary of \( D \). Suppose that \( f \) is analytic on \( D \) and that \( \text{Re}(f(z)) \to 0 \) as \( z \to \zeta \in J \). By setting \( w = z^{\pi/\alpha} \) and \( g(w) = f(z) \), we obtain an analytic function \( g \) on a semi-disc. We extend \( g \) to the disc and this extends \( f \) continuously to \( J \).
Chapter 11

Homotopy and analytic continuation

11.1 Homotopy

Let \( S \) be a path-connected topological space (i.e. any two points \( a, b \) in \( S \) can be joined by a continuous \( f : [0,1] \rightarrow S \) with \( f(0) = a, f(1) = b \)), and let \( x_0, x_1 \) be points in \( S \) (possibly the same). Let \( \gamma, \sigma \) be two paths in \( S \), both defined on \( [0,1] = I \), and both going from \( x_0 \) to \( x_1 \) i.e. \( \sigma(0) = \gamma(0) = x_0, \sigma(1) = \gamma(1) = x_1 \).

Suppose that \( S \) is a disc, or is \( \mathbb{R}^n \), or is some kind of space that can be thought of as having “no holes”. Then it’s reasonable to believe that we could continuously deform \( \gamma \) into \( \sigma \) by a family of paths in \( S \). What we mean by this is that there is a family of paths \( h_u \), \( 0 \leq u \leq 1 \), in \( S \) such that:

(i) each \( h_u \) is defined on \([0,1]\), with \( h_u([0,1]) \) contained in \( S \), and joins \( x_0 \) to \( x_1 \);

(ii) we have \( h_0 = \gamma \) and \( h_1 = \sigma \);

(iii) if \( u \) is close to \( v \) then \( h_u \) is close to \( h_v \). More precisely, if we define the function \( H(t,u) \) by \( H(t,u) = h_u(t) \) then this \( H \) will be continuous on \([0,1] \times [0,1] \) (with the usual metric on \( I^2 = [0,1] \times [0,1] \)).

\( H \) is called a homotopy function and we say that \( \gamma \) is homotopic to \( \sigma \) in \( S \).

Example 1: In \( \mathbb{C} \), let \( \gamma(t) = \cos t, 0 \leq t \leq \pi \), and let \( \sigma(t) = \cos t + i \sin t, 0 \leq t \leq \pi \). If we put \( h_u(t) = \cos t + i u \sin t, 0 \leq t \leq 1 \), we see that \( \gamma \) can be continuously deformed into \( \sigma \).

Example 2: Let \( D \) be a star domain with star centre \( w \). Let \( \gamma : [0,1] \rightarrow D \) be a closed path, with \( \gamma(0) = w \). Then \( \gamma \) is homotopic to the constant path \( \sigma \) given by \( \sigma(t) = w \). For \( h_u \) we can just take \( uw + (1 - u)\gamma(t) = w + (1 - u)(\gamma(t) - w) \).

Example 3: if \( S = \{ z : 1 < |z| < 3 \} \) then intuitively it’s easy to see that the circle \( \gamma(t) = 2e^{2\pi it} \) is not homotopic to the constant path \( \sigma(t) = 2 \) in \( S \) (although they are homotopic in \( \mathbb{C} \)).

Where no confusion might arise, we drop the phrase “in \( S \”).

Remark: we can define homotopy for paths both defined on \([a,b]\) (so that \( H \) is then defined on \([a,b] \times [0,1] \)) but the formulation above is most usual.
11.1. **Fact**

*Homotopy is an equivalence relation.*

Clearly each $\gamma$ is homotopic to itself, with homotopy function $H(t, u) = \gamma(t)$.

If $\gamma$ is homotopic to $\sigma$ with family of paths $h_u$ then $\sigma$ is homotopic to $\gamma$: just put $g_u(t) = h_{1-u}(t)$, and $G(t, u) = g_u(t)$ is continuous on $I^2$.

Finally, if $\gamma$ is homotopic to $\sigma$ with family of paths $f_u$, and $\sigma$ is homotopic to $\tau$ with family of paths $g_u$, then we form a family of paths $h_u$ which continuously deform $\gamma$ into $\tau$ just by putting $h_u = f_{2u}$ for $0 \leq u \leq 1/2$ and $h_u = g_{2u-1}$ for $1/2 \leq u \leq 1$.

Note that $h_{1/2} = f_1 = \sigma = g_0 = g_{2(1/2)-1}$.

11.1.2 **Products of paths**

Given two paths $\gamma, \sigma : [0, 1] \to S$ with $\sigma(0) = \gamma(1)$ we can define a path which is ‘$\gamma$ followed by $\sigma$’ by

$$(\gamma\sigma)(t) = \gamma(2t), \quad 0 \leq t \leq 1/2,$$

$$(\gamma\sigma)(t) = \sigma(2t - 1), \quad 1/2 \leq t \leq 1.$$

(sometimes called a “product”).

11.1.3 **Fact**

*If $\gamma_0$ is homotopic to $\gamma_1$ and $\sigma_0$ is homotopic to $\sigma_1$, with homotopy functions $F(t, u) = \gamma_u(t), G(t, u) = \sigma_u(t)$, $0 \leq t, u \leq 1$ respectively, and if the $\sigma_j$ start where the $\gamma_j$ finish, then $\gamma_0\sigma_0$ is homotopic to $\gamma_1\sigma_1$.*

Just use the paths $h_u = \gamma_u\sigma_u$. Note that $\gamma_u(1) = \sigma_u(0)$.

11.1.4 **The “inverse” path**

Let $\gamma : [0, 1] \to S$ be a path. We can define $\gamma^{-1}$ (or $\gamma$ backwards) by $\gamma^{-1}(t) = \gamma(1-t)$. Then $\gamma\gamma^{-1}(t) = \gamma(2t)$ for $0 \leq t \leq 1/2$ and $\gamma\gamma^{-1}(t) = \gamma^{-1}(2t - 1) = \gamma(2 - 2t)$ for $1/2 \leq t \leq 1$ and this is a closed curve. It is homotopic to a constant curve as follows:

Let $w = \gamma(0)$ and define $\eta(t) = w$ for $0 \leq t \leq 1$. Then $\eta$ is a constant curve, and is homotopic to $\gamma\gamma^{-1}$. Put $h_u(t) = \gamma(ut)$. Then as $t$ goes from 0 to 1, $h_u(t)$ goes along $\gamma$ as far as $\gamma(u)$. Now put $g_u = h_u h_u^{-1}$.

Clearly $g_0 = \eta, g_1 = \gamma \gamma^{-1}$ and $g_u(t)$ is continuous on $I^2$. What $g_u$ does is to go along $\gamma$ as far as $\gamma(u)$, and then retrace its steps back to $w$.

Note also that if $\gamma$ is homotopic to $\sigma$, with family of paths $p_u(t)$, then using the paths $p_u^{-1}(t) = p_u(1-t)$ we see that $\gamma^{-1}$ is homotopic to $\sigma^{-1}$.

11.1.5 **Re-scaling and homotopy**

*If $\sigma(t) = \gamma(g(t))$ where $g : [0, 1] \to [0, 1]$ is continuous, non-decreasing and onto, then $\sigma$ is homotopic to $\gamma$.****
We say that $\sigma$ is a re-scaling of $\gamma$, and a re-scaling of a path is homotopic to the original path.

To see this, just put $h_u(t) = \gamma(ut + (1-u)g(t))$ (and note that $ut + (1-u)g(t) \in [0,1]$ for $u, t \in [0,1]$).

(I prefer not to use the term re-parametrization, which is reserved for the case where $g$ above is strictly increasing.)

11.1.6 Corollary

If $\rho(t) \equiv \gamma(0)$ (constant curve) then $\rho \gamma$ ($\rho$ followed by $\gamma$) is homotopic to $\gamma$. Similarly, if $\tau(t) \equiv \gamma(1)$ then $\gamma \tau$ is homotopic to $\gamma$.

Proof: $\rho \gamma(t) = \gamma(h(t))$ and $\gamma \tau(t) = \gamma(k(t))$. Here $h(t) = 0$ for $0 \leq t \leq 1/2$ while $h(t) = 2t - 1$ for $1/2 \leq t \leq 1$. Similarly $k(t) = 2t$ for $0 \leq t \leq 1/2$ while $k(t) = 1$ for $1/2 \leq t \leq 1$.

This leads to a useful fact.

11.1.7 Fact

Let $\sigma, \tau : [0,1] \rightarrow S$ be paths from $x_0$ to $x_1$. Then $\sigma$ is homotopic to $\tau$ iff $\sigma \tau^{-1}$ (which is $\sigma$ followed by $\tau$ backwards) is homotopic to a constant path.

Why? If $\sigma$ is homotopic to $\tau$, then $\sigma \tau^{-1}$ is homotopic to $\tau \tau^{-1}$, and we know that the last path is homotopic to a constant path.

If $\sigma \tau^{-1}$ is homotopic to a constant path $\rho$, then $(\sigma \tau^{-1}) \tau$ is homotopic to $\rho \tau$ and so to $\tau$. But $(\sigma \tau^{-1}) \tau$ is a re-scaling of $\sigma (\tau^{-1} \tau)$, and so is homotopic to $\sigma \lambda$, where $\lambda$ is a constant path, and so to $\sigma$.

Now we can make a group.

11.1.8 The fundamental group

Let $S$ be a topological space, and let $x_0 \in S$. Consider the family $H$ of all closed paths $\lambda : [0,1] \rightarrow S$ starting and finishing at $x_0$. For a given $\gamma$, let $[\gamma]$ be the equivalence class of members $\lambda$ of $H$ s.t. $\lambda$ is homotopic to $\gamma$.

We can define a multiplication by $[\gamma][\sigma] = [\gamma \sigma]$ (equivalence class of the product path $\gamma$ “followed by” $\sigma$). This is well defined as, if $\gamma_1$ is homotopic to $\gamma_2$ and $\sigma_1$ is homotopic to $\sigma_2$, then $\gamma_1 \sigma_1$ is homotopic to $\gamma_2 \sigma_2$.

Define $I(t) = x_0, 0 \leq t \leq 1$. Then $[I \gamma] = [\gamma]$ and $[\gamma I] = [\gamma]$ for every $\gamma$ in $H$. Also, $[\gamma^{-1} \gamma] = [I]$ for every $\gamma$ in $H$. So we have a group, with identity $[I]$, and with $[\gamma]^{-1} = [\gamma^{-1}]$, called the fundamental group $\pi(x_0, S)$.

1. Note that this group might not be Abelian. However, the multiplication is associative, because $\rho(\sigma \tau)$ is a re-scaling of $(\rho \sigma) \tau$. 
2. If \( x_1 \in S \) then the groups \( \pi(x_0, S), \pi(x_1, S) \) are isomorphic. Choose a fixed path \( \Lambda \) from \( x_0 \) to \( x_1 \). For \([\gamma]\) in \( \pi(x_1, S) \) we define \( T([\gamma]) \) to be \([\Lambda\gamma\Lambda^{-1}] \) (which is in \( \pi(x_0, S) \)).

This is well defined. This is because, if \( \gamma \) and \( \sigma \) both start and finish at \( x_1 \) and are homotopic, then \( \Lambda\gamma\Lambda^{-1} \) is homotopic to \( \Lambda\sigma\Lambda^{-1} \).

Note that \( \Lambda\Lambda^{-1} \) is homotopic to the constant path \( I(t) \equiv x_0 \). Thus \([\Lambda\Lambda^{-1}] \) is the identity in \( \pi(x_0, S) \) and \([\Lambda^{-1}\Lambda] \) is the identity in \( \pi(x_1, S) \). Thus \( T([\sigma][\gamma]) = T([\sigma\gamma]) = [\Lambda\sigma\gamma\Lambda^{-1}] = [\Lambda\sigma\Lambda^{-1}\Lambda\gamma\Lambda^{-1}] \) and this equals \([\Lambda\sigma\Lambda^{-1}][\Lambda\gamma\Lambda^{-1}] = T([\sigma])T([\gamma]) \).

Next, \( T \) is one-one, as \( T([\gamma]) = T([\sigma]) \) implies that \( \Lambda\gamma\Lambda^{-1} \) is homotopic to \( \Lambda\sigma\Lambda^{-1} \), and so \([\gamma] = [\Lambda^{-1}\Lambda\gamma\Lambda^{-1}\Lambda] = [\Lambda^{-1}\Lambda\sigma\Lambda^{-1}\Lambda] = [\sigma] \).

Also \( T \) is onto, as \( T([\Lambda^{-1}\sigma\Lambda]) = [\Lambda\Lambda^{-1}\sigma\Lambda\Lambda^{-1}] = [\sigma] \).

So \( T \) is a group isomorphism and so we often talk just of the fundamental group \( \pi(S) \).

### 11.1.9 Fact

If \( \sigma \) and \( \tau \) are homotopic paths in \( X \), with family of paths \( h_u \) continuously deforming \( \sigma \) into \( \tau \), and \( f : X \to Y \) is continuous, then \( f(\sigma), f(\tau) \) are homotopic paths in \( Y \).

(Just use the paths \( f(h_u(t)) \).)

If \( f \) is a homeomorphism from \( X \) to \( Y \) (i.e. \( f \) is one-one and onto, and both \( f \) and the inverse \( f^{-1} \) are continuous), then paths \( \mu, \nu \) from \( x_0 \) to \( x_1 \) are homotopic iff \( f(\mu) \) and \( f(\nu) \) are.

Also, if \( f \) is a homeomorphism from \( X \) to \( Y \) then the fundamental group of \( X \) is isomorphic to the fundamental group of \( Y \) just by setting \( T([\gamma]) \) to be the class \([f(\gamma)] \). This gives \( T([\gamma])T([\lambda]) = [f(\gamma)][f(\lambda)] = [f(\gamma)f(\lambda)] = [f(\gamma\lambda)] = T([\gamma\lambda]) \).

### 11.1.10 Simple connectivity in terms of homotopy

As remarked before, if \( S \) is a suitable space with “no holes” we would expect that two paths in \( S \) starting and finishing at the same points would be homotopic.

Let \( S \) be a path-connected topological space. We say that \( S \) is HSC (homotopy simply connected) if every closed curve \( \gamma : [0, 1] \to S \) (i.e. \( \gamma(0) = \gamma(1) \)) is homotopic to the constant curve \( \eta \) which satisfies \( \eta(t) = \gamma(0) \) for all \( t \).

This is the case if and only if, for any pair of curves \( \sigma, \tau \) in \( S \) such that \( \sigma(0) = \tau(0), \sigma(1) = \tau(1) \), it is the case that \( \sigma \) is homotopic to \( \tau \). This is by Corollary 11.1.6.

This is also the same as saying that the fundamental group of \( S \) is trivial (identity only).

### 11.1.11 Lemma

A domain \( D \) in \( \mathbb{C} \) is called convex if, for every \( z, w \) in \( D \), the straight line segment \( sz+(1-s)w, 0 \leq s \leq 1 \) is contained in \( D \). Convex domains are HSC.
Proof: given $\gamma, \sigma$ such that $\gamma(0) = \sigma(0), \gamma(1) = \sigma(1)$, just set $F(t, u) = (1 - u)\gamma(t) + u\sigma(t)$.

Alternatively: convex means that any point in the domain can be used as a star centre. Use the method earlier for star domains.

11.1.12 Lemma

Let $\gamma$ be a closed curve in $\mathbb{C}$ of diameter $L > 0$. Let $w \in \mathbb{C}$ with $\text{dist}\{w, \gamma\} > 8L$. Then $\gamma$ is null-homotopic with respect to $w$.

Here the diameter of a set $E$ means $\sup\{|z - z'| : z, z' \in E\}$, and null-homotopic with respect to $w$ means homotopic to a constant in $\mathbb{C} \setminus \{w\}$.

Proof. The hypotheses imply that the variation of $\arg(z - w)$ is less than $\pi$ on $\gamma$. Hence $\gamma$ lies in a sector with vertex at $w$ (some domain $a < \arg(z - w) < b < a + \pi$) and this is a convex region.

11.1.13 Lemma

Let $\gamma$ be a closed curve in $\mathbb{C}$ which is null-homotopic with respect to $z_0 \in \mathbb{C}$, and let $\text{dist}\{z_0, \gamma\} \geq 2\delta > 0$. Let $z_1 \in D(z_0, \delta)$. Then $\gamma$ is null-homotopic with respect to $z_1$.

Proof. Assume without loss of generality that $z_0 = 0$. There is some homotopy function $H(t, u) : [0, 1]^2 \to \mathbb{C} \setminus \{0\}$, with $H(t, 0) = \gamma(t)$ and $H(t, 1)$ a constant path. There exists $\rho > 0$ such that $|H(t, u)| \geq \rho$ for all $(t, u)$ in $[0, 1]^2$. If $\rho \geq \delta$ then $H(t, u)$ never equals $z_1$ and $\gamma$ is automatically null-homotopic with respect to $z_1$.

Assume now that $\rho < \delta$, and set

$$\phi(z) = z \left| \frac{z}{2\delta} \right| > \lambda, \quad |z| < 2\delta, \quad \phi(z) = z, \quad |z| \geq 2\delta,$$

in which $\lambda > 0$ is chosen so that

$$\left(\frac{\rho}{2\delta}\right)^\lambda = \delta.$$

Then $\phi$ is a homeomorphism of $\mathbb{C}$ onto itself, and $\phi(\gamma(t)) = \gamma(t)$. Thus the composition $K(t, u) = \phi(H(t, u))$ gives a homotopy from $\gamma$ to a constant path with, for all $(t, u)$, $|K(t, u)| \geq \delta$ and so $K(t, u) \neq z_1$.

11.1.14 Theorem

Let $\gamma$ be a closed path in $\mathbb{C}$. For $z \in \mathbb{C} \setminus \gamma$, let $\phi(z) = 0$ if $\gamma$ is null-homotopic with respect to $z$, with $\phi(z) = 1$ otherwise. Then $\phi$ is continuous on $\mathbb{C} \setminus \gamma$.

Proof. We have already seen in Lemma 11.1.13 that if $\phi(z_0) = 0$ then $\phi(z) = 0$ for $z$ near $z_0$. Suppose now that $\phi(z_0) = 1$ and $\text{dist}\{z_0, \gamma\} = \delta$ (necessarily positive). Then for $z_1 \in D(z_0, \delta/8)$ we have $\text{dist}\{z_1, \gamma\} \geq 7\delta/8$ and, again by Lemma 11.1.13, we must have $\phi(z_1) = 1$. 
11.1.15 The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is HSC

This seems intuitively obvious, but complications arise when we note that a closed path may visit every point on the sphere. Let $\gamma : [0, 1] \to \mathbb{C} \cup \{\infty\}$ be any closed curve.

Case 1: the curve $\gamma$ lies entirely in $|z| \leq M < \infty$. Then $\gamma$ lies in the convex domain $D(0, 2M)$ and is homotopic to a constant curve, with some continuous homotopy function $G(t, u) : [0, 1]^2 \to D(0, 2M)$. This $G$ is continuous if we regard it as a function into $\mathbb{C} \cup \{\infty\}$.

Case 2: the curve $\gamma$ lies entirely in $|z| \geq c > 0$. Then the curve $1/\gamma$ lies entirely in $|z| \leq 1/c$ and so is homotopic to a constant, with some homotopy function $G(t, u)$. Using $1/G(t, u)$ we see that $\gamma$ is homotopic to a constant.

Case 3: $\gamma$ sometimes visits both $|z| > 2$ and $|z| < 1/2$. Without loss of generality $|\gamma(0)| \geq 1$ (else look at $1/\gamma$). Suppose we have an interval $[a, b]$ on which $|\gamma(t)| \leq 3/4$, with $|\gamma(a)| = |\gamma(b)| = 3/4$ and $|\gamma(t)| \leq 1/2$ for some $t$ with $a < t < b$. We can form a curve $\sigma : [a, b] \to \mathbb{C}$ such that $\sigma(a) = \gamma(a)$ and $\sigma(b) = \gamma(b)$ and $|\sigma(t)| = 3/4$ for every $t$. But $\sigma$ and the part of $\gamma$ for $a \leq t \leq b$ both lie in $|z| \leq 3/4$ and so using a homotopy function (modified to be defined for $a \leq t \leq b, 0 \leq u \leq 1$) we can continuously deform the restriction $\gamma : [a, b] \to \mathbb{C} \cup \{\infty\}$ into $\sigma$. We do this for each such interval $[a, b]$, and we have continuously deformed $\gamma$ into a closed path in $|z| \geq 3/4$, which is now homotopic to a constant.

11.2 Analytic continuation

11.2.1 Example

The function $L_0(z) = \log z = \log |z| + i \arg z$ is analytic in the domain $D_0$ obtained by deleting from the complex plane the non-positive real axis. Here the argument is chosen to lie in $(-\pi, \pi)$. Obviously the restriction $L_1$ of $L_0$ to the upper half plane $D_1 = \{z : \text{Im}(z) > 0\}$ is also analytic.

In the same way, the function $L_2(z) = \log |z| + i \arg z$, with the argument chosen to lie in $(0, 2\pi)$, is analytic in the domain $D_2$ obtained by deleting from the plane the non-negative real axis.

If we start at 1 and continue $L_0(z)$ counter-clockwise around the circle $|z| = 1$, the argument increases, and $L_0 = L_1 = L_2$ in the quadrant $\pi/2 < \arg z < \pi$. Following the circle further, the argument continues to increase until, on approaching 1 again, the argument tends to $2\pi$. Thus $L_0$ has been “continued” around the circle, but has not returned to its original value.

11.2.2 Analytic continuation along a path

By a function element we mean a pair $(f, D)$, in which $D$ is a domain in $\mathbb{C}^*$ and $f$ is meromorphic on $D$. As usual, meromorphic at $\infty$ means that $f(1/z)$ is meromorphic at 0.

Let $(f, D)$ be a function element and let $z_0 \in D$. Let $\gamma : [a, b] \to \mathbb{C}^*$ be a path with $\gamma(a) = z_0$ (note that continuity is with respect to the spherical metric). An analytic continuation of $(f, D)$ along $\gamma$ is a family of function elements $(f_t, D_t), a \leq t \leq b$, with the following properties.

(i) $f_a = f$ on a neighbourhood of $z_0 = \gamma(a)$.

(ii) $\gamma(t) \in D_t$ for every $t$ in $[a, b]$. 

(iii) For every $t$ in $I = [a, b]$ there exists $\rho_t > 0$ such that the following holds. For $a \leq s \leq b, |s - t| < \rho_t$ we have $\gamma(s) \in D_t$ and $f_s = f_t$ on a neighbourhood of $\gamma(s)$.

Here a neighbourhood of $z$ means an open set containing $z$. Note that in (iii) we do not require that $f_s = f_t$ on all of $D_s \cap D_t$, but this will be the case if $D_s \cap D_t$ is connected.

Strictly speaking, this is meromorphic continuation but, as this term is not normally used, we shall say that the analytic continuation is finite-valued if all the $f_t$ map their $D_t$ into $\mathbb{C}$ rather than $\mathbb{C}^*$.

If $G, H$ are domains with $G \subseteq H$ then we say that a function element $(g, G)$ admits unrestricted analytic continuation (UAC) in $H$ if $(g, G)$ can be analytically continued along every path in $H$ starting in $G$.

11.2.3 Lemma

Suppose that the function element $(f, D)$ is analytically continued along the path $\gamma : [a, b] \to \mathbb{C}^*$ by the family of function elements $(f_t, D_t)$. Then $h(t) = f_t(\gamma(t)) : [a, b] \to \mathbb{C}^*$ is a path.

Proof. We need of course to show that $h$ is continuous. Let $W$ be a neighbourhood of $f_t(\gamma(t))$ and let $U \subseteq D_t$ be a neighbourhood of $\gamma(t)$ such that $f_t(U) \subseteq W$. If $s$ is close enough to $t$ then we have $\gamma(s) \in U$ and, since $f_s = f_t$ near $\gamma(s)$, we get $f_s(\gamma(s)) \in W$.

11.2.4 Theorem

Let $(f, D)$ and $(g, D)$ both be analytically continued along the path $\gamma : [a, b] \to \mathbb{C}^*$. If there exists $u \in [a, b] = J$ such that $f_u = g_u$ on a neighbourhood of $\gamma(u)$ then for every $t$ in $J$ we have $f_t = g_t$ on a neighbourhood of $\gamma(t)$.

In particular if $f = g$ then $f_1 = g_1$ on a neighbourhood of $\gamma(1)$ and so the continuation along $\gamma$ is (locally) unique.

Proof of the theorem. Let $E$ be the set of $t$ in $J$ such that $f_t = g_t$ on a neighbourhood $U(t)$ of $\gamma(t)$. Let $t \in E$. Then there exists $\rho > 0$ such that if $s \in J$ and $|s - t| < \rho$ then $\gamma(s) \in U(t)$ and $f_s = f_t = g_t = g_s$ on a neighbourhood of $\gamma(s)$. Thus $s \in E$ and $E$ is relatively open.

Now let $t \in J \setminus E$. Then there exists a neighbourhood $U$ of $\gamma(t)$ such that for all $z$ in $V = U \setminus \{\gamma(t)\}$ we have $f_t(z) \neq g_t(z)$. If $s$ is close enough to $t$ we have $\gamma(s) \in U$ and $f_s = f_t$ and $g_s = g_t$ on a neighbourhood $W$ of $\gamma(s)$. Thus for $z$ in $W \setminus \{\gamma(t)\}$ we have $f_s(z) \neq g_z(z)$ and so $s \not\in E$. Hence $J \setminus E$ is also relatively open and so, since $E \neq \emptyset$, we have $J = E$ by connectivity.

11.2.5 Critical and asymptotic values

Let $f : \mathbb{C} \to \mathbb{C}^*$ be meromorphic. A critical point $z$ of $f$ is a multiple point of $f$ i.e. a pole of multiplicity at least 2 or a point where $f'(z) = 0$. A point $z$ is critical if and only if there is no neighbourhood of $z$ on which $f$ is one-one. The critical values of $f$ are the values taken at critical points.

Thus $\cos z$ has critical points $n\pi, n \in \mathbb{Z}$, and critical values $\pm 1$, while the only critical value of $1/(e^z - 1)^2$ is $\infty$. 
11.2. ANALYTIC CONTINUATION

An asymptotic value of \( f \) is an element \( w \) of \( \mathbb{C}^* \) such that there exists a path \( \gamma \) tending to infinity with \( f(z) \to w \) as \( z \to \infty \) on \( \gamma \).

Note that asymptotic values only have relevance for transcendental functions. If \( f \) is a rational function then \( \infty \) is a point like any other (though it may be a critical point and/or a pole), since \( f(1/z) \) is meromorphic at 0.

For example, \( e^z \) has no critical values, but it has asymptotic values \( 0, \infty \).

Iversen’s theorem (see Exercise 8.2.6) says that if \( f \) is a non-constant entire function then \( \infty \) is always an asymptotic value of \( f \).

11.2.6 Theorem

Let \( f : \mathbb{C} \to \mathbb{C}^* \) be non-constant and meromorphic. Let \( z_0 \in \mathbb{C} \) be a non-critical point of \( f \), and let \( \gamma : [0,1] \to \mathbb{C}^* \) be a path in \( \mathbb{C}^* \) starting at \( w_0 = f(z_0) \). Let \( g \) be that branch of the inverse function \( f^{-1} \) which is defined on a neighbourhood \( D \) of \( w_0 \) and maps \( w_0 \) to \( z_0 \). Let \( S \) be the supremum of \( u \) in \( [0,1] \) such that \( g = (g,D) \) admits analytic continuation along the path \( \gamma : [0,u] \to \mathbb{C}^* \), the function values \( g_t \) finite-valued. Then either (i) \( g_t(\gamma(t)) \to \infty \) as \( t \to S^- \) and \( \gamma(S) \) is an asymptotic value of \( f \), or (ii) \( g_t(\gamma(t)) \to z^* \) as \( t \to S^- \), with \( z^* \) a critical point of \( f \), and \( \gamma(S) = f(z^*) \) a critical value of \( S \), or (iii) \( g \) admits finite-valued analytic continuation along \( \gamma : [0,S] \to \mathbb{C}^* \).

If \( \gamma \) contains none of the critical and asymptotic values of \( f \), then \( g \) admits finite-valued analytic continuation along \( \gamma \).

(If \( f \) is a non-constant rational function then the same proof as below shows that continuation is possible along any \( \gamma \) avoiding critical values of \( f \), although not necessarily finite-valued).

Proof. We first note that \( S > 0 \), because for small \( t \) we can take \( D_t = D \) and \( g_t = g \).

We begin by noting a consequence of the uniqueness result Theorem 11.2.4. If \( 0 < u \leq u' \leq S \) and \( g_t \) is a continuation along \( \gamma : [0,u] \to \mathbb{C}^* \), while \( h_t \) is a continuation along \( \gamma : [0,u'] \to \mathbb{C}^* \), then for \( 0 \leq t \leq u \) we have \( g_t = h_t \) on a neighbourhood of \( \gamma(t) \). In particular, \( g_t(\gamma(t)) \) is uniquely defined, and is a continuous function from \([0,S]\) into \( \mathbb{C} \). Also, for every \( u \) with \( 0 < u < S \) the continuation is possible along \( \gamma : [0,u] \to \mathbb{C}^* \).

Next, for small \( t \) we have \( f \circ g_t(w) \equiv w \) on a neighbourhood of \( \gamma(t) \) and so, by Theorem 11.2.4 again, we have \( f(g_t(\gamma(t))) = \gamma(t) \) for \( 0 \leq t < S \). Assume that (i) and (ii) do not hold. Then it cannot be the case that \( g_t(\gamma(t)) \to \infty \) as \( t \to S^- \) and so there exists \( M > 0 \) such that \( |g_{t_n}(\gamma(t_n))| \leq M \) for a sequence \( t_n \to S^- \). Take \( N > M \) such that \( q(f(z),\gamma(S)) > \rho > 0 \) on \( |z| = N \). Then for \( t \) close to \( S \) we have \( |g_t(\gamma(t))| \neq N \), and it follows that \( |g_t(\gamma(t))| < N \) for \( t \) close to \( S \).

As \( t \to S^- \) the continuation \( g_t(\gamma(t)) \) stays in the compact region \( |z| \leq N \), and \( f(g_t(\gamma(t))) \to w_1 = \gamma(S) \). Hence there exists \( z_1 \) with \( f(z_1) = w_1 \) and \( |z_1| \leq N \) such that \( g_t(\gamma(t)) \to z_1 \) as \( t \to S^- \). Since we have assumed that (ii) does not hold, it follows that \( z_1 \) is not a critical point of \( f \) and so we can choose a small neighbourhood \( U_1 \) of \( z_1 \) on which \( f \) is one-one. Let \( h \) be the inverse function \( f^{-1} \) mapping the neighbourhood \( W_1 = f(U_1) \) onto \( U_1 \) with, obviously, \( h(w_1) = z_1 \). Since \( g_t(\gamma(t)) \to z_1 \) as \( t \to S^- \), and since \( S \) is a supremum, we may take \( u \) with \( 0 < u \leq S \) and such that \( g_u(\gamma(u)) \in U_1 \),
while \((g, D)\) admits a finite-valued analytic continuation along \(\gamma : [0, u] \to \mathbb{C}^*\).

Let \(t\) be close to \(u\), with \(t \leq u\). Then we have \(g_t(w) \in U_1\) and \(f(g_t(w)) = w \in W_1\) for \(w\) close to \(\gamma(t)\). But \(f(h(w)) = w\) on \(W_1\), and \(f\) is one-one on \(U_1\), so that \(g_t(w) = h(w)\) for \(w\) close to \(\gamma(t)\) and \(t\) close to \(u\). Thus we may use the function element \((h, W_1)\) to analytically continue \((g, D)\) all the way along \(\gamma : [0, S] \to \mathbb{C}^*\) and, if \(S < 1\), along some \(\gamma : [0, S'] \to \mathbb{C}^*\) with \(S' > S\). This proves the first assertion of the theorem.

The second assertion is easy: the assumptions rule out (i) and (ii), so that \(S\) must be 1.

### 11.2.7 Corollary

Let \(f\) be non-constant and meromorphic in \(\mathbb{C}\). Suppose that \(D\) is a domain in \(\mathbb{C}^*\) not containing any critical or asymptotic value of \(f\). Let \(z_0 \in \mathbb{C}\) and \(f(z_0) = w_0 \in D\). Then the branch of \(f^{-1}\) defined near \(w_0\) and mapping \(w_0\) to \(z_0\) admits finite-valued UAC in \(D\).

### 11.2.8 Corollary

Any local branch of the logarithm \(\log z = \log |z| + i \arg z\) admits UAC in \(\mathbb{C} \setminus \{0\}\).

### 11.3 The monodromy theorem

Suppose that we have a function element \((f, D)\) and analytic continuations \((f_t, D_t), (g_t, G_t)\) along paths \(\gamma_j : [0, 1] \to \mathbb{C}^*\) from \(z_0\) to \(z_1\). Under what circumstances will \(f_1 = g_1\) on a neighbourhood of \(z_1\)? This will certainly be the case if \(\gamma_1 = \gamma_2\) (Theorem 11.2.4) but we saw in Example 11.2.1 that is is possible to continue \(\log z\) around a closed curve and not return to the original branch.

With the \(\gamma_j\) as above we will say that the continuations \((f_t, D_t), (g_t, G_t)\) lead to the same local function element if \(f_1 = g_1\) on a neighbourhood of \(z_1\). Note that we’re not assuming that \(D_1 = G_1\).

Note also that if we continue \(f\) along \(\gamma : [0, 1] \to \mathbb{C}^*\), with function elements \((f_t, D_t)\), and then back along \(\gamma^{-1}\), the “final” function element \(f_2\) will always equal \(f_0\) near \(\gamma(0)\). This is because \((f_t, D_t), 0 \leq t \leq 1\), with \((f_{2-t}, D_{2-t}), 1 \leq t \leq 2\), gives one continuation along \(\gamma\gamma^{-1}\) and so in effect, by Theorem 11.2.4, the only continuation.

### 11.3.1 Monodromy theorem

Let \(G\) be a domain in \(\mathbb{C}^*\), and let \((f, D)\) be a function element with \(D \subseteq G\), and assume that \((f, D)\) admits UAC in \(G\). Let \(z_0 \in D\) and let \(z_1 \in D\), and let \(\gamma, \sigma\) be homotopic paths from \(z_0\) to \(z_1\) in \(G\). Then analytic continuation of \((f, D)\) along \(\gamma, \sigma\) leads to the same local function element near \(z_1\).

**Proof.** We have some homotopy function

\[
H(t, u) = h_u(t) : I^2 \to G, \quad I = [0, 1], \quad h_0 = \gamma, \quad h_1 = \sigma.
\]

Let \(\mu\) be a path in \(I^2\) from \((0, 0)\) to \((1, 1)\). Then \(H(\mu)\) is a path in \(G\) from \(z_0\) to \(z_1\), and \((f, D)\) admits analytic continuation along \(H(\mu)\). With a slight abuse of notation, we refer to this as continuation of \(f\) along \(\mu\).

We need the following idea of quadrisection of a square \(J\), which we describe only for the square
\[ J = I^2, \] but which carries over to any square, with obvious modifications. We define stepwise paths as follows:

(i) \( \mu_1(I^2) \) goes \((0,0), (0,1), (1,1)\);

(ii) \( \mu_2(I^2) \) goes \((0,0), (0,1/2), (1/2, 1/2), (1/2, 1), (1,1)\);

(iii) \( \mu_3(I^2) \) goes \((0,0), (0,1/2), (1,1/2), (1,1)\);

(iv) \( \mu_4(I^2) \) goes \((0,0), (1/2,0), (1/2,1/2), (1,1/2), (1,1)\);

(iv) \( \mu_5(I^2) \) goes \((0,0), (1,0), (1,1)\).

For a general square \( J \), the construction is the same, involving vertices and the midpoint of \( J \). Note that each \( \mu_k(J) \) has length \( 2L \), in which \( L \) is the side-length of \( J \), and \( \mu_k = \mu_{k+1} \) except on an interval of length \( L \). Also \( \mu_k \) and \( \mu_{k+1} \) together bound a square of side-length \( L/2 \).

Assume that the analytic continuations \((f_t, D_t), (g_t, G_t)\) of \((f, D)\) along \( \gamma, \sigma \) do not lead to the same local function element. Since \( h_u(0) = H(0,u) = z_0 \) and \( h_u(1) = H(1,u) = z_1 \) for all \( u \), we may continue \( f \) along \( \mu_1(I^2) \) and \( \mu_5(I^2) \) (on the vertical segments just use \( f_0, g_1 \) respectively). Thus continuation of \( f \) along \( \mu_1(I^2), \mu_5(I^2) \) does not lead to the same local function element. Set \( \gamma_0 = \mu_1(I^2), \sigma_0 = \mu_5(I^2), J_0 = I^2, K_0 = [0, 2] \).

**Claim 1:** For each non-negative integer \( n \) there exist paths \( \gamma_n, \sigma_n : [0, 2] \to I^2 \), square regions \( J_n \), and intervals \( K_n \), with the following properties:

(a) \( J_{n+1} \subseteq J_n \subseteq I^2 \) and \( J_n \) has side-length \( 2^{-n} \);

(b) \( K_{n+1} \subseteq K_n \subseteq [0, 2] \) and \( K_n \) has length \( 2^{1-n} \);

(c) as \( t \) describes the interval \( K_n \), the paths \( \gamma_n, \sigma_n \) each describe a simple arc of \( \partial J_n \) from the bottom left corner to the top right, \( \gamma_n \) clockwise, \( \sigma_n \) counter-clockwise;

(d) for \( t \notin K_n \) we have \( \gamma_{n+1} = \gamma_n = \sigma_n = \sigma_{n+1} \);

(e) for \( 0 \leq t \leq 2 \) we have

\[
|\gamma_n(t) - \gamma_{n+1}(t)| \leq 2^{-n} \sqrt{2}, \quad |\gamma_n(t) - \sigma_n(t)| \leq 2^{-n} \sqrt{2}, \quad 0 \leq t \leq 2.
\]

(f) continuation of \( f \) along \( \gamma_n, \sigma_n \) does not lead to the same local function element.

To prove Claim 1 we show how to determine \( \gamma_{n+1}, \sigma_{n+1} \) consistent with (a) to (f). We quadrisect the square \( J_n \) by paths \( \mu_k(J_n) \) as described above. Combining these with the restriction of \( \gamma_n \) to \([0, 2] \setminus K_n \) gives us five stepwise curves \( \nu_k \), joining \((0,0)\) to \((1,1)\) and travelling through \( J_n \) from the bottom left corner to the top right, such that \( \nu_1 = \gamma_n, \nu_5 = \sigma_n \) and \( \nu_k(t) = \gamma_n(t) = \sigma_n(t) \) off the interval \( K_n \). Further, \( \nu_k = \nu_{k+1} \) off an interval of length \( 2^{-n} \), and \( \nu_k, \nu_{k+1} \) together bound a square of side-length \( 2^{-n-1} \). Choosing \( k \) such that continuation of \( f \) along \( \nu_k, \nu_{k+1} \) does not lead to the same local function element, we set \( \gamma_{n+1} = \nu_k, \sigma_{n+1} = \nu_{k+1} \).
Let $x_0$ be the unique point of $\mathbb{R}^2$ lying in the intersection of the nested square regions $J_n$. Then $x_0 \in I^2$ and (abusing notation again slightly) $H(x_0) \in G$. By the construction, the paths $\gamma_n, \sigma_n$ both converge uniformly on $[0, 2]$ to a path

$$\eta : [0, 2] \to I^2, \quad \eta(t) = \gamma_0(t) + \sum_{n=1}^{\infty}(\gamma_n(t) - \gamma_{n-1}(t)).$$

Let $t_0$ be the unique point of $[0, 2]$ lying in the intersection of the $K_n$. Since $\gamma_n(t_0) \in J_n$ for all $n$ we have $\eta(t_0) = x_0$. Since $\eta$ joins $(0, 0)$ to $(1, 1)$ in $I^2$, we may continue $f$ along $\eta$ (i.e. analytically continue the function element $(f, D)$ along $H(\eta)$), using function elements $(F_t, U_t), 0 \leq t \leq 2$.

Let $n$ be large. Then we may partition $[0, 2]$ into three intervals $[0, a], [a, b], [b, 2]$ with $0 \leq a < b \leq 2$, such that $\gamma_n, \sigma_n, \eta$ agree on $[0, a]$ and $[b, 2]$, and $H(\gamma_n([a, b])), H(\sigma_n([a, b])), H(\eta([a, b]))$ all lie in $U_{t_0}$. The analytic continuations of $(f, D)$ along $H(\gamma_n) : [0, a] \to G, H(\sigma_n) : [0, a] \to G, H(\eta) : [0, a] \to G$ all lead to the same local function element, which equals $F_{t_0}$ on a neighbourhood of $H(\eta(a))$. We may then use the function element $(F_{t_0}, U_{t_0})$ to extend these analytic continuations to the interval $[0, b]$, and finally the function elements $(F_t, U_t), t \geq t_0$ extend these continuations to all of $[0, 2]$. Thus there are continuations of $f$ along $\gamma_n, \sigma_n$ leading to the same local function element near $z_1$. By Theorem 11.2.4, the same is true of any continuations of $f$ along $\gamma_n, \sigma_n$. This contradicts the way the $\gamma_n, \sigma_n$ were chosen, and proves the theorem.

### 11.3.2 Corollary

Suppose that $G$ is HSC in Theorem 11.3.1. Then $f$ extends to a meromorphic function on $G$.

### 11.3.3 Theorem

Let $D$ be a domain in $\mathbb{C}$ and let $w \in \mathbb{C} \setminus D$. Let $\gamma : [a, b] \to D$ be a closed piecewise smooth contour which is homotopic in $D$ to the constant path $\sigma(t) = \gamma(a)$. Then the winding number

$$n(\gamma, w) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - w} dz$$

is $0$. In particular, if $D$ is HSC then $D$ is simply connected in terms of winding number.

**Proof:** We can assume that $[a, b] = [0, 1]$ and $w = 0$. Thus $D \subseteq G = \mathbb{C} \setminus \{0\}$ and $\gamma$ is homotopic to a constant in $G$. We define a branch $L(z)$ of $\log z$ near $\gamma(0)$ and continue $L$ analytically along $\gamma$. Then $L_1 = L_0 = L$ on a neighbourhood of $\gamma(0)$, by Theorem 11.3.1. Since $L'(z) = \frac{1}{z}$ we have $L'_1(z) = \frac{1}{z}$ near $\gamma(t)$. For $s$ near $t$ we have $L_s = L_t$ near $\gamma(s)$, and so we have

$$\frac{d}{dt} L_t(\gamma(t)) = \lim_{s \to t} \frac{L_s(\gamma(s)) - L_t(\gamma(t))}{s - t} = \lim_{s \to t} \frac{L_t(\gamma(s)) - L_t(\gamma(t))}{s - t} = \frac{\gamma'(t)}{\gamma(t)}.$$

Thus

$$2\pi in(\gamma, 0) = L_1(\gamma(1)) - L_0(\gamma(0)) = 0.$$

### 11.3.4 Cycle reduction

Suppose that $\Gamma$ is a cycle made up of piecewise smooth contours $\gamma_1, \ldots, \gamma_n$, and that $L$ is a line segment described in one direction as part of $\gamma_j$, say from $A$ to $B$, and in the opposite direction as part of $\gamma_k$,
with \( j, k \) possibly equal. Then we may “cancel” \( L \) without changing any integral \( \int_{\Gamma} f(z)dz \), as follows. Assume for simplicity that \( \gamma_j, \gamma_k \) both start on \( L \). Follow \( \gamma_j \) from the first time it passes through \( B \) to the last time it comes to \( A \), and then follow \( \gamma_k \) from the first time it hits \( A \) to the last time it comes to \( B \). This gives a closed curve \( \lambda \) for which
\[
\int_{\lambda} f(z)dz = \int_{\gamma_j} f(z)dz + \int_{\gamma_k} f(z)dz
\]
for every continuous \( f \).

11.3.5 Lemma

Let \( A, B \) be disjoint non-empty compact subsets of \( \mathbb{C}^* \), with \( A \subseteq \mathbb{C} \). Let \( a \in A \). Then there exists a cycle \( \Gamma \) in \( \mathbb{C}^* \) such that \( \Gamma \cap (A \cup B) = \emptyset \) and \( n(\Gamma, a) = 1 \) but \( n(\Gamma, z) = 0 \) for all \( z \in B \).

Here we are using the convention that \( n(\gamma, \infty) = 0 \) for a cycle \( \gamma \) in \( \mathbb{C} \).

**Proof.** Since \( A \) and \( B \) are compact and disjoint the distance \( s \) from \( A \) to \( B \), measured in the chordal metric, is positive. Let \( r \) be small and positive, and cover the plane with a grid of closed square regions \( S_n \) of side length \( r \), pairwise disjoint except for common sides and vertices, and with \( a \) at the centre of one \( S_n \). Since \( r \) is small and \( A \) is bounded, no \( S_n \) can meet both \( A \) and \( B \).

Let \( T_n \) be the boundary curve of \( S_n \), described once counter-clockwise, and let \( \Gamma_0 \) be the cycle made up of those \( T_n \) for which \( S_n \cap A \neq \emptyset \). For these \( T_n \) it is clear that \( T_n \cap B = \emptyset \) and \( n(T_n, b) = 0 \) for all \( b \in B \), so that \( n(\Gamma_0, b) = 0 \).

Further, we have \( n(T_n, a) = 1 \) for precisely one \( n \), and so \( n(\Gamma_0, a) = 1 \).

Apply the cycle reduction process 11.3.4 repeatedly to obtain a cycle \( \Gamma \) made up of edges of \( \Gamma_0 \), in which no \( S_n \)-edge is described in both directions, and for which \( \int_{\Gamma} f(z)dz = \int_{\Gamma_0} f(z)dz \) for every continuous \( f \). We need only show that \( \Gamma \) does not meet \( A \). Suppose that \( w \in \Gamma \cap A \). If \( w \) is a vertex, then \( w \) lies on four squares \( S_n \), and there will be four edges in \( \Gamma_0 \), each described in both directions, and the cancellation of these shows that \( w \notin \Gamma \). Similarly, if \( w \) lies on a square edge, then \( w \) lies on two squares and again the edges are cancelled.

11.3.6 Theorem

Let \( D \) be a domain in \( \mathbb{C} \). The following are equivalent:

(i) \( D \) is simply connected in terms of winding number i.e. \( n(\gamma, w) = 0 \) for every cycle \( \gamma \) in \( D \) and every \( w \) not in \( D \).

(ii) \( D \) is homeomorphic to the disc \( D(0,1) \).

(iii) \( D \) is HSC.

(iv) the complement of \( D \) in \( \mathbb{C}^* \) is connected;

(v) \( \partial_\infty D \) is connected.
In particular the winding number condition (i) implies the intuitive condition of "no holes".

**Proof:** We show first that (i) implies (ii). Assume first that \( D \neq \mathbb{C} \). Then the Riemann mapping theorem tells us that \( D \) is homeomorphic to \( D(0,1) \). If \( D = \mathbb{C} \) then \( w = z/(1 + |z|) \) is a homeomorphism from \( \mathbb{C} \) to \( D(0,1) \) (although not analytic).

(ii) implies (iii). This follows from 11.1.9.

(iii) implies (i). This is by Theorem 11.3.3.

Next, (iv) implies (i). To see this, note that \( n(\gamma, w) \) is continuous off \( \gamma \) and is 0 for large \( w \). If we set \( n(\gamma, \infty) = 0 \) the resulting function is continuous on \( \mathbb{C}^* \setminus D \), and this set must be unbounded since it is connected and contains \( \infty \). So \( n(\gamma, w) \equiv 0 \) on \( \mathbb{C}^* \setminus D \), by the connectivity again.

(v) implies (iv). If \( \mathbb{C}^* \setminus D \) is not connected we form disjoint \( A, B \) as above and \( \partial_\infty D = \partial A \cup \partial B \) is disconnected.

(i) implies (v). We first prove this when \( D \) is unbounded. Assume that the compact set \( K = \partial_\infty D \) is not connected. Then we may partition \( K \) into disjoint non-empty compact subsets \( A, B \) of \( \mathbb{C}^* \) and, assuming without loss of generality that \( A \subseteq \mathbb{C}, \infty \in B \), we can find a cycle \( \Gamma \) not meeting \( K \) and such that \( n(\Gamma, a) = 1 \) for some \( a \in A \).

Let the closed piecewise smooth contours which together make up \( \Gamma \) be \( \Gamma_j \). Since

\[
n(\Gamma, a) = \sum_j n(\Gamma_j, a)
\]

there must be some \( j \) with \( n(\Gamma_j, a) = p > 0 \). We assert that \( \Gamma_j \subseteq D \). Assuming this not the case, we have \( \Gamma_j \cap D = \emptyset \), since \( \Gamma_j \) does not meet the boundary of \( D \). Since \( a \) is a boundary point of \( D \), we have \( n(\Gamma_j, c) = p \) for some \( c \in D \) and hence \( n(\Gamma_j, z) = p \) for all \( z \in D \). But \( D \) is unbounded and so there are \( z \) in \( D \) with \( n(\Gamma_j, z) = 0 \).

This contradiction proves the result when \( D \) is unbounded. If \( D \) is a bounded domain with disconnected boundary, we choose distinct \( a, b \) in \( \partial D \) and put \( G = \phi(D), \phi(z) = (z - a)/(z - b) \), noting that \( \phi \) is a homeomorphism of \( \mathbb{C}^* \). Thus \( G \) fails to satisfy (i), and so is not HSC, and nor is \( D \).
Chapter 12

Riemann surfaces and the uniformization theorem

12.0.1 Definitions

A surface $R$ is a non-empty connected Hausdorff space with a family of mappings $\phi_\alpha$ such that each $\phi_\alpha$ maps an open subset $U_\alpha$ of $R$ homeomorphically onto an open subset $V_\alpha$ of $\mathbb{C}$. The $U_\alpha$ cover $R$.

The $\phi_\alpha$ are called charts, the collection of charts is an atlas, and the $U_\alpha$ are parametric regions. If $x \in U_\alpha$ then, for sufficiently small $t$, the open set $\phi_\alpha^{-1}(D(\phi_\alpha(x), t))$ is a parametric (open) disc about $x$. A parametric closed disc is defined in the obvious analogous way i.e. as the pre-image under a chart of a closed disc of positive radius.

Suppose that $U_\alpha \cap U_\beta \neq \emptyset$. Then the function $\phi_\alpha \circ \phi^{-1}_\beta$ maps $\phi_\beta(U_\alpha \cap U_\beta)$ one-one onto $\phi_\alpha(U_\alpha \cap U_\beta)$. This map is called a transition map, and maps one open subset of $\mathbb{C}$ one-one onto another.

We say that $R$ is a Riemann surface if every such transition map is analytic (on $\phi_\beta(U_\alpha \cap U_\beta)$). In this case the $\phi_\alpha$ and $U_\alpha$ are said to define a conformal structure on $R$.

Obviously every open connected non-empty subset $D$ of $R$ is also a Riemann surface, with the obvious conformal structure (i.e. take $U_\alpha \cap D$ as parametric regions).

12.0.2 Theorem

Every surface is path-connected.

The proof is the same as the proof that connected open subsets of $\mathbb{C}$ are path-connected.

12.0.3 Analytic functions on Riemann surfaces

Let $R, S$ be Riemann surfaces, and let $f : R \to S$ be a continuous function. Suppose that $R$ has open sets and mappings $U_\alpha, \phi_\alpha$, and correspondingly $S$ has sets $W_\lambda$ and maps $\psi_\lambda$.

Let $x \in R$. Then $x$ lies in one of the open sets $U_\alpha$, and $f(x)$ lies in an open set $W_\lambda$. For $z$ in a neighbourhood of $x$, we have $f(z)$ in $W_\lambda$, by continuity. We look at the function $h = \psi_\lambda f \phi^{-1}_\alpha$ (we omit $\circ$ for convenience). This $h$ is defined near $\phi_\alpha(x)$ and maps a neighbourhood of $\phi_\alpha(x)$ into a neighbourhood of $\psi_\lambda(f(x))$, both of these sets contained in $\mathbb{C}$. Note that we have to assume in advance that $f$ is continuous in order to ensure that this composition is defined. We say that $f$ is analytic (in the
Riemann surface sense) if whenever we do this the function \( h \) we get is analytic at \( \phi_\alpha(x) \) in the usual sense that \( h'(z) \) exists on a neighbourhood of \( \phi_\alpha(x) \).

Note that for each \( x \), we only need check this for one open set \( U_\alpha \) with \( x \in U_\alpha \) and one open set \( W_\lambda \) with \( f(x) \in W_\lambda \). For, suppose that we also have \( x \in U_\beta \) and \( f(x) \in W_\mu \). Look at \( g = \psi_\mu f \phi_\beta^{-1} \).

Near to \( \phi_\beta(x) \), we have \( g = \psi_\mu f \phi_\beta^{-1} = \psi_\mu (\psi_\lambda^{-1} \psi_\lambda f \phi_\alpha^{-1} \phi_\alpha) \phi_\beta^{-1} = (\psi_\mu \psi_\lambda^{-1}) h(\phi_\alpha \phi_\beta^{-1}) \).

Now \( \phi_\alpha \phi_\beta^{-1} \) is analytic near \( \phi_\beta(x) \) (being a transition map). Also \( h(\phi_\alpha(x)) = \psi_\lambda(f(x)) \) and \( \psi_\mu \psi_\lambda^{-1} \) is analytic near this point (a transition map again). So \( g \) is analytic if \( h \) is.

Note that a constant function from \( R \) to \( S \) is always analytic. It is routine to check that the composition of analytic functions (in the Riemann surface sense) is analytic.

12.0.4 The identity theorem

Let \( f : R \to S \) be an analytic mapping between Riemann surfaces, and let \( b \in S \). Let \( E = \{ w \in R : f(w) = b \} \). If \( E \) has a limit point in \( R \) then \( E = R \).

**Proof.** Let \( F \) be the set of limit points of \( E \) in \( R \). Obviously \( R \setminus F \) is open. Let \( w \in F \), and choose charts \( \phi, \psi \) at \( w \) and \( b \) respectively. Then \( h = \psi f \phi^{-1} \) is analytic near \( \phi(w) \), and \( \phi(w) \) is a limit point of zeros of \( h(z) - b \). Looking at the Taylor series of \( h \) near \( \phi(w) \) we see that \( h(z) \equiv \psi(b) \) on a neighbourhood of \( \phi(w) \), and so \( F \) is open. The result now follows since \( R \) is connected.

12.1 Examples

12.1.1 Plane domains

Any plane domain \( D \) can be made into a Riemann surface by taking just one parametric region \( U_\alpha = D \), with \( \phi_\alpha \) the identity.

12.1.2 The Riemann sphere

The extended plane \( \mathbb{C} \cup \{ \infty \} = \mathbb{C}^* \) is made into a Riemann surface as follows. Set \( U_1 = \mathbb{C} \) and \( U_2 = \mathbb{C}^* \setminus \{ 0 \} \) and \( \phi_1(z) = z, \phi_2(z) = 1/z \). Both \( \phi_1 \phi_2^{-1} \) and \( \phi_2 \phi_1^{-1} \) are defined on \( \mathbb{C} \setminus \{ 0 \} = U_1 \cap U_2 \) and both are just \( z \to 1/z \), which is analytic there.

It follows easily that if \( D \) is a plane domain and \( f : D \to \mathbb{C}^* \) is meromorphic (i.e. analytic apart from isolated poles) then \( f \) is an analytic function from \( D \) into the Riemann surface \( \mathbb{C}^* \) (with the standard conformal structure above). The converse is also true, except that the function which is identically \( \infty \) is not normally regarded as meromorphic.

Since \( R(1/z) \) is a rational function when \( R(z) \) is, it’s also easy to see that rational functions are analytic functions from \( \mathbb{C}^* \) into itself.

12.1.3 Theorem

Let \( f : \mathbb{C}^* \to \mathbb{C}^* \) be analytic and non-constant. Then \( f \) is a rational function.
12.1. **Examples**

Proof. We can assume $f(\infty) \neq \infty$ (because $z \to 1/z$ is an analytic function on $\mathbb{C}^*$). So there is some $R > 0$ such that $f(z) \neq \infty$ for $|z| > R$. The set of $z$ in $\mathbb{C}$ with $f(z) = \infty$ has no limit point $w$ in $\mathbb{C}$ and so $f^{-1}(\{\infty\}$ is finite, since $\{z \in \mathbb{C} : |z| \leq R\}$ is a compact set.

If $f(a) = \infty$ with $a$ finite, then near $a$ we use Laurent’s theorem to write $f(z) = (z - a)^{-n}H(z) = S_a(z) + H_1(z)$, where $H$ and $H_1$ are analytic at $a$, and $S_a$ is a polynomial in $1/(z - a)$. Note that $S_a(z) \to 0$ as $z \to \infty$.

Now we just set $S(z) = \sum S_a$, in which the sum is over the finitely many $a$ for which $f(a) = \infty$. Then $f(z) - S(z)$ stays bounded as $z$ approaches each such $a$, and so $f(z) - S(z)$ is an entire function. Since $S(\infty) = 0$ and $f(\infty)$ is finite, $f - S$ is a bounded entire function and so constant.

12.1.4 Lifting a conformal structure to a covering space

Let $R$ be a Riemann surface and let $X$ be a path-connected Hausdorff topological space with a mapping $\psi : X \to R$ which is continuous and locally one-one, and maps open sets to open sets. Then $X$ is called a covering space of $\psi(X)$, and $X$ inherits a conformal structure from $R$ as follows.

Let the open sets and mappings of $R$ be $U_\alpha, \phi_\alpha$. Then we know that $\phi_\alpha^{-1}$ is analytic on $\phi_\beta(U_\alpha \cap U_\beta)$ whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Let $x$ be in $X$. Then $\psi(x)$ lies in some $U_\alpha$. Take a neighbourhood $V_\alpha$ of $x$ on which $\psi$ is one-one and such that $\psi(V_\alpha) = U_\alpha^*$ is contained in $U_\alpha$. The open sets for $X$ are just these $V_\alpha$, and these cover $X$. The mappings are just the compositions $\psi_\alpha = \phi_\alpha \psi$. If $V_\alpha \cap V_\beta$ is non-empty, then $\psi_\beta(V_\alpha \cap V_\beta) = \phi_\beta(U_\alpha \cap U_\beta) \subseteq \phi_\beta(U_\alpha \cap U_\beta)$. On $\psi_\beta(V_\alpha \cap V_\beta)$ we have $\psi_\alpha \psi_\beta^{-1} = \phi_\alpha \phi_\beta^{-1}$, which is analytic.

In particular any space homeomorphic to a Riemann surface inherits a conformal structure. Thus a sphere can be made into a Riemann surface, and so can the exterior of a cube.

12.1.5 Multiply-valued functions

The best known application of Riemann surfaces is as the “natural” domain of definition of certain multiply-valued functions.

The logarithm

The complex logarithm $\log z = \ln |z| + i \arg z$, with any choice of the argument, is analytic on the cut plane $D_0 = \{z = re^{i\theta}, r > 0, -\pi < \theta < \pi\}$, but is discontinuous as $z$ approaches the negative real axis.

To get around this difficulty, take countably many copies $G_n$ of $D_0$, and glue them together along the interval $(-\infty, 0)$ so that as we leave $G_n$ travelling counter-clockwise, we move up to $G_{n+1}$. On $G_n$, define $\log z = \ln |z| + i \arg z$ with the argument chosen to lie in $((2n-1)\pi, (2n+1)\pi)$. On the resulting ‘spiral’ surface $R$, the function $\log z$ so assembled is continuous, and maps $R$ onto $\mathbb{C}$. The charts on $R$ are just projection onto $\mathbb{C}$ in the obvious way.

A slightly more formal approach is to take the surface $S = \{(r \cos t, r \sin t, t) : r > 0, t \in \mathbb{R}\}$ with local charts $(r \cos t, r \sin t, t) \to (r \cos t, r \sin t)$, and ln $r + it$ maps $S$ onto $\mathbb{C}$.
12.1.6 The square root

The square root \( f_0(z) = z^{1/2} = \sqrt{r e^{i\theta}}/2 \), with \( z = re^{i\theta}, r > 0, -\pi < \theta < \pi \), is again analytic on \( D_0 \) but discontinuous at the negative real axis. Set \( f_1(z) = -f_0(z) \). For \( w \) on \((-\infty, 0)\) we have

\[
\lim_{z \uparrow w} f_j(z) = \lim_{z \downarrow w} f_{1-j}(z),
\]

Take two copies \( G_0, G_1 \) of the Riemann sphere, slit along the open interval \((-\infty, 0)\), and glue them together so that as we leave \( G_j \) across \((-\infty, 0)\) we enter \( G_{1-j} \). Define \( f \) to be \( f_j \) on \( G_j \), and extend it continuously to the edges. The resulting surface is homeomorphic under \( f \) to \( \mathbb{C}^* \).

12.1.7 Example

The solutions of algebraic equations may be defined as single valued functions on suitable Riemann surfaces: see Ahlfors’ Complex Analysis [2] for details. We describe here just one example, which leads to a surface which is not simply connected. Define \( w \) by

\[
w^2 = z(z-1)(z-2).
\]

A basic fact from complex analysis states that if \( F \) is analytic and zero-free on a simply connected domain \( G \), then \( F \) has an analytic square root on \( G \). Hence we may form analytic solutions \( w = f_0(z), w = f_1(z) = -f_0(z) \) on the domain formed by cutting the plane along the interval \([0, \infty)\). Let

\[
A = (0, 1), \quad B = (1, 2), \quad C = (2, \infty).
\]

Then we have

\[
\lim_{z \uparrow w} f_j(z) = \lim_{z \downarrow w} f_{1-j}(z), \quad w \in A.
\]

On the other hand, writing

\[
w = z(1 - 1/z)^{1/2}(z - 2)^{1/2}
\]

we see that the \( f_j \) extend analytically to \( 1 < |z| < 2 \).

Take two copies \( G_0, G_1 \) of the Riemann sphere, each slit along the open intervals \( A, C \), and again join them across the cuts. Let \( f(z) \) be \( f_j(z) \) on the interior of \( G_j \), and extend \( f \) continuously to the resulting surface (which is topologically a torus). Apart from at \( 0, 1, 2, \infty \), the local charts are just projection onto \( \mathbb{C} \), but at the four “branch points” we need to be more careful, since projection is not locally one-one there. However, on a neighbourhood of \( 0 \) we may use \( z^{1/2} \), as in the previous example, and we do the same at \( 1, 2, \infty \).

12.1.8 The uniformization theorem

The majority of this chapter is devoted to presenting a proof of this important result, which states that if \( R \) is a simply connected Riemann surface then \( R \) is conformally equivalent to precisely one of the following (in each case with the standard conformal structure): the open plane \( \mathbb{C} \); the extended plane \( \mathbb{C}^* \); the unit disc \( D(0, 1) \).

It follows from Liouville’s theorem and compactness that no two of \( \mathbb{C}, \mathbb{C}^*, D(0, 1) \) can be conformally equivalent. The fact that \( R \) is conformally equivalent to one of these requires the theory of subharmonic functions on Riemann surfaces. The proof presented here is modified from one given by W. Abikoff [1] in the American Math. Monthly, October 1981.
12.2 Subharmonic functions and Perron families

12.2.1 Lemma

Let $u$ be subharmonic on a plane domain $D$, and let $f : D \to \mathbb{C}$ be conformal. Then $u \circ f^{-1}$ is subharmonic on $f(D)$.

This follows immediately from Theorem 8.2.4. With this result we can define subharmonic functions on Riemann surfaces as follows.

12.2.2 Definition

Let $R$ be a Riemann surface, and let $u : R \to [-\infty, \infty)$ be continuous. We say (initially) that $u$ is subharmonic on $R$ if $u \circ \phi^{-1}$ is subharmonic on $\phi^{-1}(U)$ for every chart $\phi$.

12.2.3 Lemma

Let $u$ be subharmonic on the Riemann surface $R$, let $N$ be a closed parametric disc (with local chart $\phi$), and suppose that $u \equiv -\infty$ on a subset $E$ of $\partial N$ such that $\phi(E)$ has positive angular measure. Then $u \equiv -\infty$ on $R$.

Proof. By the theory of subharmonic functions in the plane (Poisson’s formula) we get $u \equiv -\infty$ on $N$. Let $F$ be the (obviously open) subset of $R$ defined by the property that $w \in F$ iff $u \equiv -\infty$ on a neighbourhood of $w$. We claim that $F$ is closed, and this holds since if $w_n \in F$ tend to $w \in R$ then $u \equiv -\infty$ on a parametric disc centred at $w$. So $F = R$ by connectedness.

Henceforth we consider only subharmonic functions which are not $\equiv -\infty$. We recall Harnack’s theorem from 7.3.10.

12.2.4 Harnack’s theorem

Let $D$ be a domain in $\mathbb{C}$ and let $u_n$ be functions harmonic on $D$ such that $u_1 \leq u_2 \leq \ldots$. Let $v(z) = \lim_{n \to \infty} u_n(z)$ for each $z \in D$. Then either $v \equiv \infty$ on $D$, or $v$ is harmonic on $D$.

12.2.5 Definitions

Let $u$ be subharmonic on $R$ and let $D$ be a closed parametric disc in $R$. Then we can form a subharmonic function $u_D$ which satisfies $u \leq u_D$ on $R$, is harmonic on the interior of $D$ (using Lemma 12.2.3), and equals $u$ off the interior of $D$. We call $u_D$ the Poisson modification of $u$.

By a Perron family $P$ we mean a non-empty collection of functions $u$ subharmonic on $R$, such that:

(i) if $u \in P$ then $u_D \in P$ for every closed parametric disc $D$ in $R$;

(ii) if $u, v \in P$ then $\max\{u, v\}$ is in $P$.

12.2.6 Theorem

Let $P$ be a Perron family, and for each $p \in R$ define $g(p) = \sup\{u(p) : u \in P\}$. Then either $g \equiv +\infty$ on $R$, or $g$ is harmonic on $R$. 
Proof. Take \( p_0 \in R \), and a closed parametric disc \( D_0 \) centred at \( p_0 \).

Suppose first that \( g(p_2) = \infty \) for some \( p_2 \) in the interior \( U_0 \) of \( D_0 \). Then there exist \( v_n \in P \) such that \( v_n(p_2) \to \infty \), and we may assume that \( v_n \leq v_{n+1} \) and each \( v_n \) is harmonic on \( U_0 \) (if not, first take maximums so that \( v_n \leq v_{n+1} \) and then replace each \( v_n \) by its Poisson modification). Then Harnack’s theorem tells us that \( v_n \to \infty \) on \( U_0 \) and so \( g \equiv +\infty \) on \( U_0 \).

Suppose now that \( g(p) < \infty \) for every \( p \in U_0 \). Choose \( p_1 \in U_0, p_1 \neq p_0 \). We can take \( u_n \in P, v_n \in P \) such that \( u_n(p_0) \to g(p_0), v_n(p_1) \to g(p_1) \), and we may assume that \( u_n \) and \( v_n \) satisfy \( u_n \leq u_{n+1}, v_n \leq v_{n+1} \) and are harmonic on \( U_0 \). We may also take \( w_n \in P \), harmonic on \( U_0 \), such that \( w_n \geq \max\{u_n, v_n\} \) on \( R \).

Harnack’s theorem gives us \( u_n \to h, w_n \to k \), with \( h, k \) harmonic on \( U_0 \). Since \( g(p_0) \geq w_n(p_0) \geq u_n(p_0) \to g(p_0) \), we get \( h(p_0) = k(p_0) = g(p_0) \). On the other hand, since \( u_n \leq w_n \) we get \( h \leq k \) on \( U_0 \). The maximum principle now tells us that \( h = k \) on \( U_0 \).

It follows in particular that \( h(p_1) = k(p_1) \). Since \( g(p_1) \geq w_n(p_1) \geq v_n(p_1) \to g(p_1) \) we get \( g(p_1) = k(p_1) = h(p_1) \). Hence \( g = h \) on \( U_0 \), and \( g \) is harmonic on \( U_0 \).

The result now follows by connectedness.

12.3 Green’s function

12.3.1 Definition

Let \( R \) be a Riemann surface, and let \( p_0 \in R \). Let \( \phi \) be a chart near \( p_0 \). Consider the Perron family \( V_{p_0} \) of all functions \( v \) which are subharmonic on \( R \setminus \{p_0\} \), and with the following properties:

(i) there exists a compact \( K_v \) such that \( v = 0 \) off \( K_v \);

(ii) \( \limsup_{p \to p_0} v(p) + \log |\phi(p) - \phi(p_0)| < \infty \).

Condition (ii) is independent of the particular chart \( \phi \), because if \( \psi \) is another chart then \( h = \psi \circ \phi^{-1} \) is locally conformal and

\[
|\psi(p) - \psi(p_0)| = |h(\phi(p)) - h(\phi(p_0))| \leq c|\phi(p) - \phi(p_0)|
\]

as \( p \to p_0 \), for some constant \( c \). Obviously \( 0 \in V_{p_0} \). Set

\[
g(p, p_0) = \sup\{v(p) : v \in V_{p_0}\} \geq 0.
\]

Then either \( g \equiv \infty \), or \( g \) is harmonic on \( R \setminus \{p_0\} \). In the second case, we call \( g \) the Green’s function for \( R, p_0 \).

12.3.2 Lemma

Let \( \phi \) be a chart at \( p_0 \), mapping \( p_0 \) to \( z_0 \) and a neighbourhood of \( p_0 \) onto \( D(z_0, r) \). Let \( 0 < r_1 < r_2 \), and define closed parametric discs \( K_1, K_2 \) by

\[
K_j = \phi^{-1}(\{z : |z| \leq r_j\}).
\]
Let $v \geq 0, v \in V_{p_0}$, and let

$$m_j = \max \{ v(p) : p \in \partial K_j \}. \quad (12.2)$$

Then

$$m_1 + \log r_1 \leq m_2 + \log r_2. \quad (12.3)$$

If $g(p, p_0)$ exists then we also have

$$\max \{ g(p, p_0) : p \in \partial K_1 \} + \log r_1 \leq \max \{ g(p, p_0) : p \in \partial K_2 \} + \log r_2. \quad (12.4)$$

**Proof.** We may assume that $z_0 = 0$. Let $\varepsilon > 0$. The function

$$h(p) = v(p) + (1 + \varepsilon) \log |\phi(p)|$$

tends to $-\infty$ as $z \to p_0$, and is subharmonic on a neighbourhood of $K_2$. Hence

$$h(p) \leq \max \{ h(q) : q \in \partial K_2 \}, \quad p \in K_2.$$ 

In particular

$$m_1 + (1 + \varepsilon) \log r_1 \leq m_2 + (1 + \varepsilon) \log r_2$$
and (12.3) follows on letting $\varepsilon \to 0$. We now get, for $w \in V_{p_0}, w \geq 0$,

$$\max \{ w(p) : p \in \partial K_1 \} + \log r_1 \leq \max \{ g(p, p_0) : p \in \partial K_2 \} + \log r_2$$
and the last assertion of the lemma follows.

**12.3.3 Lemma**

If $g(p, p_0)$ exists then $g(p, p_0) + \log |\phi(p) - \phi(p_0)|$ has a harmonic extension to a neighbourhood of $p_0$.

**Proof.** Assume that $\phi(p_0) = 0$. By (12.4), $g(p, p_0) + \log |\phi(p)|$ is bounded above as $p \to p_0$. Now take a small positive $r_0$ and define $v$ by

$$v(p) = \log r_0 - \log |\phi(p)|, \quad |\phi(p)| < r_0,$$

with $v(p) = 0$ otherwise. Then $v$ is subharmonic on $R \setminus p_0$ and is in $V_{p_0}$, and so

$$g(p, p_0) \geq v(p) \geq -\log |\phi(p)| - O(1), \quad p \to p_0.$$

**12.3.4 Lemma**

If $g(p, p_0)$ exists then $g$ is non-constant and positive and $c = \inf \{ g(p, p_0) : p \in R \}$ satisfies $c = 0$.

**Proof.** Lemma 12.3.3 shows that $g$ is non-constant, and thus $g$ is positive, by the maximum principle on $R \setminus \{ p_0 \}$. Clearly $c \geq 0$. Take $\varepsilon > 0$ and $v \in V_{p_0}$, and set

$$k(p) = (1 - \varepsilon)v(p) - g(p, p_0) + c.$$ 

Outside a compact set we have $k(p) \leq 0$. Also, as $p \to p_0$ we have

$$v(p) \leq g(p, p_0), \quad g(p, p_0) \to \infty, \quad k(p) \to -\infty.$$ 

Hence $k(p) \leq 0$ on $R \setminus \{ p_0 \}$. Letting $\varepsilon \to 0$ we get

$$v(p) \leq g(p, p_0) - c$$
and taking the supremum over $v$ gives $c = 0$. 

12.3.5 Existence of the Green’s function

If $R$ is compact, then Green’s function cannot exist, because $g$ would have a minimum on $R$, contradicting the fact that $g$ would be harmonic and non-constant on $R \setminus \{p_0\}$.

We consider next necessary conditions, and sufficient conditions, for Green’s function to exist on a non-compact surface.

12.3.6 Definition

Let $K$ be a compact subset of $R$. We say that the maximum principle fails for $K$ if there exists a function $h$, subharmonic and bounded above on $R \setminus K$, with $\limsup_{p \to K} h(p) \leq 0$ and $h(p) > 0$ for some $p \in R \setminus K$.

Here $\limsup_{p \to K}$ means $\limsup_{p \to \partial K, p \in R \setminus K}$.

12.3.7 Examples

The function $\log |z|$ shows that the maximum principle fails for $\{z : |z| \leq 1\}$, with respect to the surface $D(0,2)$.

On the other hand, the maximum principle holds for $\{z : |z| \leq 1\}$, with respect to the surfaces $\mathbb{C}^*$ and $\mathbb{C}$, in the latter case because the singularity at $\infty$ is removable for subharmonic functions which are bounded above.

Note that Green’s function $g(p,0)$ does not exist for $\mathbb{C}$, since $\log^+ R/|z|$ is in $V_0$ for every $R > 0$.

12.3.8 Lemma

Suppose that $g(p,p_0)$ exists, and that $K$ is a compact subset of $R$, properly containing $\{p_0\}$. Then the maximum principle fails for $K$.

Proof. The function $h(p) = -g(p,p_0) \leq 0$ has a maximum $m$ on $K$, and this is taken at some $p_1 \in K, p_1 \neq p_0$. If we had $h(p) \leq m$ on $R \setminus K$ then $h$ would have a maximum on $R \setminus \{p_0\}$, which violates the ordinary maximum principle.

In the converse direction, we have:

12.3.9 Theorem

Let $K$ be a compact subset of $R$, and let $p_0$ be an interior point of $K$. Suppose that the maximum principle fails for $K$. Then $g(p,p_0)$ exists.

Proof. Choose a neighbourhood of $p_0$, contained in $K$, mapped onto $D(z_0,r)$ by a chart $\phi$. We may assume that $z_0 = 0$. Let $0 < r_1 < r_2 < r$. Let $K_1$ be as in (12.1). Let $V$ be the family of subharmonic functions $v : R \setminus K_1 \to [0,1]$, such that if $\varepsilon > 0$ then $v(w) < \varepsilon$ for all $w$ outside some compact $L_{v,\varepsilon}$ (thus $v(w) \to 0$ as $w$ “tends to infinity” in $R \setminus K_1$). Set $u(p) = \sup\{v(p) : v \in V\}$. Then $u$ is harmonic on $R \setminus K_1$.

We claim that $u(p) < 1$ for all $p \in R \setminus K_1$. We know that there exists a function $h$, subharmonic
and bounded above by 1 on $R \setminus K$, with $\limsup_{p \to K} h(p) \leq 0$ and $h(p_1) > 0$ for some $p_1 \in R \setminus K$. Let $v \in V$. Then $v + h$ is subharmonic on $R \setminus K$, and
\[
\limsup_{p \to K} (v(p) + h(p)) \leq 1, \quad \limsup_{p \to \infty} (v(p) + h(p)) \leq 1.
\]
The ordinary maximum principle now gives $v(p_1) \leq 1 - h(p_1) < 1$, which gives $u(p_1) < 1$. Since we now have $u \not\equiv 1$, applying the maximum principle proves the claim.

Now take any $v \in V_{p_0}$. We may assume that $v \geq 0$, since otherwise we replace $v$ by $\max\{v, 0\}$. Define $m_j$ by (12.2). Since $v$ vanishes off a compact set we have $v(p) \leq m_1$ on $R \setminus K_1$. It follows that $v/m_1 \in V$. Hence we get
\[
v(p) \leq m_1 u(p), \quad p \in R \setminus K_1.
\]
In particular we have
\[
m_2 \leq m_1 M_2, \quad M_2 = \max\{u(p) : p \in \partial K_2\} < 1.
\]
Combining this with (12.3) leads to
\[
m_1 \leq m_2 + \log r_2/r_1 \leq m_1 M_2 + \log r_2/r_1
\]
and so
\[
m_1 = \max\{v(p) : p \in \partial K_1\} \leq (1 - M_2)^{-1} \log r_2/r_1.
\]
Since $v$ is an arbitrary non-negative element of $V_{p_0}$, while $u$ is fixed, we deduce that $g(p, p_0)$ is finite for $p \in \partial K_1$, and so the Green’s function exists.

**12.3.10 Corollary**

If $g(p, p_0)$ exists for some $p_0$ in $R$ then $g(p, p_1)$ exists for every $p_1$ in $R$.

To see this, take a compact set, the interior of which contains $p_0, p_1$.

A non-compact Riemann surface $R$ is called **hyperbolic** if Green’s function exists, and **parabolic** otherwise.

**12.3.11 Lemma**

Let $R$ be a non-compact Riemann surface such that there exists a function $u$ non-constant, subharmonic and bounded above on $R$. Then $R$ is hyperbolic.

The converse is also true, because we can take $-g(p, p_0)$.

**Proof.** Take $p_0 \in R$. Since $u$ is non-constant, there exists $p_1 \in R$ with $u(p_1) > u(p_0)$. Take a compact subset $K$ of $R$, with $p_0$ an interior point of $K$, and with $p_1 \not\in K$, and such that $u(p) < u(p_1)$ on $K$. Thus the maximum principle fails for $K$.

**12.3.12 Lemma**

Suppose that $S$ is a domain on $R$, and that the boundary of $S$, relative to $R$, contains a path $\gamma$ joining distinct points of $R$. Then $S$ has a Green’s function.
Proof. Take a parametric disc \( U \), the closure of which is contained in \( R \), such that \( U \) contains a subpath \( \sigma \) of \( \gamma \) joining distinct points of \( U \). Solving the Dirichlet problem on the image of \( U \) in \( \mathbb{C} \) we get a function \( u \) harmonic on \( U \setminus \sigma \), with

\[
\lim_{p \to \sigma} u(p) = 1, \quad \lim_{p \to \partial U} u(p) = 0.
\]

Extend \( u \) to be 0 on \( S \setminus U \). Then \( u \) is non-constant and subharmonic, but bounded above, on \( S \).

12.4 The uniformization theorem: the hyperbolic case

12.4.1 Theorem

Let \( R \) be a hyperbolic Riemann surface (and so open). Then \( R \) is conformally equivalent to \( D(0,1) \) (with the standard conformal structure).

Proof. Take some \( p_0 \) on \( R \) and form Green’s function \( g(p, p_0) \). Let \( \phi \) be a chart near \( p_0 \), without loss of generality mapping \( p_0 \) to 0. Now \( g(p, p_0) + \log |\phi(p)| \) has a harmonic extension \(-u(p)\) to a neighbourhood of \( p_0 \), on which we define a harmonic conjugate \( v(p) \) of \( u \), and \( f \) by

\[
f(p) = \phi(p) \exp(u(p) + iv(p)), \quad \log |f| = u + \log |\phi| = -g.
\]

Since \(-g\) has a harmonic conjugate in a neighbourhood of each point of \( R \setminus \{p_0\} \), we may analytically continue \( f \) subject to \( \log |f| = -g \) throughout \( R \), and by the monodromy theorem this defines an analytic function \( f : R \to D(0,1) \).

It suffices to show that \( f \) is univalent, because \( f(R) \) will then be a simply connected subdomain of \( D(0,1) \) and so conformally equivalent to \( D(0,1) \). Let \( p_1 \in R \setminus \{p_0\} \), let \( a = f(p_1) \), and let

\[
T(w) = \frac{w - a}{1 - \overline{a}w},
\]

so that \( T(a) = 0 \). Note that \( T(0) = -a \).

Let \( \varepsilon > 0 \) and let \( v_1 \in V_{p_1} \). Let

\[
h(p) = v_1(p) + (1 + \varepsilon) \log |T(f(p))|.
\]

Outside a compact set we have \( v_1 = 0 \) and so \( h < 0 \). Next, let \( \psi \) be a chart at \( p_1 \), without loss of generality mapping \( p_1 \) to 0. Then \( Tf\psi^{-1} \) is analytic at 0 and as \( p \to p_1 \) we have

\[
\log |T(f(p))| \leq \log |\phi(p)| + O(1)
\]

and hence \( h(p) \to -\infty \). Thus \( h \) is subharmonic and negative throughout \( R \). Letting \( \varepsilon \to 0 \) we get

\[
v_1(p) \leq -\log |T(f(p))|, \quad g(p, p_1) \leq -\log |T(f(p))|.
\]

But

\[
|T(f(p_0))| = |T(0)| = |a| = |f(p_1)|
\]

and so we get

\[
g(p_0, p_1) \leq -\log |f(p_1)| = g(p_1, p_0).
\]
By symmetry, we get
\[ g(p_1, p_0) = g(p_0, p_1), \]
and so
\[ g(p_0, p_1) = -\log |f(p_1)| = -\log |T(f(p_0))|. \]
Since \( g(p, p_1) + \log |T(f(p))| \) is subharmonic and, by (12.5), non-positive on \( R \setminus \{p_1\} \), we deduce that
\[ g(p, p_1) = -\log |T(f(p))|, \quad p \in R, \]
in which both sides are infinite at \( p_1 \).

Now suppose that \( f(p_2) = f(p_1) \). Then \( T(f(p_2)) = 0 \), and so \( g(p_2, p_1) = \infty \). But this gives \( p_2 = p_1 \), and \( f \) is univalent as required.

12.5 The non-hyperbolic case

12.5.1 Divergent curves

Let \( R \) be a non-compact Riemann surface. A divergent curve on \( R \) is a simple path \( \gamma : [0, \infty) \to R \) such that
\[ \lim_{t \to \infty} \gamma(t) = \infty, \]
by which we mean that if \( K \) is a compact subset of \( R \) then there exists \( t_0 \geq 0 \) such that \( \gamma(t) \not\in K \) for all \( t \geq t_0 \).

Since deleting a point from \( R \) gives a set which is still connected, \( \gamma \) cannot pass through every point of \( R \).

Obviously if \( R \) is compact and \( p_0 \in R \) then \( R^0 = R \setminus \{p_0\} \) is not compact (take open sets \( U_n = R \setminus K_n \), in which \( K_n \) are compact neighbourhoods of \( p_0 \) decreasing to \( \{p_0\} \)), and \( R^0 \) has a divergent curve (since \( R \) is Hausdorff any compact \( K \subseteq R^0 \) fails to meet some open neighbourhood of \( p_0 \)).

12.5.2 Lemma

Let \( \gamma \) be a divergent curve on the simply connected non-compact Riemann surface \( R \). For \( t \geq 0 \) set \( R_t = R \setminus \gamma([t, \infty)) \). Then \( R_t \) is an open set. Further, if \( U \) is an open parametric disc centred at \( \gamma(s) \) and \( s_1, s_2 \) are sufficiently close to \( s \) then there is a homeomorphism \( f \) of \( R_{s_2} \) onto \( R_{s_1} \) which is the identity outside \( U \).

**Proof.** Assume without loss of generality that \( s_1 \leq s_2 \). Let \( s_3 = \inf \{t \geq s : \gamma(t) \not\in U\} \). Then \( V = U \setminus \gamma([s_1, s_3]) \) is a domain (by the Jordan curve theorem, using the fact that a disc is homeomorphic to \( \mathbb{C} \)) and is simply connected (via the homeomorphism to \( \mathbb{C} \) and a winding number argument). The Riemann mapping theorem gives a homeomorphism \( g \) of \( V \) onto \( W = D(0, 1) \setminus [\frac{1}{2}, 1] \), and we then take a homeomorphism of \( W \) which is the identity outside \( D(0, \frac{3}{4}) \).

12.5.3 Lemma

\( R_t \) is a simply connected domain, and a hyperbolic Riemann surface.

**Proof.** We prove first that \( R_t \) is connected. To see this, assume without loss of generality that \( t = 0 \). Let \( a, b \in R_0 \) and let \( s = \inf A \), where \( A \) is the set of \( t > 0 \) such that there exists a path from \( a \) to \( b \) in \( R_t \). Then \( s < \infty \) since \( R \) is connected.
Take an open parametric disc $U$ centred at $\gamma(s)$, and not containing $a, b$. Take $s_1, s_2$ close to $s$, with $s_1 \leq s < s_2$ and $s_2 \in A$. By assumption, there exists a path $\rho$ from $a$ to $b$ in $R_{s_2}$. Let $f$ be as in Lemma 12.5.2. Then $f(a) = a, f(b) = b$, and $f(\rho)$ is a path from $a$ to $b$ in $R_{s_1}$. It follows that $s \in A$ and $s = 0$.

We now show that $R_t$ is simply connected. Assume again that $t = 0$, and let $\Gamma$ be a closed curve in $R_0$. Since $R$ is simply connected, $\Gamma$ is homotopic in $R$ to a constant curve, via a homotopy function $F$. Let $s = \inf B$, with $B$ the set of $t > 0$ such that $\Gamma$ is homotopic to a constant in $R_t$. Then $s < \infty$, because for large $s$ we can take $F$.

Take an open parametric disc $U$, centred at $\gamma(s)$, such that $U$ does not meet $\Gamma$. Take $s_1 \leq s < s_2 < \infty$, with $|s_j - s|$ small, such that $s_2 \in B$. Thus $\Gamma$ is homotopic to a constant in $R_{s_2}$, via a homotopy function $F_2$. Take $f$ as in Lemma 12.5.2 again, so that $f(\Gamma(t)) = \Gamma(t)$. It follows that $f \circ F_2$ is a homotopy in $R_{s_1}$, deforming $\Gamma$ to a constant path.

Since the boundary of $R_t$ in $R$ contains a simple path, each $R_t$ is hyperbolic.

12.5.4 Theorem

Let $R$ be a simply connected Riemann surface, with a divergent curve, and having no Green’s function. Then $R$ is conformally equivalent to $\mathbb{C}$.

Note that $\mathbb{C}$ is homeomorphic to $D(0, 1)$, and thereby inherits a conformal structure with a Green’s function. So the hypothesis that $R$ has no Green’s function is not redundant.

Proof. Fix $p_0 \in R_0$, and a chart $\phi$ at $p_0$. We may assume that $\phi(p_0) = 0$. Each $R_n$ is hyperbolic, and so there is a conformal map $G_n : R_n \to D(0, 1)$ with the standard conformal structure on $D(0, 1)$, and with $G_n(p_0) = 0$. Let $g_n = G_n \circ \phi^{-1}$, and let $g_n'(0) = 1/c_n$. Let $F_n = c_n G_n$. Then $F_n$ maps $R_n$ conformally onto $B_n = D(0, |c_n|)$, and $f_n = F_n \circ \phi^{-1}$ has $f_n'(0) = 1$.

For $n \geq m$, we have $R_m \subseteq R_n$, and so $F_n \circ F_m^{-1}$ maps $B_m$ conformally into $B_n$, with $0$ mapped to $0$. Since $F_n \circ F_m^{-1} = f_n \circ f_m^{-1}$ near $0$, we see that the derivative of $F_n \circ F_m^{-1}$ at $0$ is $1$.

Now the family of functions $f$ analytic and univalent on a fixed disc $D(0, r)$, with the normalization $f(0) = f'(0) - 1 = 0$, is a normal family, by Koebe’s distortion theorem. The limit function of any convergent sequence in this family is analytic and, by Hurwitz’ theorem, univalent.

We apply the diagonalization process. Take a subsequence $F_{1n}$ of $F_n$ such that as $n \to \infty$ the sequence $F_{1n} \circ F_{1n}^{-1}$ converges LU on $B_1$ to a function $H_1$ analytic and univalent there. Take a subsequence $F_{2n}$ of $F_{1n}$ such that, as $n \to \infty$, the sequence $F_{2n} \circ F_{2n}^{-1}$ converges LU on $B_2$, to $H_2$. We repeat this. Note that $H_k \circ F_k$ is a conformal map of $R_k$ into $\mathbb{C}$.

Now let $P_n = F_{nn}$. For each $k$, this sequence $P_n$ is eventually a subsequence of $F_{kn}$ and so $P_n \circ F_k^{-1}$ converges LU on $B_k$ to $H_k$.

Let $k \leq m$. Then $R_k \subseteq R_m$ and we have, on $R_k$,

$$H_k \circ F_k = \lim(P_n \circ F_k^{-1}) \circ F_k = \lim(P_n \circ F_k^{-1} \circ F_m^{-1} \circ F_n \circ F_k^{-1}) \circ F_k = \lim(P_n \circ F_k^{-1}) \circ F_m = H_m \circ F_m.$$ 

Since the union of the $R_k$ is $R$, this defines a conformal map $H$ of $R$ onto a simply connected domain $D$ in $\mathbb{C}$. If $D \neq \mathbb{C}$ then there exists a conformal map $\psi$ of $D$ onto $D(0, 1)$, so that $\psi \circ H$ maps $p_0$ to $0$. 

12.6. THE CASE OF NO DIVERGENT CURVES

But then the function \( \log |\psi \circ H| \) is subharmonic, non-constant and bounded on \( R \), contradicting the assumption that \( R \) is not hyperbolic.

To handle the remaining cases of the uniformization theorem, we need the following lemma, the proof of which we postpone till the next section.

12.5.5 Lemma

Let \( R \) be a simply connected Riemann surface and let \( p_0 \in R \). Let \( R^0 = R \setminus \{p_0\} \). If \( R \) has no divergent curves, then \( R^0 \) is simply connected.

In particular \( R^0 \) is simply connected if \( R \) is simply connected and compact.

12.5.6 Theorem

Let \( R \) be a simply connected Riemann surface with no divergent curves. Then \( R \) is conformally equivalent to \( \mathbb{C}^* \) (and so compact).

Proof. Take \( p_0 \in R \) and form the punctured surface \( R^0 \), which by Lemma 12.5.5 is simply connected. Obviously \( R^0 \) has a divergent curve.

If \( R^0 \) is parabolic then \( R^0 \) is conformally equivalent to \( \mathbb{C} \), via some conformal map \( f \), and a simple argument shows that \( f(p) \to \infty \) as \( p \to p_0 \). Thus \( R \) is conformally equivalent to \( \mathbb{C}^* \).

It remains only to show that \( R^0 \) cannot be hyperbolic. Assuming that \( R^0 \) has a Green’s function, we obtain a conformal mapping \( f \) of \( R^0 \) onto \( D(0,1) \). The singularity at \( p_0 \) is removable, and the maximum principle gives \( f(p_0) \in D(0,1) \) and so \( f(p_0) = f(p_1) \) for some \( p_1 \in R^0 \). But then, by the open mapping theorem, all values near \( f(p_1) \) are taken by \( f \) near \( p_0 \) and near \( p_1 \), contradicting the univalence of \( f \) on \( R^0 \).

12.6 The case of no divergent curves

In this section we prove Lemma 12.5.5. We puncture \( R \) to form \( R^0 = R \setminus \{p_0\} \). Let \( N = N_0 \) be a closed parametric disc centred at \( p_0 \), and let \( R^1 = R \setminus N_0 \). Since an annulus is homeomorphic to a punctured disc, \( R^1 \) is homeomorphic to \( R^0 \), and it will therefore suffice to prove that \( R^1 \) is simply connected. Fix \( p_1 \in R^1 \), and form the Green’s function \( g(p,p_1) \). This exists, by Lemma 12.3.12.

12.6.1 Lemma

We have \( g(p,p_1) \to 0 \) as \( p \to \partial N \) from \( R^1 \).

Proof. Take a closed parametric disc \( N_1 \) centred at \( p_0 \), such that \( N_0 \) lies in the interior of \( N_1 \). Solve the Dirichlet problem with boundary values \( g(p) \) on \( \partial N_1 \) and 0 on \( \partial N_0 \). Note that to do this we only need the Dirichlet problem for a plane annulus. Let the resulting function be \( h \). If \( u \) is a function in the Perron family defining \( g \), then \( \lim \sup_{p \to \partial N_1} (u(p) - h(p)) \leq 0 \). Thus \( u \leq h \) on \( N_1 \setminus N_0 \) by the maximum principle, giving \( g \leq h \) there. Since \( h(p) \to 0 \) as \( p \to \partial N_0 \) we get \( \lim \sup_{p \to \partial N_0} g(p) \leq 0 \). Since \( g \geq 0 \), the result follows.
12.6.2 Lemma

Let \( \infty > T > 0 \) and let \( Y \) be a component of the set \( \{ p \in R^1 : g(p) > T \} \). Then \( g \) is not bounded above on \( Y \) and there exists a path \( \gamma : [0, \infty) \to Y \) such that \( g(\gamma(t)) \to \infty \) as \( t \to \infty \).

Proof. Suppose first that \( g \) is bounded above on \( Y \). Obviously \( g(p) = T \) on \( \partial Y \), which does not meet \( N \). Define \( v(p) = g(p) \) for \( p \in Y \), with \( v(p) = T \) for \( p \in R \setminus Y \). Then \( v \) is subharmonic and bounded above on \( R \), contradicting Lemma 12.3.11 and the assumption that \( R \) is not hyperbolic.

The construction of the path is now standard. Fix \( y_0 \in Y \). Let \( Y_0 = Y \) and, assuming that \( y_n, Y_n \) have been defined, let \( Y_{n+1} \) be a component of the set \( \{ p : g(p) > T + n + 1 \} \) such that \( Y_{n+1} \subseteq Y_n \), and choose \( y_{n+1} \in Y_{n+1} \). Join \( y_n \) to \( y_{n+1} \) by a path in \( Y_n \).

Since \( R \) has no divergent curves, \( \gamma(t) \) must tend to \( p_1 \) and we get immediately:

12.6.3 Lemma

Let \( \infty > T \geq 0 \). Then the set \( \{ p \in R^1 : g(p) > T \} \) has a unique component.

12.6.4 Lemma

Let \( \infty > T > 0 \), and suppose that \( p_3 \in R^1 \) with \( g(p_3) \leq T \). Then \( p_3 \) can be joined to \( \partial N \) by a path \( \sigma(t), 0 \leq t \leq 1 \), such that \( g(\sigma(t)) < T \) for \( 0 < t < 1 \).

Proof. Assume that \( g(p_3) < T \) (if not, first join \( p_3 \) to \( p_4 \) with \( g(p_4) < T \)). Let \( Y \) be that component of the set \( \{ p : g(p) < T \} \) which contains \( p_3 \). We assert that \( \partial Y \) (the boundary of \( Y \) with respect to \( R \)) meets \( \partial N \), from which the existence of the required path is immediate. Suppose that \( \partial Y \) does not meet \( \partial N \). Then \( \partial Y \subseteq R^1 \) and \( g(p) = T \) on \( \partial Y \). Let \( v(p) = -g(p) \) for \( p \in Y \), with \( v(p) = -T \) for \( p \in R \setminus Y \). Then \( v \) is subharmonic, non-constant and bounded above on \( R \), contradicting Lemma 12.3.11.

12.6.5 Critical points

If \( \phi \) is a local parameter near \( p_2 \in R^1, p_2 \neq p_1 \), then \( p_2 \) is a critical point of \( g \) if \( G = g \circ \phi^{-1} \) has a critical point (i.e. \( G_x = G_y = 0 \)) at \( \phi(p_2) \). This property is independent of the choice of \( \phi \), by the chain rule. The critical points of \( g \) are isolated, and if \( f \) is analytic on a neighbourhood \( U \) of \( p_2 \) with \( \log |f| = -g \), then critical points of \( g \) coincide with critical points of \( f \).

12.6.6 The local behaviour of \( g \)

Let \( p_2 \in R^1, p_2 \neq p_1 \), and assume that \( p_2 \) is not a critical point of \( g \). Then there is a unique curve \( C \) through \( p_2 \) with the following property. If \( f \) is analytic near \( p_2 \) with \( \log |f| = -g \) then \( \arg f \) is constant on \( C \). Further, \( g \) is strictly monotone on \( C \). To see this note that if \( f \) and \( f^* \) are both analytic with \( \log |f| = -g \) then \( f^*/f \) is constant. The curve \( C \) is a level curve of the function \( \text{Im}(\log f) \), this function a harmonic conjugate of \( -g \).

Suppose next that \( p_2 \) is a critical point of \( g \). Then there are \( n \geq 2 \) curves through \( p_2 \) on each of which \( \arg f \), for \( f \) as above, is constant. To see this, choose \( n \) so that \( (f(p) - f(p_2))^1/n \) has a simple zero at \( p_2 \). This allows us to write \( f(p) = G(F(p)^n) \) with \( G \) and \( F \) locally one-one. Choose a curve for \( G \) through \( F(p_2)^n \); then this curve has \( n \) pre-images through \( F(p_2) \).
Indeed, let \( q_2 = \log f(p_2) \) and draw a horizontal and a vertical straight line through \( q_2 \). This forms four quadrants which we label 1, 2, 3, 4 counter-clockwise, starting at the top right in the usual way. There are \( 4n \) curves emanating from \( p_2 \), in which \( n \) is the multiplicity of the zero of \( f(p) - f(p_2) \) at \( p_2 \), and \( 4n \) “sector-like” regions with vertex at \( p_2 \). On \( 2n \) of these curves we have \( \arg f \) constant, while \( \log |f| \) is constant on the other \( 2n \). There are \( n \) curves emanating from \( p_2 \) on which \( \arg f \) is constant and \( \log |f| \) decreases, and as we cross one of these in the counter-clockwise sense, \( \arg f \) decreases (here counter-clockwise is interpreted with respect to the image under a local chart).

12.6.7 The main step

Define a function \( f_1 \), analytic near \( p_1 \), with \( \log |f_1| = -g \). Then \( f_1 \) has a simple zero at \( p_1 \). Let \( h \) be the inverse function of \( f_1 \), defined on a neighbourhood of 0. Let \( r \) be the supremum of positive \( s \) such that \( h \) extends to be analytic on \( D(0, s) \). Then \( h \) is analytic on \( D(0, r) \). Since \( f_1 \) may be analytically continued along any path in \( R^1 \), starting at \( p_1 \), and since \( f_1 \circ h \) is the identity near 0, it follows that

\[
g(h(w)) = \log \frac{1}{|w|}, \quad w \in D(0, r).
\]  

(12.6)

We see now that \( H = h(D(0, r)) \) is a connected set on which \( g(p) > \log 1/r \).

Let \( p_n \to p^* \), with \( p_n \in H \) and \( p^* \in \partial H \). Then we may write \( p_n = h(w_n) \) and without loss of generality \( w_n \to w^* \) with \( |w^*| \leq r \). Suppose that \( |w^*| < r \). Then \( p_n \to h(w^*) \), and \( p^* = h(w^*) \) is an interior point of \( H \), by the open mapping theorem. This is a contradiction, and hence \( |w^*| = r \) and \( g(p_n) = \log 1/|w_n| \to \log 1/r \). We deduce that \( H \) is a component, and so by Lemma 12.6.3 the unique component, of the set \( \{ p \in R^1 : g(p) > \log 1/r \} \).

12.6.8 Lemma

\( h \) is locally univalent on \( D(0, r) \), and \( H = h(D(0, r)) \) contains no critical point of \( g \).

Proof. Take \( w_1 \in D(0, r) \), and analytically continue \( f_1 \) along the image under \( h \) of the line segment from 0 to \( w_1 \). This gives a function \( f_2 \) analytic near \( w_1 \), with \( \log |f_2| = -g \), and \( f_2(h(w)) = w \) near \( w_1 \). So \( h \) must be one-one near \( w \), and \( f_2 \) must be one-one near \( h(w) \).

12.6.9 Lemma

\( h \) is univalent on \( D(0, r) \).

Proof If \( |w_2| < r \) and \( h(w_1) = h(w_2), w_1 \neq w_2 \), then by (12.6) we have \( |w_1| = |w_2| \). Suppose now that \( 0 \leq \theta_1 < \theta_2 < 2\pi \) and set

\[
E = \{ t \in (0, 1) : h(te^{i\theta_1}) = h(te^{i\theta_2}) \}.
\]

Obviously \( (0, 1) \setminus E \) is open, and this set is non-empty since \( h \) is one-one near 0. Suppose \( t_1 \in E \), and let \( p_3 = h(w_1), w_j = t_1e^{i\theta_j} \). Let \( T_j \) be the image under \( h \) of the ray \( \arg w = \theta_j \), and analytically continue \( f_1 \) along the curves \( T_j \). This gives \( F_1, F_2 \) analytic on a parametric disc \( V \) centred at \( p_3 \), and \( F_j \circ h(w) = w \) near \( w_j \). Thus \( \arg F_j \) is constant on an arc of \( T_j \) passing through \( p_3 \).

By 12.6.6 there is a unique curve \( C \) passing through \( p_3 \) on which \( \arg F_j \) is constant, and \( g \) is strictly monotone on \( C \). Since \( g(h(te^{i\theta_j})) = \log 1/t \), we have \( h(te^{i\theta_1}) = h(te^{i\theta_2}) \) for \( t \) close to \( t_1 \). Hence \( E \) is open and so empty, by connectedness.

The next lemma is now obvious.
12.6.10 Lemma

\( f_1 \) extends to be analytic and univalent, with \( f_1 = h^{-1} \), on the simply connected domain \( H = h(D(0, r)) \).

If \( r = 1 \) then we have finished, and we assume henceforth that \( r < 1 \). Since \( H = h(D(0, r)) \) is the unique component of the set \( \{ p \in \mathbb{R}^1 : g(p) > \log 1/r \} \), the closure of \( H \) does not meet \( \partial N \), by Lemma 12.6.1.

12.6.11 Lemma

Suppose that \( \zeta \in \partial H \), and that \( \zeta \) is not a critical point of \( g \). Then \( f_1 \) extends analytically and univalently to a neighbourhood \( V \) of \( \zeta \).

Proof. Let \( G \) be analytic near \( \zeta \), with \( \Re(G) = g \). Then \( G \) is one-one near \( \zeta \), and we let \( V \) be the pre-image under \( G \) of a disc centred at \( G(\zeta) \). Then \( V \cap H \) is connected, since its image under \( G \) is a half-disc. Choose \( f_2 \) analytic near \( \zeta \), with \( \log |f_2| = -g \). Thus \( f_2 \) is univalent near \( \zeta \), and \( f_2 / f_1 \) is constant on \( V \cap H \). Multiplying \( f_2 \) by a constant gives the required extension. Since \( |f_2(p)| \geq r \) on \( V \setminus H \) the extended function remains univalent.

12.6.12 Lemma

Let \( \theta \in [0, 2\pi) \), and define \( \gamma_\theta(t) = h(te^{i\theta}) \) for \( 0 \leq t < r \). Then there exists \( \zeta \in \partial H \) such that \( \gamma_\theta(t) \to \zeta \) as \( t \to r \).

Proof. Since \( R \) has no divergent curves, the curve \( \gamma(t) = \gamma_\theta(t) \) visits some compact set through a sequence tending to \( r \). Thus there exists \( \zeta \in \mathbb{R}^1 \) such that \( \gamma(t_n) \to \zeta \) through a sequence \( t_n \to r \), and \( \zeta \) is a boundary point of \( H = h(D(0, r)) \), since \( g(\gamma(t_n)) = \log 1/t_n \to \log 1/r \). Finally, \( \gamma(t) \to \zeta \) by 12.6.6, since \( \arg f_1 \) is constant on \( \gamma \).

12.6.13 Lemma

\( g \) has a critical point on \( \partial H \).

Proof. Assume not. Then for each \( \theta \in [0, 2\pi) \), the curve \( \gamma_\theta(t) \) tends to \( \zeta = \zeta_\theta \in \partial H \), and \( f_1 \) extends analytically and univalently to a neighbourhood \( V \) of \( \zeta \). We may assume that \( W = f_1(V) \) is a disc.

We have \( f_1(\zeta) = re^{i\theta} \), since 
\[
 f_1(\zeta) = \lim f_1(\gamma(t_n)) = \lim t_ne^{i\theta}, 
\]
in which \( t_n \) increases with limit \( r \). Let \( h^* \) be the inverse function of \( f_1 \), mapping \( W = f_1(V) \) onto \( V \). On 
\[
 W \cap D(0, r) = f_1(V \cap H) 
\]
we have \( h = f_1^{-1} \) and \( h^* = f_1^{-1} \), and so \( h^* \) extends \( h \) to \( D(0, r) \cup W \).

We do this for each \( \theta \), and obtain an extension of \( h \) to a disc \( W_\theta \) centred at \( re^{i\theta} \). Since the intersection of any two \( W_\theta \) is connected, and meets \( D(0, r) \) unless the intersection is void, it follows by compactness that this permits us to extend \( h \) analytically to a larger disc \( D(0, r') \), contradicting the choice of \( r \).
12.6.14 A closed curve

Choose a critical point $\zeta$ of $g$ on $\partial H$. There exist (at least) two curves $\eta_j, j = 1, 2$ emanating from $\zeta$, on which $g$ increases and $\arg f$ is constant, for any any $f$ analytic on a neighbourhood $V$ of $\zeta$ with $\log |f| = -g$. Note that $f_1/f$ is constant on every connected subset of $V \cap H$. These curves lie, apart from their starting point, in $H$, and so by the constancy of $\arg f_1$ on $\eta_j$ we see that each $\eta_j$ is the image $T_j$ under $h$ of a ray $\arg w = \theta_j$. Mark “lower” and “upper” sides of $T_j$ as $T_j^l, T_j^u$, so that $\arg f_1(p)$ increases as $p$ crosses from $T_j^l$ to $T_j^u$. By 12.6.6, we go from $T_j^u$ to $T_j^l$ as we cross $T_j$ moving counter-clockwise around $\zeta$.

Thus the union of $T_1, T_2, p_1$ and $\zeta$ gives a closed curve $\sigma$ on $R$ with two well defined “edges”, which we will label “positive” and “negative”.

12.6.15 Lemma

$R \setminus \sigma$ is path-connected.

Proof. We shall show that every $p \in R \setminus \sigma$ can be joined to $p_0$ by a path avoiding $\sigma$. This is true if $p \in N$ or if $g(p) \leq \log 1/r$, by Lemma 12.6.4. Suppose now that $g(p) > \log 1/r$, so that $p \in H$. Now only finitely many curves $\gamma(t)$ can land at $\zeta$, and so we first move from $p$ to a point $p'$ not lying on any of these. Following a curve $\arg f_1 = c$ from $p'$ we land at $\zeta' \in \partial H, \zeta' \neq \zeta$, and since $g(\zeta') \leq \log 1/r$ we can continue on to $p_0$.

12.6.16 Lemma

There exists a continuous function from $R$ to $\mathbb{C} \setminus \{0\}$, not having a continuous logarithm.

Proof. To the cut surface $R \setminus \sigma$ we adjoin two copies of $\sigma$, labelled $\sigma^+, \sigma^-$, corresponding to the positive and negative edges of $\sigma$. Let the resulting space be $X$. We construct a continuous function $q : X \to [0, 1]$, with $q = 0$ on $\sigma^-$ and $q = 1$ on $\sigma^+$. The function $Q = \exp(2\pi i q)$ will then be well-defined and continuous on $R$, but does not have a continuous logarithm $q^*$ on $R$, because for any such $q^*$ the function $q^* - q$ would be constant on $R \setminus \sigma$.

To construct $q$, cover $\sigma$ by finitely many closed parametric discs $P_j$, each contained in a small open parametric disc $D_j$. Let $F_j : R \to [0, 1]$ be continuous, with $F_j = 0$ on $P_j$, and $F_j = 1$ off $D_j$. We then define $G_j$ on $X$ to be the same as $F_j$, except that $G_j = 1$ on $\sigma^+$ and on all points of $D_j$ on the “positive” side of $\sigma$. Thus $G_j : X \to [0, 1]$ is continuous, with $G_j = 0$ on $\sigma^- \cap P_j$, and $G_j = 1$ on $\sigma^+$. Finally set $q(x) = \min \{F_j(x)\}$.

This result contradicts the following standard lemma, and the proof of Lemma 12.5.5 is complete.

12.6.17 Lemma

Let $S$ be any simply connected Riemann surface, and let $Q : S \to \mathbb{C} \setminus \{0\}$ be continuous. Then $Q$ has a continuous logarithm on $S$.

Proof. Fix $a \in S$ and assume without loss of generality that $Q(a) = 1$. Let $\sigma_1, \sigma_2$ be paths in $S$ joining $a$ to $b$. Then $Q(\sigma_1), Q(\sigma_2)$ are homotopic paths in $\mathbb{C} \setminus \{0\}$ starting at 1, and by the ordinary monodromy theorem the continuations of $\log w$ along these paths agree near $Q(b)$, so that $\log Q(b)$ is well defined.
Chapter 13

The Phragmén-Lindelöf principle

13.1 Introduction

This represents a refinement of the maximum principle for subharmonic and analytic functions. The classical proofs have largely been supplanted by use of harmonic measure. We begin with:

13.1.1 Lemma

Let \( D \) be a domain in \( \mathbb{C} \) and let \( u \) be subharmonic and bounded above on \( D \), with

\[
\limsup_{z \to \zeta, z \in D} u(z) \leq 0
\]

for all finite boundary points of \( D \). Then \( u(z) \leq 0 \) on \( D \).

This follows at once from Lemma 8.2.7. The next lemma is a refinement of Lemma 13.1.1 for functions having slow growth as \( z \) tends to infinity in \( D \).

13.1.2 Lemma: the classical Phragmén-Lindelöf principle

Let \( D \) be a domain in \( \mathbb{C} \) and let \( u \) be subharmonic on \( D \), such that

\[
\limsup_{z \to \zeta, z \in D} u(z) \leq 0
\]

for all finite boundary points \( z \) of \( D \). Suppose further that there exists \( v(z) \) harmonic on \( D \), with

\[
\liminf_{z \to \zeta, z \in D} v(z) \geq 0
\]

for every finite boundary point \( z \) of \( D \), and such that for every \( \delta > 0 \) we have

\[
\limsup_{z \to \infty, z \in D} (u(z) - \delta v(z)) \leq 0.
\]

Then \( u(z) \leq 0 \) on \( D \).

Proof. Fix \( w \) in \( D \). For \( \delta > 0 \) the maximum principle gives

\[
u(w) - \delta v(w) \leq 0
\]

and we just let \( \delta \to 0 \).
13.1.3 Corollary

Let $R > 0$ and $M > 0$ and let $-\pi \leq a < b \leq \pi$. Let $f$ be analytic on the domain

$$D = \{ z : \; |z| > R, \; a < \arg z < b \},$$

with

$$\limsup_{z \to \zeta, z \in D} |f(z)| \leq M < \infty$$

for all finite boundary points $\zeta$ of $D$. Assume that

$$\log |f(z)| < |z|^s \tag{13.1}$$

for all large $z$ in $D$, in which $s < S < \pi/(b - a)$. Then $|f(z)| \leq M$ in $D$.

Proof. We may clearly assume that $M = 1$ (otherwise replace $f$ by $f(z)/M$, which does not affect the existence of an $s$ as in (13.1)). By considering $f(ze^{it})$ in place of $f$, for some fixed $t$, we may assume that $b > 0, a = -b$. Thus $s < S < \pi/2b$.

Take $u(z) = \log |f(z)|$ and

$$v(z) = |z|^S \cos(S \arg z) = \Re(z^S).$$

For $z$ in $D$ we have

$$|S \arg z| \leq Sb < \pi/2, \quad \cos(S \arg z) \geq \cos Sb = \mu > 0$$

and so

$$v(z) \geq |z|^S \mu.$$ 

Thus, for every $\delta > 0$ we have

$$u(z) - \delta v(z) \to -\infty$$

as $z \to \infty$ in $D$. By Lemma 13.1.2, we get $u(z) \leq 0$ on $D$.

This result is sharp: to see this, take $0 < b \leq \pi$ and $a = -b$ and $f(z) = \exp(z^\pi/(b-a))$. Then $f$ is bounded on the finite boundary of $D$ but unbounded in $D$.

Thus the narrower the sectorial region $D$ is, the faster $f$ has to grow in $D$ in order to not be bounded. We will see a far-reaching generalization of this idea in the section on the Carleman-Tsuji estimate for harmonic measure.

13.2 Applications

The next two results are among the most useful applications of this strand of ideas.

13.2.1 Theorem

Let $D$ be an unbounded simply connected domain in $\mathbb{C}$, not the whole plane. Let $f$ be analytic and bounded on $D$, and continuous on $D \cup \partial D$. Assume that $f(z) \to 0$ as $z$ tends to infinity on $\partial D$. Then $f(z) \to 0$ as $z$ tends to infinity in $D$. 
Proof. Assume without loss of generality that \(|f(z)| \leq 1\) on \(D\). Let \(0 < \delta < 1\) and let \(E\) be a closed subset of \(\partial D\) such that \(|f(z)| < \delta\) on \(\partial D \setminus E\). Since \(\infty\) is a regular point of \(X = \partial_{\infty}D\), by §8.1.4, we get
\[
\omega(z, E, D) \to 0, \quad z \to \infty,
\]
using §9.1.9. But now the two-constants theorem gives
\[
|f(z)| \leq \delta + \omega(z, E, D)
\]
and the result follows.

13.2.2 Theorem

Let \(D\) be a simply connected domain as in Theorem 13.2.1, such that the boundary of \(D\) consists of two simple curves \(C_1, C_2\) both tending to infinity, and disjoint apart from their common starting point \(a \in \partial D\). Let \(f\) be analytic and bounded in the domain \(D\), and continuous in \(D \cup C_1 \cup C_2\). Assume that \(f(z) \to a_j\) as \(z\) tends to infinity on \(C_j\). Then \(a_1 = a_2\).

Proof. It is clear that the \(a_j\) are finite, since \(f\) is bounded. Assume \(a_1 \neq a_2\) and apply Theorem 13.2.1 to \(g(z) = (f(z) - a_1)(f(z) - a_2)\). Thus \(g(z) \to 0\) as \(z\) tends to infinity in the closure of \(D\). Let \(\varepsilon > 0\), and take \(M > 0\) such that \(|g(z)| < \varepsilon\) for \(z \in D, |z| > M\).

We now use the fact that \(J = C_1 \cup C_2 \cup \{\infty\}\) is a Jordan curve on the Riemann sphere (in particular \(J\) cannot contain a disc), and so a rotation of \(D\) is a Jordan domain in \(C\). Take a curve \(I\) which lies in the closure of \(D\) and joins \(C_1\) to \(C_2\), with \(|z| > M\) for all \(z\) on \(I\). Such a curve exists by Theorem 10.5.3: take the Riemann mapping \(h\) from \(D(0, 1)\) to \(D\) and extend it to a homeomorphism on \(|z| \leq 1\). The curve \(I\) is then the image of an arc of a circle centred at \(h^{-1}(\infty)\). We have \(|g(z)| < \varepsilon\) on \(I\) and so, by connectedness, either \(f(z) - a_1\) is small on all of \(I\) or \(f(z) - a_2\) is small on all of \(I\). This contradicts the fact that \(f(z) - a_j\) is small for large \(z\) on \(C_j\).
Chapter 14

The Carleman-Tsuji estimate for harmonic measure

14.1 The Carleman-Tsuji estimate

14.1.1 Parseval’s formula for a continuous function

Let \( f \) be a continuous real-valued function on \([-\pi, \pi]\). Define the Fourier coefficients

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.
\]

These are uniformly bounded. As shown in the section on Poisson’s formula (7.2.2),

\[
F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{e^{it} + z}{e^{it} - z} \, dt = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt + \frac{1}{2\pi i} \int_{|w|=1} \frac{2f(\arg w)}{w - z} \, dw
\]

is analytic in \( D(0, 1) \) with \( c_0 = F(0) = \frac{1}{\pi} a_0 \). Also \( u = \text{Re}(F) \) is bounded and, as \( z \to e^{is}, -\pi < s < \pi \), we have \( u(z) \to f(s) \).

Differentiation gives, for \( n > 0 \),

\[
F^{(n)}(z) = \frac{n!}{2\pi i} \int_{|w|=1} \frac{2f(\arg w)}{(w - z)^{n+1}} \, dw
\]

and

\[
c_n = \frac{F^{(n)}(0)}{n!} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt = a_n - ib_n.
\]

Thus Taylor’s theorem applied to \( F \) gives

\[
u(re^{it}) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos nt + b_n \sin nt),
\]

the series uniformly convergent on each closed disc \(|z| \leq r < 1\). The orthogonality of the trigonometric functions gives

\[
I(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} u(re^{it})^2 \, dt = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} r^{2n} (a_n^2 + b_n^2).
\]
Since \( u(re^{it}) \) is uniformly bounded and tends pointwise to \( f(t) \) on \((-\pi, \pi)\) the dominated convergence theorem gives Parseval’s formula
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)^2 dt = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).
\]

14.1.2 Wirtinger’s inequality

Suppose that \( f \) is a real-valued function such that \( f' \) is continuous on \([a, b] \) and \( f(a) = f(b) = 0 \). Then
\[
\int_{a}^{b} f'(x)^2 dx \geq \frac{\pi^2}{(b-a)^2} \int_{a}^{b} f(x)^2 dx.
\]

Proof. It suffices to prove this when \( a = 0, b = \pi \). Extend \( f \) to an odd function on \([-\pi, \pi]\). In the Fourier expansion of \( f \) we have \( a_n = 0 \) and
\[
b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \ dx.
\]

Further, \( f' \) can be extended to an even function \( h \) on \([-\pi, \pi]\) with \( \int_{-\pi}^{\pi} h(x) dx = 2 \int_{0}^{\pi} f'(x) dx = f(\pi) - f(0) = 0 \). The Fourier expansion of \( h \) has no \( \sin nx \) terms and has
\[
A_n = \frac{2}{\pi} \int_{0}^{\pi} f'(x) \cos nx \ dx = nb_n,
\]
using integration by parts. Parseval’s formula gives
\[
\frac{2}{\pi} \int_{0}^{\pi} f(t)^2 dt = \sum_{n=1}^{\infty} b_n^2 \leq \sum_{n=1}^{\infty} n^2 b_n^2 = \sum_{n=1}^{\infty} A_n^2 = \frac{2}{\pi} \int_{0}^{\pi} f'(t)^2 dt.
\]

14.1.3 Definition

For \( 0 < t < \infty \) and a domain \( D \) in \( \mathbb{C} \) we define \( \theta_D^*(t) \) as follows. If \( D \) contains the whole circle \( |z| = t \) then \( \theta_D^*(t) = \infty \). If \( D \cap \{ z : |z| = t \} \) is not the whole circle \( |z| = t \) then it consists of countably many open arcs, and we define \( \theta_D^*(t) \) to be the angular measure of the longest of these (if one has angular measure \( s > 0 \) then at most finitely many can have angular measure \( > s \)). Note that if \( \theta_D^*(t) > y \), then \( D \cap \{ z : |z| = t \} \) contains a closed arc \( A \) of angular measure \( y \), and \( D \) contains a neighbourhood of \( A \). Thus \( \theta_D^*(t') > y \) for \( t' \) close to \( t \) and so \( \theta_D^*(t) \) is measurable (we’ve shown that \( -\theta_D^*(t) \) is upper semi-continuous i.e. \( \theta_D^*(t) \) is LSC).

Obviously if \( D, U \) are domains with \( D \subseteq U \) then \( \theta_D^*(t) \leq \theta_U^*(t) \).

14.1.4 The Carleman-Tsuji estimate: a special case

Let \( 0 < r < \infty \) and let \( D \) be a domain in \( \mathbb{C} \) with \( 0 \in D \) such that \( D \) meets the circle \( |z| = r \). Assume that there exist a positive increasing sequence \( \rho_n \to \infty \) and a finite subset \( \{ \theta_j \} \) of \([0, 2\pi]\) such that \( \partial D \) consists of:

\( (i) \) arcs of circles \( |z| = \rho_n \), each such circle contributing at most finitely many arcs;
(ii) radial line segments $z = se^{i\theta}, \rho_n \leq s \leq \rho_{n+k}$;

Let $D_r$ be the component of $D \cap D(0, r)$ containing 0. Let $\theta_r = \partial D_r \setminus \partial D$.

Then $\theta_r$ is a subset of the circle $|z| = r$, since $w \in \theta_r$ implies that $|w| \leq r$ and that $w$ is a limit point of $D$ and so in $D$.

Let

$$u(z) = \omega(z, \theta_r, D_r)$$

and extend $u$ to a function $v$ subharmonic in $D(0, r)$, by setting $v = 0$ in $D(0, r) \setminus D_r$. To do this, note that if $w \in D(0, r) \cap \partial D_r$ then $u(z) \to 0$ as $z \to w$ with $z$ in $D_r$.

Since $v \geq 0$ we see that $v^2$ is upper semi-continuous. Also for $|z_0| < r$ and small $s > 0$, Cauchy-Schwarz gives

$$v(z_0)^2 \leq \left(\frac{1}{2\pi} \int_0^{2\pi} v(z_0 + se^{i\theta}) d\theta\right)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} v(z_0 + se^{i\theta})^2 d\theta$$

and so $v^2$ is subharmonic. For $0 < \rho < r$ let

$$m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} v(\rho e^{i\theta})^2 d\theta = \frac{1}{2\pi} \int_{\theta_{\rho}} u(\rho e^{i\theta})^2 d\theta$$

in which $\theta_{\rho} = D_r \cap \{z : |z| = \rho\}$ for $0 < \rho < r$. Then, by Theorem 8.2.1, $m(\rho)$ is a convex non-decreasing function of $\log \rho$ on $(0, r)$ and in particular $m$ is continuous. Also, since $u$ is harmonic and so continuous at 0, we have $\lim_{\rho \to 0+} m(\rho) = u(0)^2$.

By (1.1) the derivative $\mu = \frac{\partial m}{\partial \log \rho}$ exists on $J = (0, r) \setminus E_0$, where $E_0$ is a countable set, and $\mu$ is non-decreasing on $J$.

Claim 1: $\mu$ is positive on $J$.

To prove the claim we note that $u$ is harmonic and non-constant near the origin, using the identity theorem for harmonic functions. So near the origin $u$ is the real part of a non-constant analytic function and there are constants $a_n, b_n$ such that we can write, for small $\rho$,

$$u(\rho e^{i\theta}) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta)$$

and

$$m(\rho) = \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \rho^{2n} (a_n^2 + b_n^2)$$

so that $m'(\rho) > 0$. This proves the claim.

Let $\rho_n < \rho < \rho_{n+1}$. Then $\theta_{\rho}$ consists of finitely many open arcs of $|z| = \rho$. On $P_n = \{z \in \partial D_r : \rho_n < |z| < \rho_{n+1}\}$ we have $u = 0$, and $P_n$ consists of finitely many open radial segments, across which $u$ can be extended by the Schwarz reflection principle 10.7.1. So all partial derivatives of $u$ extend continuously up to $P_n$.

Let $t = \log \rho$. For $\rho_n < \rho < \rho_{n+1}$,

$$m_t = \frac{1}{\pi} \int_{\theta_{\rho}} u_t d\theta.$$  

(14.2)
Also, writing \( u \) locally as a harmonic function of \( \log z = t + i\theta \),
\[
m_{tt} = \frac{1}{\pi} \int_{\theta_\rho} (u_t)^2 + uu_{tt} \, d\theta = \frac{1}{\pi} \int_{\theta_\rho} (u_t)^2 - uu_{\theta \theta} \, d\theta \tag{14.3}
\]
and so integration by parts gives
\[
m_{tt} = \frac{1}{\pi} \int_{\theta_\rho} (u_t)^2 + (u_\theta)^2 \, d\theta \geq 0. \tag{14.4}
\]
By (14.2) and Cauchy-Schwarz,
\[
(m_t)^2 \leq \frac{1}{\pi} \int_{\theta_\rho} u^2 \, d\theta \int_{\theta_\rho} (u_t)^2 \, d\theta \tag{14.5}
\]
and so
\[
(m_t)^2 \leq \frac{1}{2m} \int_{\theta_\rho} (u_t)^2 \, d\theta. \tag{14.6}
\]
Define \( \theta^*(\rho) \) for \( \rho_n < \rho < \rho_{n+1} \) as follows. If \( \theta_\rho \) consists of the whole circle \( |z| = \rho \) then put \( \theta^*(\rho) = \infty \). If \( \theta_\rho \) is not the whole circle \( |z| = \rho \) then it consists of finitely many open arcs \( \theta_\rho' \). Then \( \theta^*(\rho) = \theta^*_D(\rho) \) is the angular length of the longest of these.

In the second case we get by Wirtinger’s inequality 14.1.2
\[
\int_{\theta_\rho} (u_\theta)^2 \, d\theta \geq \frac{\pi^2}{|\theta_\rho'|^2} \int_{\theta_\rho} u^2 \, d\theta \geq \frac{\pi^2}{\theta^*(\rho)^2} \int_{\theta_\rho'} u^2 \, d\theta,
\]
since \( u \) vanishes at the end-points, and summing gives
\[
\frac{1}{\pi} \int_{\theta_\rho} (u_\theta)^2 \, d\theta \geq \frac{2\pi^2}{\theta^*(\rho)^2} m(\rho), \quad \rho \neq \rho_n. \tag{14.7}
\]
Thus (14.4), (14.6) and (14.7) give
\[
m_{tt} \geq \frac{(m_t)^2}{2m} + \frac{1}{2} \left( \frac{2\pi}{\theta^*(\rho)} \right)^2 m(\rho), \quad \rho \neq \rho_n. \tag{14.8}
\]
Put
\[
t = \log \rho, \quad t_n = \log \rho_n, \quad M(t) = m(\rho), \quad h(t) = \frac{2\pi}{\theta^*(\rho)}.
\]
Then (14.8) becomes
\[
M'' \geq \frac{(M')^2}{2M} + \frac{1}{2} h^2 M, \quad t_n < t < t_{n+1}. \tag{14.10}
\]
In particular, using Claim 1,
\[
M'(t) > 0, \quad M''(t) > 0, \quad t \neq t_n. \tag{14.11}
\]
Thus
\[
L'' + (L')^2 \geq \frac{1}{2} (L')^2 + \frac{1}{2} h^2,
\]
and
\[
2L'' + (L')^2 \geq h^2, \quad L = \log M. \tag{14.12}
\]
This gives
\[
(M''/M')^2 = (L''/L' + L')^2 \geq (L')^2 + 2L'' \geq h^2.
\]
14.1. THE CARLEMAN-TSUJI ESTIMATE

and so, using (14.11),

\[ M''/M' \geq h, \quad t_n < t < t_{n+1}. \] (14.13)

So for \( t_n < s < s' < t_{n+1} \) we have

\[ M'(s') \geq M'(s) \exp(\int_s^{s'} h(t)dt). \]

Iterating this and using the fact that \( M'(s) \leq M'(s') \) for \( s, s' \not\in \{t_n\} \) with \( s < s' \), since \( M \) is convex, we get

\[ M'(\tau) \geq M'(t) \exp(\int_t^\tau h(s)ds), \quad -\infty < t < \tau < \log r, \quad t, \tau \not\in \{t_n\}. \] (14.14)

Now put

\[ t = \log \rho, \quad \sigma = e^\tau, \quad t^* = \log r \] (14.15)

and assume that \( k \in (0, 1) \). Suppose that \( 0 < \rho < kr \). Then, since \( M \) is continuous, non-negative and non-decreasing,

\[ 1 \geq \lim_{\tau \to t} M(\tau) \geq \int_t^{t^*} M'(\tau)d\tau, \]

we get, using (14.14),

\[
1 \geq M'(t) \int_t^{t^*} \exp\left( \int_t^\tau h(s)ds \right) d\tau \\
= M'(t) \int_\rho^{e^\tau} \exp\left( \int_\sigma^{e^\tau} \frac{2\pi dx}{x^{\theta^*(x)}} \right) \frac{d\sigma}{\sigma} \\
\geq M'(t) \int_\rho^{e^\tau} \exp\left( \int_\sigma^{e^\tau} \frac{2\pi dx}{x^{\theta^*(x)}} \right) \frac{d\sigma}{\sigma} \\
\geq M'(t) \int_{kr}^{e^\tau} \frac{2\pi dx}{x^{\theta^*(x)}} \\
\geq (1-k)M'(t) \exp\left( \int_{kr}^{e^\tau} \frac{2\pi dx}{x^{\theta^*(x)}} \right)
\]

since

\[
\int_{kr}^{e^\tau} \frac{d\sigma}{\sigma} = \int_k^1 \frac{d\sigma}{\sigma} \geq \int_k^1 d\sigma = 1 - k.
\]

This gives

\[ M'(t) \leq (1-k)^{-1} \exp\left( -\int_{kr}^{e^\tau} \frac{2\pi dx}{x^{\theta^*(x)}} \right), \quad 0 < \rho = e^t < kr, \quad 0 < k < 1. \] (14.16)

Now (14.10) gives

\[ M''(t) \geq \frac{1}{2} h(t)^2 M(t) \geq \frac{1}{2} h(t) M(t) = \frac{\pi m(\rho)}{\theta^*(\rho)}. \] (14.17)

Let \( t < \tau \) with \( t, \tau \not\in \{t_j\} \) and let \( t_\nu < \ldots < t_n \) be those \( t_j \) lying in \((t, \tau)\). Since \( M' \) is non-decreasing,

\[ M'(\tau) \geq M'(t) + M'(t_\nu^+) + \ldots + M'(t_n^+) \geq \int_t^{\tau} M''(s)ds. \]
and so (14.17) gives
\[
M'(\tau) \geq \pi \int_{\rho}^{\sigma} \frac{m(x)dx}{x^{\theta^*(x)}} \geq \pi m(\rho) \int_{\rho}^{\sigma} \frac{dx}{x^{\theta^*(x)}}.
\] (14.18)

Using (14.16), with \( t \) replaced by \( \tau = \log \sigma \), and (14.18), we now obtain, for \( \rho < \sigma < kr, 0 < k < 1 \),
\[
\pi m(\rho) \int_{\rho}^{\sigma} \frac{dx}{x^{\theta^*(x)}} \leq M'(\tau) \leq (1 - k)^{-1} \exp\left(- \int_{\rho}^{kr} \frac{2\pi dx}{x^{\theta^*(x)}} + \int_{\rho}^{\sigma} \frac{2\pi dx}{x^{\theta^*(x)}} \right).
\] (14.19)

and in particular
\[
\pi m(\rho) \int_{\rho}^{\sigma} \frac{dx}{x^{\theta^*(x)}} \leq (1 - k)^{-1} \exp\left(- \int_{\rho}^{kr} \frac{2\pi dx}{x^{\theta^*(x)}} \right).
\] (14.20)

If \( 0 < \rho < kr \) and
\[
\int_{\rho}^{kr} \frac{2\pi dx}{x^{\theta^*(x)}} > 1
\] (14.21)

then we choose \( \sigma \in (\rho, kr) \) with
\[
\int_{\rho}^{\sigma} \frac{2\pi dx}{x^{\theta^*(x)}} = 1
\]
and (14.19) gives
\[
m(\rho) \leq \frac{2e}{(1 - k)^2} \exp\left(-2\pi \int_{\rho}^{kr} \frac{dx}{x^{\theta^*(x)}} \right).
\] (14.22)

On the other hand if (14.20) fails then the RHS of (14.21) is at least \( 2/(1 - k) > 2 > m(\rho) \). Thus (14.21) always holds.

Letting \( \rho \to 0+ \) we get \( m(\rho) \to u(0)^2 \) and so
\[
u(0) = \omega(0, \theta_r, D_r) \leq \frac{(2e)^{1/2}}{(1 - k)^{1/2}} \exp\left(-\pi \int_{0}^{kr} \frac{dt}{t^{\theta^*(t)}} \right), \quad 0 < k < 1.
\] (14.22)

14.1.5 The Carleman-Tsuji estimate: the main step

Let \( 0 < r < \infty \) and let \( D \) be a semi-regular domain containing 0 and meeting the circle \( |z| = r \). Let \( D_r \) be the component of \( D \cap D(0, r) \) containing 0, and let \( H_r = \partial D_r \setminus \partial D \), so that \( H_r \subseteq D \cap \{z : |z| = r\} \).

Note that \( H_r \) is a relatively open subset of \( \partial D_r \). Note also that \( D_r \) is semi-regular (if \( x \) is a boundary point of \( D \) and \( D_r \) then a barrier for \( x, D \) will serve for \( x, D_r \), while if \( x \) is in \( H_r \) then \( x \) satisfies the condition 8.1.8). Let \( E_0 \) be a compact subset of \( \partial D_r \) such that \( E_0 \subseteq H_r \) and let \( u(z) = \omega(z, E_0, D_r) \).

Let \( \nu = \theta_r \) be a positive integer, and define building blocks of the \( n \)th stage to be the sets
\[
\{z : |z| \leq 2^{-n}r\}, \quad \{z : p2^{-n}r \leq |z| \leq (p + 1)2^{-n}r \}, \quad \pi q 2^{-n} \leq \arg z \leq \pi (q + 1)2^{-n}, \quad p, q \in \mathbb{N}.
\]

Let \( D_n^* \) be the union of all blocks of the \( n \)th stage which are contained in \( D \), and let \( D_n \) be that component of the interior of \( D_n^* \) of \( D_n \) which contains 0. Obviously \( D_n^* \subseteq D_{n+1}^* \) and so \( D_n^* \subseteq D_{n+1}^* \) and \( D_n \subseteq D_{n+1} \).

Claim 1: \( D = \bigcup_{n=1}^{\infty} D_n \).

To see this, take \( z \in D \) and join 0 to \( z \) by a path \( \gamma \) in \( D \). If \( n \) is large then \( 2^{-n} \) is small compared to \( \text{dist}(\gamma, \partial D) \) and so a neighbourhood of \( \gamma \) lies in \( D_n^* \). This proves Claim 1.
Note that it follows that $D_n$, for large $n$, meets the circle $|z| = r$.

For each $n$, let $D_n(r)$ be the component of $D_n \cap D(0, r)$ containing 0, and define $\theta_n(r) = \partial D_n(r) \setminus \partial D_n$. Since $D_n \subseteq D_{n+1}$ we clearly have $D_n(r) \subseteq D_{n+1}(r)$.

Claim 2: $\bigcup_{n=1}^{\infty} D_n(r) = D_r$.

Since $D_n \subseteq D$ we have $D_n(r) \subseteq D_r$. Now join 0 to $z$ in $D_r$ by a path $\gamma$ in $D_r$. Then for large $n$ we have $\gamma \subseteq D^*_n$ and so $\gamma \subseteq D_n$.

Claim 3: let $x \in E_0$; then there exists $\delta > 0$ such that, for all sufficiently large $n$, we have $D(x, \delta) \cap D(0, r) \subseteq D_n(r)$ and $\{z : |z| = r\} \cap D(x, \delta) \subseteq \theta_n(r)$.

To see this, note that since $x \in H_r$ we have $x \in D$. Thus by Claim 1 there exists $\delta > 0$ such that $D(x, \delta)$ is in $D_p$ for all sufficiently large $p$. Thus $V = D(x, \delta) \cap D(0, r) \subseteq D_p \cap D(0, r)$ for all large $p$.

Since $x$ is a boundary point of $D_r$, there exists $y \in V \cap D_r$. By Claim 2, we have $y \in D_n(r)$ for all large $n$, and so $V \subseteq D_n(r)$ for large $n$, since $V \subseteq D_n \cap D(0, r)$ and $V$ is connected. Since $D(x, \delta) \cap \{z : |z| = r\} \subseteq C(\{D_n(r)\}) \cap D_n$, the second assertion of Claim 3 follows.

It follows from compactness that the same $n$ will serve for all $x \in E_0$, if sufficiently large.

Let $u_n(z) = \omega(z, \theta_n(r), D_n(r))$ for large $n$. Since $D_n(r) \subseteq D_{n+1}(r) \subseteq D$, §14.1.4 gives

$$u_n(0) = \omega(0, \theta_n(r), D_n(r)) \leq \frac{(2e)^{1/2}}{(1-k)^{1/2}} \exp \left(-\pi \int_0^{kr} \frac{dt}{t\theta_D^*(t)}\right).$$

(14.23)

Further, $u_n \leq u_{n+1}$, by the comparison principle, and

$$v(z) = \lim_{n \to \infty} u_n(z)$$

is harmonic in $D_r$, by Harnack’s theorem, using Claim 2 and the fact that $u_n \leq 1$.

We compare $u$ to $v$. By Claim 3, if $x \in E_0$ then $x$ is an interior point of $\theta_n(r)$ for large $n$. Again by Claim 3, there is some $\delta > 0$ such that $D(0, r) \cap D(x, \delta) \subseteq D_n(r)$ and as $z \to x$ with $z \in D(0, r)$ we have $1 \geq v(z) \geq u_n(z) \to 1$ (note here that $D_n(r)$ is regular).

If $x \in \partial D_r \setminus E_0$ then $x \in \partial D$ or $|x| = r$. Since $D$ is semi-regular, it follows that, with finitely many exceptions, $u(z) \to 0$ as $z \to x$ from within $D_r$. Thus we get $u \leq v$ on $D_r$. Since $E_0$ is an arbitrary compact subset of $H_r$ we have, using (14.23),

$$\omega(0, H_r, D_r) \leq \frac{(2e)^{1/2}}{(1-k)^{1/2}} \exp \left(-\pi \int_0^{kr} \frac{dt}{t\theta_D^*(t)}\right).$$

(14.24)

14.1.6 The Carleman-Tsuji estimate

Let $D$ be a semi-regular domain in $\mathbb{C}$ and let $z \in D$. Let $0 < r < \infty$, $0 < k < 1$ and $2|z| \leq kr$. Let $D_r$ be the component of $D \cap D(0, r)$ containing $z$. Then with $S(0, r)$ the circle $|z| = r$,

$$u(z) = \omega(z, S(0, r), D_r) \leq \frac{3(2e)^{1/2}}{(1-k)^{1/2}} \exp \left(-\pi \int_{2|z|}^{kr} \frac{dt}{t\theta_D^*(t)}\right).$$

(14.25)
Proof. We should more precisely write \( S(0, r) \cap \partial D_r \) in place of \( S(0, r) \) on the LHS of (14.25). However, the statement here is slightly stronger than that in Tsuji’s book, in which only \( \omega(z, S(0, r) \cap D, D_r) \) is considered.

Assume first that \( D \) meets \( S(0, r) \) and let \( H_r = \partial D_r \setminus \partial D \). Let \( U = D \cup D(0, 2|z|) \) and let \( U_r \) be the component of \( U \cap D(0, r) \) containing \( z \). Since \( D \subseteq U \) we have \( D_r \subseteq U_r \) and \( H_r \subseteq \partial U_r \setminus \partial U \). Now

\[
\theta^*_U(t) = \theta^*_D(t), \quad (2|z| < t < r),
\]

and \( \theta^*_U(t) = \infty \) for \( 0 < t < 2|z| \). Thus, using the comparison principle and Harnack’s inequality,

\[
\omega(z, H_r, D_r) \leq \omega(z, L_r, U_r) \leq 3\omega(0, L_r, U_r)
\]

and (14.24) gives

\[
\omega(z, H_r, D_r) \leq \frac{3(2e)^{1/2}}{(1 - k)^{1/2}} \exp \left( -\pi \int_{2|z|}^{kr} \frac{dt}{t \theta^*_D(t)} \right). \tag{14.26}
\]

Now take \( s \) with \( r - s \) small and positive. Let \( G \) be the component of \( D \cap D(0, s) \) containing \( z \), and let \( L = \partial G \setminus \partial D \). Obviously \( G \subseteq D_r \). Also \( \theta^*_G(t) \leq \theta^*_D(t) \) and (14.26) gives

\[
v(z) = \omega(z, L, G) \leq \frac{3(2e)^{1/2}}{(1 - k)^{1/2}} \exp \left( -\pi \int_{2|z|}^{ks} \frac{dt}{t \theta^*_D(t)} \right). \tag{14.27}
\]

We compare \( u(w) \) to \( v(w) \) on \( G \). If \( w \to x \in L \), then \( v(w) \to 1 \), since \( L \) is a relatively open subset of \( S(0, s) \). On the other hand, if \( w \to x \in \partial G \setminus L = \partial G \cap \partial D \) then \( |x| \leq s \). Thus \( x \) is in the closure of \( D_r \) but not in \( D \), and so \( x \) is in the relatively open set \( \partial D_r \setminus S(0, r) \). Provided \( x \) is a regular boundary point of \( D_r \) it follows that \( u(w) \to 0 \) as \( w \to x \), and we have already seen in §14.1.5 that \( D_r \) is semi-regular since \( D \) is. Hence, with finitely many exceptions, \( \limsup_{w \to x \in \partial G} (u(w) - v(w)) \leq 0 \) and so \( u(z) \leq v(z) \). Since \( s \) is arbitrary in (14.27), we get (14.25).

Remark. With the above notation let \( Y_r \) be the part of \( \partial D \) lying in \( |z| \geq r \). Using the comparison theorem we get

\[
\omega(z, D(0, r) \cap \partial D_r, D_r) \leq \omega(z, D(0, r) \cap \partial D_r, D) \leq \omega(z, D(0, r) \cap \partial D, D),
\]

since evidently \( D(0, r) \cap \partial D_r \subseteq D(0, r) \cap \partial D \). Taking complements we get

\[
\omega(z, Y_r, D) \leq \omega(z, S(0, r) \cap \partial D_r, D_r)
\]

and (14.25) can be applied again.

The next theorem is a typical application of this estimate 14.1.6 and is a powerful refinement of Lemma 13.1.1 and Corollary 13.1.3.

14.1.7 Theorem

Let \( v \) be subharmonic on the semi-regular domain \( D \) in \( \mathbb{C} \), and assume that

\[
\limsup_{z \to \zeta, z \in D} v(z) \leq 0
\]
for every finite boundary point $\zeta$ of $D$. Assume further that $r_n \to \infty$ and
\[
B(r_n, v) \exp \left( -\pi \int_1^{r_n/2} \frac{dt}{t\theta_D^*(t)} \right) \to 0
\]
as $n \to \infty$, in which
\[
B(r, v) = \sup \{ v(z) : z \in D, |z| = r \}.
\]
Then $v(z) \leq 0$ on $D$.

Proof. Assume that $v(z) > 0$ for some $z$ in $D$. The two-constants theorem gives
\[
v(z) \leq B(r_n, v)\omega(z, S(0, r_n), D_{r_n})
\]
and applying the Carleman-Tsuji estimate 14.1.6 the RHS tends to 0.

Note that if $D$ is a sectorial region $\{ z : |z| > R, a < \arg z < b \}$ then
\[
\theta_D^*(t) = (b - a), \quad \exp \left( -\pi \int_1^{e^{r/2}} \frac{dt}{t\theta_D^*(t)} \right) = (r/2)^{-\pi/(b - a)},
\]
and so Corollary 13.1.3 is a special case of this result.

14.1.8 Boundary behaviour of harmonic measure: revisited

Let $D$ be a semi-regular domain in $\mathbb{C}$ with $0 \in X = \delta_{\infty}D$. Let $E \subseteq X$ be closed, with $0 \not\in E$. If $0$ is regular for $D$ then by §9.1.9 we know that $\omega(z, E, D) \to 0$ as $z \to 0, z \in D$, whereas if $0$ is not regular then Example 9.1.10 shows that this may fail.

On the other hand if $S(0, t)$ meets the complement of $D$ for every $t > 0$ then $\theta_D^*(t) \leq 2\pi$ for $t > 0$ and the Carleman-Tsuji estimate shows that for a given component $D_r$ of $D \cap D(0, r)$ we have $\omega(z, S(0, r), D_r) \to 0$ as $z \to 0$. Indeed, for this to hold it is only necessary that $S(0, t)$ meet $\mathbb{C} \setminus D$ for a sufficiently “thick” set of $t$ tending to 0.
Chapter 15

Two fundamental results on asymptotic values

15.1 Transcendental singularities of the inverse function

Let $f$ be non-constant and meromorphic in the plane, let $a \in \mathbb{C}$ and $t > 0$, and let $C(t)$ be a (non-empty) component of the set \( \{ z \in \mathbb{C} : |f(z) - a| < t \} \).

Then \( 1/(f - a) \) must be unbounded on $C(t)$. To see this, assume without loss of generality that $t = 1$, and suppose that \( 1/(f - a) \) is bounded on $C(1) = D$. Then \( 1/(f - a) \) is analytic on $D$, and setting
\[
u(z) = \log \frac{1}{|f(z) - a|} \quad (z \in D), \quad u(z) = 0 \quad (z \notin D),
\]
defines a non-constant bounded subharmonic function in $\mathbb{C}$. This is a contradiction.

Assume now that we have a family of such components $C(t)$, $0 < t < t_0$, with the property that $C(t) \subseteq C(s)$ for $t < s$. Then there are two cases to consider.

Case 1: there exists $z_0 \in \bigcap_{0 < t < t_0} C(t)$.

Then evidently $f(z_0) = a$. Let $s > 0$ and pick $r$ with $0 < r < s$ such that $f - a$ has no zeros in $0 < |z - z_0| \leq r$. Let $T = \min\{|f(z) - a| : |z - z_0| = r\}$. Then $C(t) \subseteq D(z_0, r) \subseteq D(z_0, s)$ for $t < T$.

In particular, $\bigcap_{0 < t < t_0} C(t) = \{z_0\}$ and $C(t)$ is bounded for small positive $t$.

Case 2: $\bigcap_{0 < t < t_0} C(t) = \emptyset$.

For each large positive integer $n$, choose $z_n \in C(n)$ and a path $\gamma_n$ from $z_n$ to $z_{n+1}$ in $C(n)$. The union of these gives a path $\gamma(t)$ such that $f(\gamma(t)) \to a$ as $t \to \infty$.

We assert that $\gamma(t) \to \infty$ as $t \to \infty$. Assume not. Then there exist $t_m \to \infty$ with $|\gamma(t_m)| \leq M < \infty$ for all $m \in \mathbb{N}$, and without loss of generality $\gamma(t_m) \to w \in \mathbb{C}$ as $m \to \infty$. Since $f(\gamma(t_m)) \to a$ we must have $f(w) = a$. Let $L$ be a positive real number. Then $|f(z) - a| < L$ on some open neighbourhood $U_L$ of $w$. Since $\gamma(t_m) \to w$ and $\gamma(t_m) \in C(L)$ for large $m$, we have $U_L \cap C(L) \neq \emptyset$ and hence $w \in U_L \subseteq C(L)$. This contradicts the assumption that the $C(t)$ have empty intersection.

We have thus shown in Case 2 that $f(z)$ tends to $a$ along a path tending to infinity. In particular, if $f$ is transcendental then $a$ is an asymptotic value of $f$ and the $C(t)$ are said to determine a transcen-
15.2. THE DENJOY-CARLEMAN-AHLFORS THEOREM

15.2.1 Lemma

Let \( n \geq 2 \) be an integer. Let \( D_j, j = 1, \ldots, n \) be pairwise disjoint domains, and let \( u_j \) be non-constant subharmonic functions such that \( u_j \) vanishes outside \( D_j \). Assume that \( h(r) \) is a positive function such that, for each \( j \), we have \( B(r, u_j) \leq O(h(r)) \) as \( r \to \infty \). Then we have

\[
\liminf_{r \to \infty} \frac{h(r)}{r^{n/2}} > 0.
\]

Proof. Since each \( u_j \) is non-constant and vanishes outside \( D_j \), each domain \( D_j \) must be unbounded. Let \( \theta_j(t) = \theta_{D_j}(t) \) be defined as in 14.1.3. Note that if \( t \) is large then \( \theta_j(t) < 2\pi \), because \( n \geq 2 \) and the circle \( |z| = t \) meets \( D_k \) for \( k \neq j \). Theorem 14.1.7 implies that for each \( j \), as \( r \to \infty \),

\[
\pi \int_1^r \frac{dt}{t\theta_j(t)} \leq \log B(2r, u_j) + O(1) \leq \log h(2r) + O(1).
\]  

(15.1)

But, since the \( D_j \) are pairwise disjoint, the Cauchy-Schwarz inequality gives

\[
n^2 = \left( \sum_{j=1}^n \theta_j(t)^{1/2} \theta_j(t)^{-1/2} \right)^2 \leq \sum_{j=1}^n \theta_j(t)^{n} \sum_{j=1}^n \theta_j(t)^{-1} \leq 2\pi \sum_{j=1}^n \theta_j(t)^{-1}
\]

if \( t \) is large. Thus for large \( r \) we have, using (15.1),

\[
n^2 \log r - O(1) \leq 2 \pi \sum_{j=1}^n \int_1^r \frac{dt}{t\theta_j(t)} \leq 2n \log h(2r) + O(1),
\]
and this proves the lemma.

15.2.2 Theorem (Denjoy-Carleman-Ahlfors)

Suppose that $f$ is transcendental and meromorphic in the plane, and that the inverse function $f^{-1}$ has $n \geq 2$ direct transcendental singularities, lying over $a_1, \ldots, a_n$ (not necessarily distinct). Then

$$\liminf_{r \to \infty} \frac{T(r, f)}{r^{n/2}} > 0. \quad (15.2)$$

In particular, the lower order of $f$ is at least $n/2$.

Moreover, if $F$ is transcendental and meromorphic in the plane and $F^{-1}$ has a direct transcendental singularity over $\infty$ and $F(z)$ is bounded on a path tending to infinity then

$$\liminf_{r \to \infty} \frac{T(r, F)}{r^{1/2}} > 0.$$

Proof. To prove the first part assume that all the $a_j$ are finite. Thus there exists $\delta > 0$ such that for each $j = 1, \ldots, n$ we can find a non-empty component $D_j$ of the set \( \{ z \in \mathbb{C} : |f(z) - a_j| < \delta \} \), such that $f(z) \neq a_j$ on $D_j$ and such that the $D_j$ are pairwise disjoint (if $a_j = a_k$ then $D_j$, $D_k$ are distinct and so disjoint components).

For each $j$ we define a non-constant subharmonic function $u_j$ by

$$u_j(z) = \log \left| \frac{\delta}{f(z) - a_j} \right| \quad (z \in D_j), \quad u_j(z) = 0 \quad (z \notin D_j).$$

By Theorem 7.3.5,

$$B(r, u_j) = \sup \left\{ u_j(z) : |z| = r \right\}$$

satisfies

$$B(r, u_j) \leq 3 \int_0^{2\pi} u_j(2re^{it})dt \leq 3m(2r, 1/(f - a_j)) + O(1) \leq 3T(2r, f) + O(1).$$

The result then follows by applying Lemma 15.2.1 with $h(r) = T(2r, f)$.

To prove the second part, apply the first part to $f(z) = F(z^2)$, which has two direct singularities over $\infty$.

The theorem is sharp, since $e^z$ has order 1 and two direct transcendent singularities (over $0, \infty$).

15.2.3 Corollary

Let $f$ be a transcendental entire function with $n \geq 1$ finite asymptotic values. Then $f$ satisfies (15.2).

Proof. We can assume that there are simple paths $\gamma_j \to \infty, j = 1, \ldots, n$, pairwise disjoint except that each starts at 0, and such that $f(z) \to a_j \in \mathbb{C}$ as $z \to \infty$ on $\gamma_j$. (To ensure that each path is simple we can first approximate by a stepwise curve and then delete any repeated segments of the curve).
Let $\gamma_{n+1} = \gamma_n$. In the region $D_j$ between $\gamma_j$ and $\gamma_{j+1}$ the function $f$ must be unbounded, by Theorem 13.2.2. This gives us $n$ direct transcendental singularities over $\infty$, with $f$ bounded on the intermediate paths, and proves the result.

This theorem is also sharp, as

$$f(z) = \int_0^z \frac{\sin t}{t} dt$$

has order 1, and two finite asymptotic values.

## 15.3 Two lemmas needed for the Bergweiler-Eremenko theorem

The Bergweiler-Eremenko theorem is a striking result from [19] connecting the critical and asymptotic values of a meromorphic function, which has subsequently found widespread application in value distribution theory. The result shows that if $f$ is a transcendental meromorphic function of finite order then any direct transcendental singularity of $f^{-1}$ must be a limit point of critical values. We will present the subsequent modification by Hinchliffe [45], which shows that the result remains true for functions of finite lower order.

The proof will require the following lemma [57, p.287] on isolated singularities of the inverse function.

### 15.3.1 Lemma

Let $f$ be transcendental and meromorphic in the plane, and let $0 < S < \infty$, and let $C$ be a component of the set $\{z : S < |f(z)| \leq \infty\}$. Let $z_0 \in C$ with $w_0 = f(z_0)$ finite and $f'(z_0) \neq 0$, and let $g$ be that branch of the inverse function $f^{-1}$ which maps $w_0$ to $z_0$. Suppose that $g$ admits unrestricted analytic continuation in the annulus $S < |w| < \infty$, starting at $w_0$. Then $C$ is simply connected, and contains either one pole (possibly multiple) of $f$, or no pole of $f$ but instead a path $\sigma$ tending to infinity on which $f(z) \to \infty$.

**Proof.** We may assume that $S = 1$. Choose $v_0$ such that $e^{v_0} = w_0 = f(z_0)$. Then, starting at $v_0$,

$$h(v) = g(e^v) = f^{-1}(e^v)$$

admits unrestricted analytic continuation in the half-plane $U$ given by $\text{Re}(v) > 0$. By the monodromy theorem, $h$ then extends to an analytic function on $U$, with $f(h(v)) = e^v$.

Next, $h$ maps $U$ into $C$. Indeed, $h(U) = C_0 = \{z \in C : f(z) \neq \infty\}$, for if $z_1 \in C_0$ we can choose a simply connected domain $C_1$ with $\{z_0, z_1\} \subseteq C_1 \subseteq C_0$. Since $f$ maps $C_0$ into $1 < |w| < \infty$, we may define an analytic branch of $F = \log f$ on $C_1$, mapping $C_1$ into $U$. Further, $e^F = f$ maps $z_0$ to $w_0$ and $h(F)$ is the identity near $z_0$, and this remains the case throughout $C_1$ by the identity theorem. Thus $z_1 = h(F(z_1)) \in h(F(C_1)) \subseteq h(U)$.

There are now two possibilities to consider.

**Case 1:** suppose that $h$ is univalent on $U$. In this case the image under $z = h(v)$ of $\text{Re}(v) = 1$ is a simple curve $L$, on which $|f(z)| = e$. We assert that $L$ must tend to infinity in both directions. Since $f(h(1 + k\pi i)) = e$ for every integer $k$, it is clear that $h(1 + iy)$ must be unbounded as $y \to +\infty$, and as $y \to -\infty$, in both cases with $y$ real. If we have $|h(1 + iy)| \leq M < \infty$ for arbitrarily large $|y|$, with $y$ real, then there must be infinitely many points on the circle $|z| = 2M$ with $|f(z)| = e$. This is impossible, since $f$ is transcendental, and so $h(1 + iy) \to \infty$, as asserted.
Next, the function $H(v) = 1/(h(v) - h(1/2))$ is bounded on $\text{Re}(v) \geq 1$, by the open mapping theorem and the assumption that $h$ is univalent. Thus, by the Phragmén-Lindelöf principle, we have $H(v) \to 0$, and $h(v) \to \infty$, as $v \to \infty$ with $\text{Re}(v) \geq 1$. It follows that $C_0$ is an unbounded simply connected domain and for the path $\sigma$ we may take $h : [1, \infty) \to C_0$.

We deduce that $C$ cannot contain a pole of $f$. To see this, suppose $z_2$ is a pole of $f$ in $C$, and take a sequence $u_n$ in $C_0$, with $u_n \to z_2$, and $s_n \in U$ such that $h(s_n) = u_n$. Since $e^{s_n} = f(h(s_n)) = f(u_n) \to \infty$, we get $s_n \to \infty$, $\text{Re}(s_n) > 1$, and so $u_n = h(s_n) \to \infty$, which is a contradiction. Thus $C = C_0$ and $C$ is simply connected; further, $F = \log f$ may be defined on $C$, mapping $z_0$ to $v_0$, and $h(F)$ is the identity near $z_0$ and so throughout $C$, while $F(h)$ is the identity near $v_0$ and so on $U$. This completes the proof of the lemma in this case.

**Case 2:** Suppose that we have $v_1, v_2 \in U$ with $v_1 \neq v_2$, $h(v_1) = h(v_2)$. Then $e^{v_1} = f(h(v_1)) = f(h(v_2)) = e^{v_2}$ and so $v_2 = v_1 + m2\pi i$ for some integer $m$. If $v$ is close to $v_1$ then the open mapping theorem tells us that $h$ takes the value $h(v)$ at some $v'$ close to $v_2$, and we must have $(v' - v)/2\pi i \in \mathbb{Z}$ and so $v' = v + m2\pi i$. Thus $h$ has period $m2\pi i$ near $v_1$ and so throughout $U$.

Let $k$ be the smallest positive integer such that $h$ has period $k2\pi i$. In this case the function $G(\zeta) = h(k \log \zeta) = g(\zeta^k)$ is analytic and univalent in $W = \{\zeta : 1 < |\zeta| < \infty\}$, and maps $W$ onto $C_0$. To see this, just note that $G$ can be analytically continued along any path in $W$, and continuation in $W$ once around 0 leads back to the same function element, by the periodicity of $h$. Since $G$ is univalent, $z_1 = \lim_{\zeta \to \infty} G(\zeta)$ exists.

Suppose that $z_1 = \infty$. If $\tau$ is large, then $G$ takes the value $\tau$ at $\zeta$ with $\zeta$ large, and this gives $f(\tau) = f(G(\zeta)) = f(g(\zeta^k)) = \zeta^k$ so that $f(\tau)$ is large. But this gives $\lim_{\tau \to \infty} f(\tau) = \infty$, contradicting the fact that $f$ is transcendental. Thus $z_1$ is finite. The same argument shows that $f(z)$ is large for $z$ close to $z_1$, and so $z_1$ is a pole of $f$.

We now see that $G$ is univalent on $W^* = W \cup \{\infty\}$, mapping $W^*$ onto $C_0 \cup \{z_1\}$, which is therefore simply connected, so that $z_1$ is the only pole of $f$ in $C$. This may also be seen as follows: if $u_n \in C_0$ and $f(u_n) \to \infty$ take $\zeta_n \in W$ with $G(\zeta_n) = u_n$. Then $\zeta_n^k = f(g(\zeta_n^k)) = f(G(\zeta_n)) = f(u_n) \to \infty$ and so $\zeta_n \to \infty$ and $u_n = G(\zeta_n) \to z_1$.

Finally, we note that since $z = G(\zeta) = g(\zeta^k)$ is univalent on $W^*$, it follows that $\zeta = G^{-1}(z) = f(z)^{1/k}$ is meromorphic and univalent on $C$, so that $z_1$ is a pole of multiplicity $k$. This completes the proof of Lemma 15.3.1.

The next lemma [53] gives an estimate for the length of level curves of a meromorphic function, and is a slightly more precise version of [54, Lemma 2].

**15.3.2 Lemma**

Let $G$ be transcendental and meromorphic in the plane, and let $\alpha \in (1, \infty)$. For $w \in \mathbb{C}$ and positive $r$ and $R$, let $L(r, w, R, G)$ denote the length of the level curves $|G(z)| = R$ lying in $D(w, r)$, and set $L(r, R, G) = L(r, 0, R, G)$. Let $\psi(t)$ be continuous, positive and non-decreasing on $[1, \infty)$ such that

$$
\int_1^\infty \frac{1}{t \psi(t)} dt < \frac{\log \alpha}{4}.
$$

Then if the positive constant $S$ is large enough there exist uncountably many $R \in (S, 2S)$ such that

$$
L(r, R, G)^2 \leq \alpha r^2 \psi(\alpha r)(T(\alpha r, G) + \log S), \quad r \geq 1,
$$

(15.4)
in which \( c \) is a positive constant depending only on \( \alpha \).

Proof We use the length-area inequality as in [38, Theorem 2.1, p.29] (see also [70, p.44]). Let \( \Delta \) be an open disc in \( \mathbb{C} \) of area \( A \). Then

\[
\int_S^{2S} \frac{L(\Delta, R, G)^2}{p(\Delta, R, G)R} dR \leq 2\pi A,
\]

in which \( L(\Delta, R, G) \) is the length of the curves \(|G(z)| = R\) in \( \Delta \) and

\[
p(\Delta, R, G) = \frac{1}{2\pi} \int_0^{2\pi} n(\Delta, Re^{i\phi}, G) d\phi, \tag{15.6}
\]

where \( n(\Delta, a, G) \) is the number of roots of \( G(z) = a \) in \( \Delta \), counting multiplicity.

Denote by \( c_j \) positive constants depending only on \( \alpha \). Set \( \beta = \sqrt{\alpha} \) and \( r_q = \beta^q \), \( q = 0, 1, 2, \ldots \). Then (15.3) gives

\[
\sum_{q=1}^{\infty} \frac{1}{\psi(r_q)} \leq \frac{1}{\log \beta} \int_1^{\infty} \frac{1}{t \psi(t)} dt < \frac{1}{2}. \tag{15.7}
\]

Assume that \( S \) is large. Then for \( \phi \) real and for \( S \leq R \leq 2S \) we have \( \infty \geq |G(0) - Re^{i\phi}| \geq 1 \). This gives, for \( r \geq 1 \),

\[
n(D(0, r), Re^{i\phi}, G) \leq n(r, Re^{i\phi}, G) \\
\leq c_0 N(\beta r, 1/(G - Re^{i\phi})) \\
\leq c_0 T(\beta r, G - Re^{i\phi}) + C_1^* \\
\leq c_0 T(\beta r, G) + c_0 \log R + c_0 \log 2 + C_1^* \\
\leq c_0 T(\beta r, G) + c_1 \log S.
\]

Here the constant \( C_1^* \) only arises if \( G(0) = \infty \). Substituting this estimate into (15.5) and (15.6) gives, for \( r \geq 1 \),

\[
\int_S^{2S} \frac{L(r, R, G)^2}{R} dR \leq c_2 r^2 (T(\beta r, G) + \log S) .
\]

Hence if the positive constant \( c_3 \) is chosen large enough then for each \( q \in \mathbb{N} \) there exists a subset \( E_q \) of \((S, 2S)\) with

\[
\int_{E_q} \frac{dR}{R} < \frac{\log 2}{\psi(r_q)} \tag{15.8}
\]

such that for all \( R \in (S, 2S) \setminus E_q \) and for \( r_{q-1} < r \leq r_q \) we have

\[
L(r, R, G)^2 \leq L(r_q, R, G)^2 \\
\leq c_3 r_q^2 \psi(r_q) (T(\beta r_q, G) + \log S) \\
\leq c_4 r^2 \psi(\alpha r) (T(\alpha r, G) + \log S).
\]

Since (15.7) and (15.8) give

\[
\int_{\bigcup_{q=1}^{\infty} E_q} \frac{dR}{R} < \frac{\log 2}{2}, \tag{15.9}
\]

(15.4) follows.
15.4 The Bergweiler-Eremenko theorem: preliminaries

Following [19, 45], the key step is to prove the following proposition.

15.4.1 Proposition

Let $f$ be transcendental and meromorphic of finite lower order in the plane, such that $f^{-1}$ has an indirect transcendental singularity over $0$. Let the components $C(t)$ be as in §15.1. Then for every $t > 0$ the component $C(t)$ contains infinitely many zeros of $f'$.

The proof of Proposition 15.4.1 will take up the whole of this section. Assume throughout that $f$ is transcendental and meromorphic of finite lower order in the plane, and that $f^{-1}$ has an indirect transcendental singularity over $0$, such that $C(\varepsilon)$, for some $\varepsilon > 0$, contains finitely many zeros of $f'$. By reducing $\varepsilon$, if necessary, it may be assumed that $C(\varepsilon)$ contains no zeros of $f'$.

15.4.2 Lemma

Let $0 < \delta < \varepsilon$. Let $z_1 \in C(\delta)$, with $f(z_1) = 0$. Then there exist $a$ with $0 < |a| = r < \delta$ and a simply connected domain $D \subseteq C(\delta)$, such that $f$ maps $D$ univalently onto $D(0, r)$, and $D$ contains a path $\sigma$ tending to infinity on which $f(z) \to a$ as $z \to \infty$, mapped by $f$ onto the line segment $w = ta, 0 \leq t < 1$.

Proof. Let $g$ be that branch of the inverse function $f^{-1}$ which maps $0$ to $z_1$. Next, let $r$ be the supremum of positive real $t$ such that $g$ extends to be analytic in $D(0, t)$. We have $r > 0$, since $f$ is univalent on a neighbourhood of $z_1$. Further, $g$ is analytic on $D(0, r)$, and univalent there, since $g(w_1) = g(w_2)$ gives $w_1 = f(g(w_1)) = f(g(w_2)) = w_2$. Moreover, $D = g(D(0, r))$ is a simply connected domain and $|f(z)| \to r$ as $z$ tends to the finite boundary $\partial D$, and so $D$ is that component of the set $\{z : |f(z)| < r\}$ which contains $z_1$. It follows that $r < \delta$, for otherwise we would have $C(\delta) \subseteq D$, which contradicts the fact that $C(\delta)$ contains infinitely many zeros of $f$. In particular, we now have $D \subseteq C(\delta)$.

Now suppose that, for every $a$ with $|a| = r$, the branch $g$ of $f^{-1}$ can be analytically continued along the line segment $w = ta, 0 \leq t \leq 1$. Then each such continuation defines an extension $h_a$ of $g$ to a disc $U_a = D(a, d_a), d_a > 0$. If $U_a \cap U_b \neq \emptyset$, then $h_a = h_b = g$ on the non-empty intersection $U_a \cap U_b \cap D(0, r)$. Since $U_a \cap U_b$ is connected we get $h_a = h_b$ on $U_a \cap U_b$. But the circle $|w| = r$ is compact, and can be covered by finitely many such $U_a$, from which it follows that $g$ extends analytically to a disc $D(0, r_1), r_1 > r$, and this is a contradiction.

It follows that there is some $a$ with $|a| = r$ such that $g$ does not admit analytic continuation along the path $w = ta, 0 \leq t \leq 1$. Now the path $g(ta), 0 \leq t < 1$, lies in $D$ and so in $C(\delta)$, and so does its closure in the finite plane, since $r < \delta$. It follows that, as $t \to 1-$, $g(ta)$ must tend either to infinity or to a critical point of $f$, and the latter is ruled out since $f$ has no critical points in $C(\delta)$. Thus we obtain the path $\sigma$.

15.4.3 Lemma

There exist points $z_j \to \infty$, $z_j \in C(\varepsilon)$, and distinct complex numbers $a_j$ with $0 < |a_j| < \varepsilon / 2$, and pairwise disjoint simply connected domains $D_j \subseteq C(\varepsilon)$, with $0 \notin D_j$, with the following properties. First, $f$ maps $D_j$ univalently onto $D(0, r_j)$, with $f(z_j) = 0$. Second, each $D_j$ contains a path $\sigma_j \to \infty$ on which $f(z) \to a_j$ as $z \to \infty$, and the path $\sigma_j$ is mapped by $f$ onto the line segment $w = ta_j, 0 \leq t < 1$.

Proof. The $z_j, a_j$ will be defined inductively. Take $z_1 \in C(\frac{1}{2} \varepsilon)$ with $f(z_1) = 0$, and let $a = a_1, D =$
15.4.1 \(D_1, \sigma = \sigma_1\) be as in Lemma 15.4.2. Assuming that \(z_{n-1}, D_{n-1}\) have already been determined, we need only take \(z_n \in C(\frac{1}{2} r_{n-1})\), with \(f(z_n) = 0\) and \(z_n \neq z_j, 1 \leq j \leq n - 1\), and determine \(D_n, r_n, a_n, \sigma_n\) as in Lemma 15.4.2. We assert that the \(D_j\) are pairwise disjoint. If \(m < n\) and \(D_n\) meets \(D_m\) then, since \(D_n\) is a component of the set \(\{ z : |f(z)| < r_n \}\), we have \(D_n \subseteq D_m\). But this is a contradiction since \(z_n \neq z_m\) and \(f\) is univalent on \(D_m\). It now follows that the \(D_j\) may be chosen so that \(0 \notin D_j\), by deleting one of the \(D_j\) if necessary.

15.4.4 Lemma

Let the \(z_j, a_j\) and \(D_j\) be as in Lemma 15.4.3. For \(t > 0\), let \(t\theta_j(t)\) be the length of the longest open arc of \(|z| = t\) which lies in \(D_j\). As \(z\) tends to infinity on \(\sigma_j\), we have

\[
\log \frac{r_j}{|f(z) - a_j|} \geq \int_{|z|=t} |z| \frac{dt}{t\theta_j(t)} - \log 2. \tag{15.10}
\]

**Proof.** The function \(h_j(z) = f(z)/r_j\) maps \(D_j\) univalently onto \(\Delta_j\), with \(z_j\) mapped to 0. By §6.2.3 and Lemma 6.2.5 we then have

\[
\log \left| \frac{1 + |h_j(z)|}{1 - |h_j(z)|} \right| = [z_j, z]_{\partial D_j} \geq \int_{|z|=t} |z| \frac{dt}{t\theta_j(t)}. \tag{15.11}
\]

But \(f\) maps \(\sigma_j\) onto the line segment \(w = ta_j, 0 \leq t < 1\), and so \(1 - |h_j(z)| = |f(z) - a_j|/r_j\). Since \(\log 2 \geq \log(1 + |h_j(z)|)\), (15.10) now follows from (15.11).

15.4.5 Lemma

Let \(u\) lie on \(\sigma_j\). Then there exists \(v\) on \(\sigma_j\), with \(|u| \leq |v| \leq |u| + 1\), such that

\[
\max\{|f(v) - a_j|, |f'(v)|\} \leq |f(u) - a_j|.
\]

**Proof.** Starting at \(u\), follow \(\sigma_j\) in the direction in which \(|f(z) - a_j|\) decreases. Then \(\sigma_j\) describes an arc \(\gamma\) joining the circle \(|z| = |u|\) and \(|z| = |u| + 1\). Then the inverse function \(g = f^{-1}\) maps a sub-segment \(I\) of \([f(u), a_j]\) onto \(\gamma\), and so

\[
1 \leq \left| \int_I g'(\zeta) d\zeta \right| \leq |f(u) - a_j| \max\{|g'(\zeta)| : \zeta \in I\}.
\]

15.4.6 A sequence on which \(T(r, f')\) grows slowly

Since \(f\) has finite lower order, so has \(f'\), and hence there exist a real number \(M > 12\) and a sequence \((s_n)\) tending to infinity such that

\[
T(s_n^5, f') + T(s_n^5, 1/f') \leq s_n^M. \tag{15.12}
\]

Let \(N, K\) and \(L\) be integers, with \(N/M, K/N\) and \(L/K\) large. Set

\[
G(z) = z^N f'(z), \tag{15.13}
\]
and apply Lemma 15.3.2 to $1/G$, with $\alpha = e^8$ and $\psi(t) = t$. This gives a small positive $\eta$ such that $G$ has no critical values $w$ with $|w| = \eta$ and such that

$$L(r, \eta, G)^2 = O(r^3T(\alpha r, G)) = O(r^3T(\alpha r, f'))$$

as $r \to \infty$. In particular, since $M > 12$,

$$L(s^4_n, \eta, G) = O(s^6_nT(\alpha s^4_n, f')^{1/2}) = O(s^6_n + M/2) \leq s^M_n$$

(15.14)

as $n \to \infty$.

15.4.7 Lemma

For each large $n$ there exist $t_n, T_n$ satisfying

$$s^1_n - 1 \leq t_n \leq s^1_n, \quad s^2_n \leq T_n \leq s^2_n + 1,$$  

(15.15)

such that

$$\max \{|\log |f'(z)|| : z \in S(0, t_n) \cup S(0, T_n)\} \leq s^M_n + 1.$$  

(15.16)

Proof. By (15.12) and standard estimates (see §3.2.7), the number of zeros and poles of $f'$ in $s^4_n/4 \leq |z| \leq s^4_n$, counting multiplicity, is

$$q_n \leq n(s^4_n, f') + n(s^4_n, 1/f') \leq s^M_n.$$

Label these zeros and poles as $w_1, \ldots, w_q$ and let $U_n$ be the union of the discs $D(w_j, s^{-M-1}_n)$, $j = 1, \ldots, q_n$. Then the discs of $U_n$ have sum of radii at most $s^{-1}_n$ and so for large $n$ it is possible to choose $t_n, T_n$ satisfying (15.15) and such that the circles $S(0, t_n), S(0, T_n)$ do not meet $U_n$. But then standard estimates based on the Poisson-Jensen formula give, for $z \in S(0, t_n) \cup S(0, T_n)$,

$$|\log |f'(z)|| \leq O(T(s^3_n, f')) + O(q_n \log s_n) \leq s^M_n + 1$$

if $n$ is large enough.

15.4.8 Lemma

Let $\tau > 0$ and for large $n$ let $t_n$ and $T_n$ be as in Lemma 15.4.7. Provided the positive integer $N$ was chosen large enough, any component $C_n$ of the set

$$\{z \in \mathbb{C} : t_n < |z| < T_n, |G(z)| < \eta\}$$

satisfies

$$\text{diam } f(C) < \tau.$$

Proof. Fix $z^* \in C$ and choose any $z \in C$. Join $z^*$ to $z$ by a path $\lambda$ in the closure of $C$ consisting of part of the ray $\arg u = \arg z^*, |u| > t_n$, part of the circle $|u| = |z|$, and part of $\partial C$. Since

$$|f'(u)| \leq \eta |u|^{-N} \leq \eta s_n^{-N/4}$$

on $\lambda$ this gives, using (15.14),

$$|f(z) - f(z^*)| \leq 2\pi \eta |z|^{1-N} + \eta \int_{t_n}^{\infty} t^{-N} dt + s_n^M \eta s_n^{-N/4} = o(1),$$

which proves the lemma.
15.4.9 Lemma

Let $Q \in \mathbb{N}, Q \geq 4L$. Let $n \in \mathbb{N}$ be large and let $E_1, \ldots, E_Q$ be pairwise disjoint domains such that for each $j$ and each $t > 0$ the circle $S(0, t)$ is not contained in $E_j$. For $t > 0$ let $\phi_j(t)$ be the angular measure of $S(0, t) \cap E_j$.

Then at least $Q - 2L$ of the domains $E_1, \ldots, E_Q$ are such that

$$
\pi \int_{[4s_n^{1/2}, s_n^2/4]} \frac{dt}{t \phi_j(t)} > K \log s_n \quad \text{and} \quad \pi \int_{[4s_n, s_n^2/4]} \frac{dt}{t \phi_j(t)} > K \log s_n. \tag{15.17}
$$

Proof. Assume that at least $L$ of the $E_j$, without loss of generality $D_1, \ldots, E_L$, are such that the second inequality of (15.17) fails (in particular, the closure of each of these $E_j$ must therefore meet $S(0, 4s_n)$ and $S(0, s_n^2/4)$, because otherwise the given integral would evidently be infinite, and hence each of these $E_j$ must meet $S(0, t)$ for $4s_n < t < s_n^2/4$). By our assumption,

$$
\pi \int_{[4s_n, s_n^2/4]} \sum_{j=1}^L \frac{dt}{t \phi_j(t)} \leq LK \log s_n. \tag{15.18}
$$

But the Cauchy-Schwarz inequality gives, for $t \in (4s_n^{1/2}, s_n^2/4)$,

$$
L^2 \leq \left( \sum_{j=1}^L \phi_j(t) \right) \left( \sum_{j=1}^L \frac{1}{\phi_j(t)} \right) \leq 2\pi \left( \sum_{j=1}^L \frac{1}{\phi_j(t)} \right)^2.
$$

On integrating from $4s_n$ to $s_n^2/4$ and using (15.18), this leads to

$$
L^2 \log(s_n/16) \leq 2LK \log s_n,
$$

an obvious contradiction if $n$ is large enough, since $L$ and $K$ were chosen in §15.4.6 with $L/K$ large. The same argument shows that it is not possible for the first inequality of (15.17) to fail for $L$ of the $E_j$.

15.4.10 Completion of the proof of Proposition 15.4.1

Let $a_1, \ldots, a_{6L}$ be as in Lemma 15.4.3, and choose $\tau$ such that

$$
0 < \tau < \varepsilon/4, \quad 4\tau < \min \{|a_j - a_{j'}| : 1 \leq j < j' \leq 6L\}. \tag{15.19}
$$

Let $n \in \mathbb{N}$ be large, and apply Lemma 15.4.9 to the domains $D_1, \ldots, D_{6L}$ corresponding to $a_1, \ldots, a_{6L}$ as in Lemma 15.4.3. Let $u_j \in \sigma_j$ with $|u_j| = s_n$. Then applying Lemma 15.4.4 gives

$$
\log \frac{r_j}{|f(u_j) - a_j|} \geq K \log s_n - O(1),
$$

by (15.10) and (15.17) for at least $4L$ of the $a_j$, which after re-labelling we may assume are $a_1, \ldots, a_{4L}$. Applying Lemma 15.4.5 now gives $v_j$ satisfying

$$
v_j \in D_j, \quad s_n \leq |v_j| \leq s_n + 1, \quad \max \{|f(v_j) - a_j|, |f'(v_j)|\} \leq s_n^{1-K}, \quad j = 1, \ldots, 4L. \tag{15.20}
$$

Since $K/M$ is large this gives, in particular, $|G(v_j)| < \eta$, where $G$ and $\eta$ are as in (15.13) and (15.14). Let $t_n$ and $T_n$ be as in Lemma 15.4.7, and let $C_j$ be that component of the set

$$
\{z \in \mathbb{C} : t_n < |z| < T_n, |G(z)| < \eta\}
$$
which contains \( v_j \). Then the diameter of \( f(C_j) \) is at most \( \tau \) by Lemma 15.4.8, and

\[
|f(z) - a_j| \leq \tau + s_n^{1-K} \leq \tau + o(1)
\]

for all \( z \in C_j \). In particular this implies using (15.19) that the following hold:
(a) \( C_1, \ldots, C_{4L} \) are pairwise disjoint;
(b) each of \( C_1, \ldots, C_{4L} \) lies in \( C(\varepsilon) \) and so contains no zeros of \( f' \);
(c) for \( 1 \leq j < 4L \) the component \( C_j \) does not meet the path \( \sigma_j \) of Lemma 15.4.3, since
\[
f(z) = a_j + o(1)
\]
for \( z \in \sigma_j, |z| > t_n \), and in particular \( C_j \) cannot contain a circle \( S(0, t), t > 0 \).

Lemma 15.4.9 may now be applied again, this time with \( E_j = C_j \) and \( \phi_j(t) \) the angular measure of \( C_j \cap S(0, t) \), and it may be assumed without loss of generality that (15.17) holds for \( j = 1, \ldots, 2L \).

The Carleman-Tsuji estimate for harmonic measure (§14.1.6) and the conformal invariance of harmonic measure now give

\[
\omega(v_j, C_j, S(0, T_n) \cup S(0, t_n)) \leq c_1 \exp \left( -\pi \int_{2\|v_j\|}^{T_n/2} \frac{dt}{t\phi_j(t)} \right) + c_1 \exp \left( -\pi \int_{2\|v_j\|}^{t_n/2} \frac{dt}{t\phi_j(t)} \right) \leq c_2 s_n^{-K},
\]

using (15.15), (15.17) and (15.20), in which \( c_1, c_2 \) are positive constants independent of \( j \) and \( n \).

Since

\[
|f'(z)| = \eta|z|^{-N} \geq \eta T_n^{-N} \geq \frac{1}{2} \eta s_n^{-2N} \quad \text{for} \quad z \in \partial C_j \setminus (S(0, t_n) \cup S(0, T_n)),
\]

the two constants theorem 9.2.10 may be applied to \( \log |1/f'(z)| \), which is subharmonic on \( C_j \) by (b).

This gives, using (15.16), (15.20) and (15.21),

\[
(K - 1) \log s_n \leq \log \frac{1}{|f'(v_j)|} \leq c_2 s_n^{M+1-K} + 2N \log s_n + O(1),
\]

a contradiction if \( n \) is large enough, since \( K \) and \( N \) were chosen in §15.4.6 with \( K/N \) large.

### 15.5 Statement and proof of the Bergweiler-Eremenko theorem

#### 15.5.1 Theorem

Let \( f \) be transcendental and meromorphic of finite lower order in the plane, with an indirect transcendental singularity over \( a \in \mathbb{C} \). Then for every \( t > 0 \), the corresponding component \( C(t) \) contains infinitely many critical points \( z \) of \( f \) with \( f(z) \neq a \).

In particular, if \( f \) has finite lower order and finitely many critical values then every every asymptotic value of \( f \) corresponds to a direct transcendental singularity of the inverse function \( f^{-1} \).

**Proof.** Assume the contrary. Then there is some \( \varepsilon > 0 \) such that the only critical points of \( f \) in \( C(\varepsilon) \) are zeros of \( f - a \). Assume without loss of generality that \( a = 0 \). Since \( f \) has finite lower order, \( f \) can have only finitely many direct transcendental singularities, by the Denjoy-Carleman-Ahlfors theorem, and we assume that \( \varepsilon \) is so small that there is no \( w \) with \( 0 < |w| < \varepsilon \) such that \( f^{-1} \) has a direct transcendental singularity over \( w \).

Take \( z_0 \in C(\varepsilon) \), with \( f(z_0) = w_0 \neq 0 \), and a path \( \gamma : [0, 1] \to \{ w : \delta \leq |w| \leq \varepsilon - \delta \} \), with \( \delta \) positive but small compared to \( |w_0| \), such that \( \gamma \) starts at \( w_0 \). Let \( g \) be that branch of \( f^{-1} \) mapping \( w_0 = f(z_0) \) to \( z_0 \), and suppose that analytic continuation of \( g \) along \( \gamma \) is not possible. Then there exists
$S \in [0,1]$ such that as $t \to S^-$, $z = g(\gamma(t))$ either tends to infinity or to a critical point $z_1$ of $f$ with $\delta \leq |f(z_1)| \leq \varepsilon - \delta$. But the latter may be excluded since $g(\gamma(t)) \in C(\varepsilon)$ for $0 \leq t < S$, which implies, since $|f(z_1)| \leq \varepsilon - \delta$, that $z_1 \in C(\varepsilon)$, which is impossible by assumption. It follows that the path $\sigma$ given by $z = g(\gamma(t)), 0 \leq t < S$, is a path tending to $\infty$, and lying in $C(\varepsilon)$, on which $f(z) \to w_1$ as $z \to \infty$, with $\delta \leq |w_1| \leq \varepsilon - \delta$. But then an unbounded subpath of $\sigma$ lies in a component $C'$ of the set $\{z : |f(z) - w_1| < \delta/2\}$, and $C' \subseteq C(\varepsilon)$. Hence $f'$ has no zeros on $C'$. Further, the singularity over $w_1$ must be indirect, since we have excluded direct singularities with $0 < |w| < \varepsilon$, and this contradicts Proposition 15.4.1.

Since $\delta$ may be chosen arbitrarily small, we now see that $g$ admits unrestricted analytic continuation in $0 < |w| < \varepsilon$. But, using Lemma 15.3.1, this implies that $C(\varepsilon)$ is simply connected, and contains at most one zero of $f$, which contradicts the definition of an indirect singularity.

15.5.2 Theorem

Let $f$ be transcendental and meromorphic in the plane.

(a) Suppose that $f'$ has finitely many zeros. Then

$$\liminf_{r \to \infty} \frac{T(r, f)}{r} > 0.$$  \hspace{1cm} (15.22)

(b) Suppose that $f'/f$ has finitely many zeros. Then

$$\liminf_{r \to \infty} \frac{T(r, f)}{r^{1/2}} > 0.$$  \hspace{1cm} (15.23)

If, in addition, $f$ has finitely many poles, then (15.22) holds.

Theorem 15.5.2 is Hinchliffe’s refinement [45] of results from [24, 25]. The elementary examples $\tan z, \tan^2 \sqrt{z}$, as well as examples of larger order constructed in [24] using Riemann surfaces, show that both parts of the theorem are sharp. The proof here will be based on the unified approach given in [19], and in particular on the following lemma.

15.5.3 Lemma

Let $f$ be transcendental and meromorphic in the plane such that $f$ has infinitely many zeros but no asymptotic values in $A = \{w \in \mathbb{C} : 0 < |w| < \infty\}$. Then $f$ has infinitely many critical points $z$ with $f(z) \in A$.

Proof. Let $a \in \mathbb{C}$ with $f(a) = 0$ and let $m \in \mathbb{N}$ be the order of the zero of $f$ at $a$. Let $g(z) = f(z)^{1/m}$ near $a$ and let $h$ be the branch of $g^{-1}$ mapping $0$ to $a$. Let $t$ be the supremum of positive $t$ such that $h$ admits unrestricted analytic continuation in $|w| < t$. Then $h$ extends to be analytic on $D(0, r)$ and hence $r$ must be finite, since otherwise $h$ is a univalent entire function and so a linear function, from which it follows that $f$ is a rational function, which is a contradiction.

A compactness argument then gives $b$ with $|b| = r$ such that $h$ cannot be analytically continued along the closed line segment $[0, b]$. As $t \to 1^-$ with $t \in (0, 1)$ the preimage $z = h(tb)$ cannot tend to infinity, because otherwise we obtain a path tending to infinity on which $g(z)$ tends to $b$ and $f(z)$ tends to $b^m \in A$, a contradiction. Hence there exist a sequence $t_m \in (0, 1)$ with $t_m \to 1$ such that $z_m = h(t_m b)$ tends to $z^* \in \mathbb{C}$, and $g(z^*) = b, f(z^*) = b^m \in A$. Thus $z^*$ must be a critical point of $f$, because otherwise $h$ could be continued along $[0, b]$ all the way to $b$.

Since $f$ has infinitely many zeros and since a critical point of $f$ can be associated as above to at most finitely many zeros of $f$, it follows that $f$ has infinitely many such critical points $z^*$ with $f(z^*) \in A$. 
15.5.4 Proof of Theorem 15.5.2

Let \( f \) be transcendental and meromorphic in the plane such that that \( f'/f \) has finitely many zeros (which is obviously the case if \( f' \) has finitely many zeros) but \( f \) does not satisfy (15.22). Then \( f \) has finitely many critical values and by Theorem 15.5.1 every asymptotic value of \( f \) corresponds to a direct transcendental singularity of the inverse function \( f^{-1} \). By the Denjoy-Carleman-Ahlfors theorem, \( f \) has at most one asymptotic value.

Assume first that \( f' \) has finitely many zeros. Choose \( a \in \mathbb{C} \) such that \( f - a \) has infinitely many zeros. Applying Lemma 15.5.3 shows that \( f \) has a finite asymptotic value \( b \neq a \). Thus \( \infty \) is not an asymptotic value of \( f \), and so by Iversen's theorem \( f \) must have infinitely many poles. Hence the function

\[
g(z) = \frac{1}{f(z) - b}
\]

has infinitely many zeros and asymptotic value \( \infty \). By Lemma 15.5.3 the function \( g \) also has a finite non-zero asymptotic value, contradicting the fact that \( f \) has at most one asymptotic value. This proves part (a).

To prove part (b) we may assume without loss of generality that \( f \) has infinitely many zeros, since otherwise the result follows from part (a). Hence Lemma 15.5.3 shows that \( f \) has a finite asymptotic value \( b \neq 0 \), and again \( \infty \) is not an asymptotic value of \( f \), and \( f \) has infinitely many poles, which proves the last assertion of part (b).

Now choose \( \delta > 0 \) such that \( f \) has no critical or asymptotic values in \( 0 < |w - b| < 2\delta \). Then there exists a component \( C \) of the set \( \{ z \in \mathbb{C} : |f(z) - b| < \delta \} \) containing a path tending to infinity on which \( f(z) \) tends to \( b \), and by Lemma 15.3.1 the component \( C \) is simply connected. By the choice of \( \delta \) the boundary of \( C \) is a union of simple curves each tending to infinity in both directions, and so \( 1/(f(z) - b) \) is bounded on a path tending to infinity. Thus \( f \) satisfies (15.23) by the Denjoy-Carleman-Ahlfors theorem. This completes the proof of part (b).
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