The Spin Foam Lectures
1: Introduction and Spin Networks

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v2

Spin foam models

- Quantum gravity without matter
- Not realistic physics
- Models quantum space-time (technology, concepts)
- Observables
- Planck scale structure

Planck scale

- Planck area $= G\hbar$ is only scale
- Discrete structure at Planck scale (superpositions)
- Discreteness compatible with symmetries (c.f. angular momentum)
- General relativity in $G\hbar \to 0$ limit
- Continuum quantum picture? May not exist

3d QG: History

- Ponzano, Regge 1968 (3d gravity state sum model, SU(2))
- Penrose 1970 (Spin networks, SU(2))
- Witten 1989 (3d gravity functional integral)
- Turaev, Viro 1991 (3d gravity $\Lambda > 0$ ssm, $\mathrm{U}_q\mathrm{sl2}$)
- JWB 2002 (3d gravity with observables)

Spin networks

Representations of a group/Hopf algebra G

$$X, Y, \dots, X \otimes Y, \dots$$

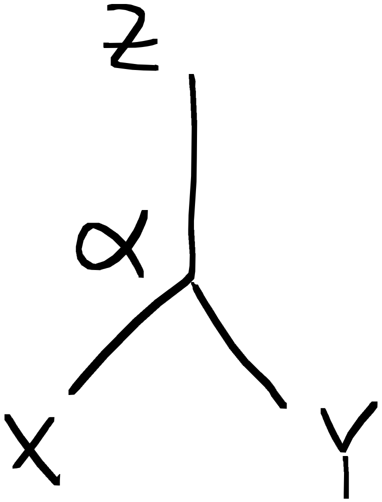
Intertwiners

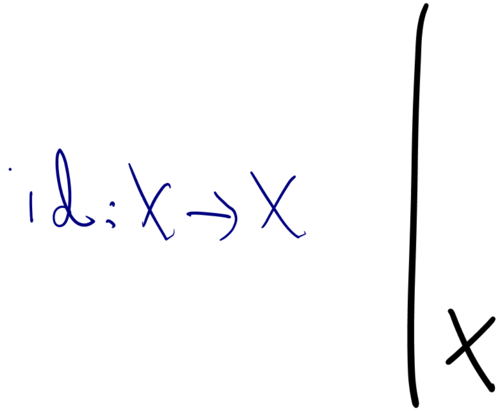
$$\alpha \colon X \to Y$$

$$\alpha(gx) = g\alpha(x), \qquad g \in G, x \in X.$$

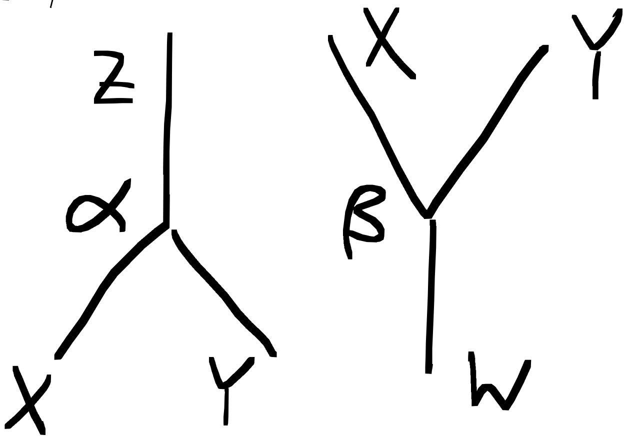
Diagrams

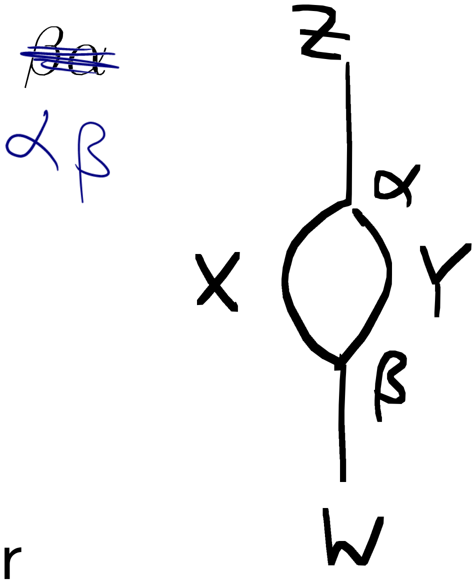
$\alpha\colon X \otimes Y \to Z$





$\alpha \otimes \beta$



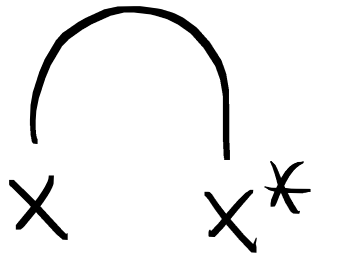


Equivalence of diagrams... see later

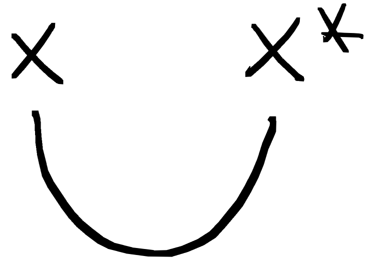
Duals

Data: for any X, a dual representation X^*, and maps

$$X \otimes X^* \xrightarrow{\epsilon} \mathbb{C}$$

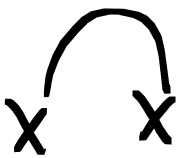


$$\mathbb{C} \xrightarrow{\iota} X \otimes X^*$$



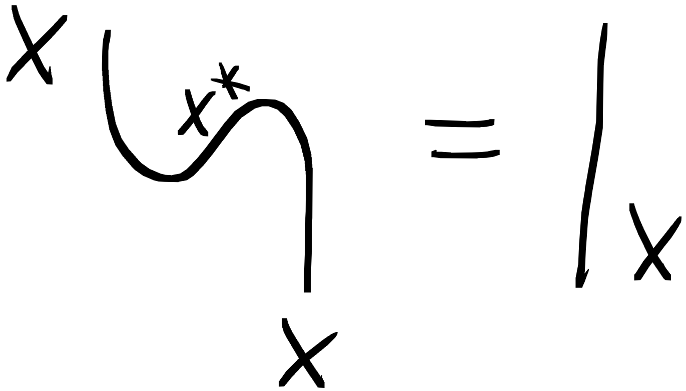
Always, $X^{**} = X$.

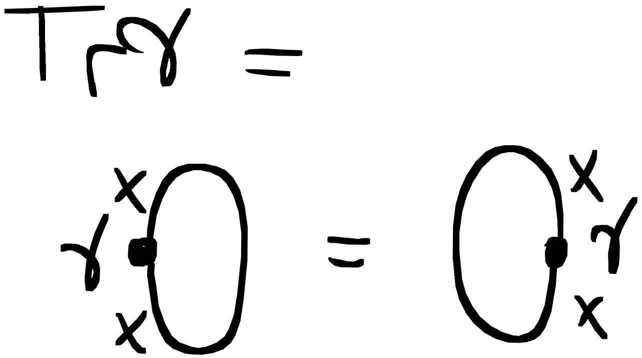
Examples

- $X^* =$ canonical dual
- $X = X^*$,  $=$ inner product

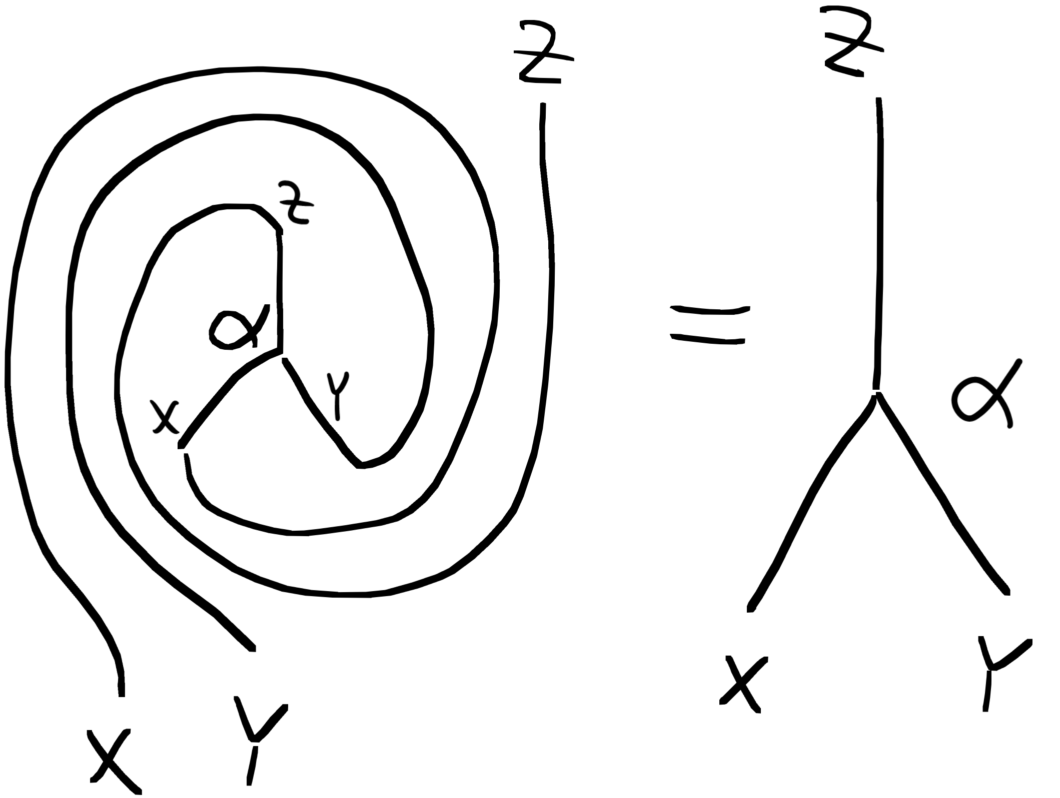
Spherical symmetry

Symmetries: diffeomorphisms of S^2. Examples:



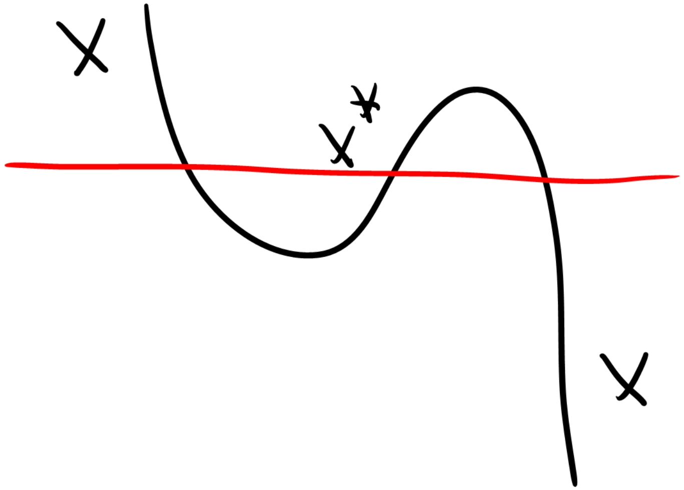


Spherical condition

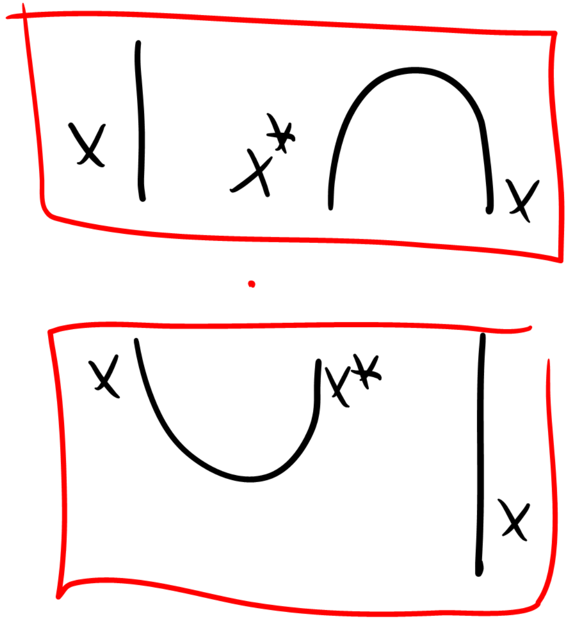


Pivotal condition

Snake



$=$

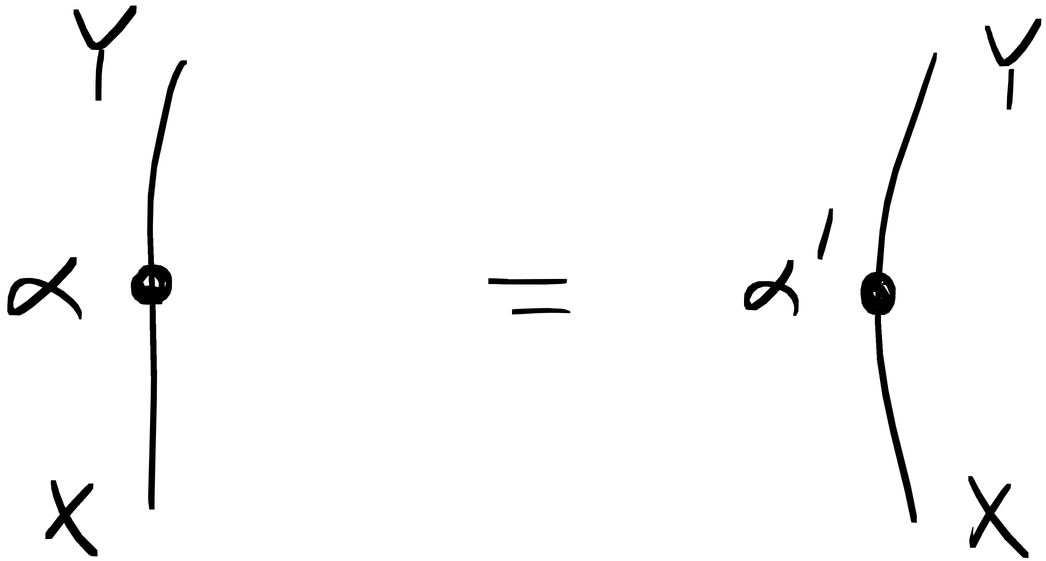


$X \otimes X^* \otimes X \to X$

$X \to X \otimes X^* \otimes X$

$$= (1_X \otimes \varepsilon_X) \cdot (\eta_X \otimes 1_X) = 1_X$$

Equivalence of diagrams



if $\mathrm{Tr}(\beta\alpha) = \mathrm{Tr}(\beta\alpha')$ for all $\beta\colon Y \to X$.

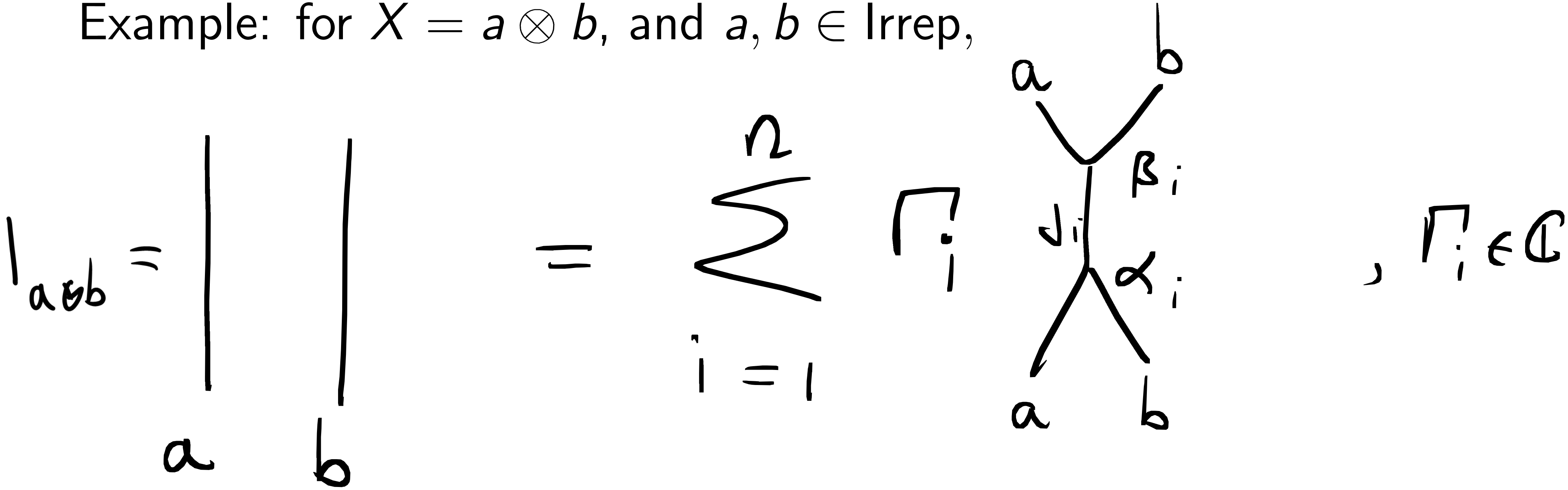
- Diagrams are equivalence classes of intertwiners
- A diagram is 0 if any closed diagram containing it is 0.
- Equivalence clear if only closed diagrams used

Semisimple condition

There is a list of irreducible representations $j_1, j_2, \ldots$
For any X,

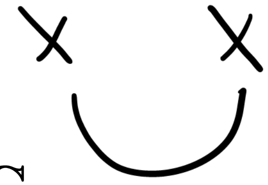
$$X \cong \bigoplus_{i=1}^{n} j_i$$

Example: for $X = a \otimes b$, and $a, b \in \mathrm{Irrep}$,

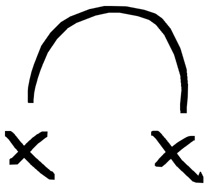


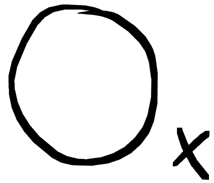
- Quantum group: semsimple after equivalence

Exercises

1. If $X = X^* = \mathbb{C}^2$, with basis u and d. Suppose 

is $1 \to Au \otimes d - A^{-1} d \otimes u$, for a constant $A \in \mathbb{C}$.

Calculate 

and the number 

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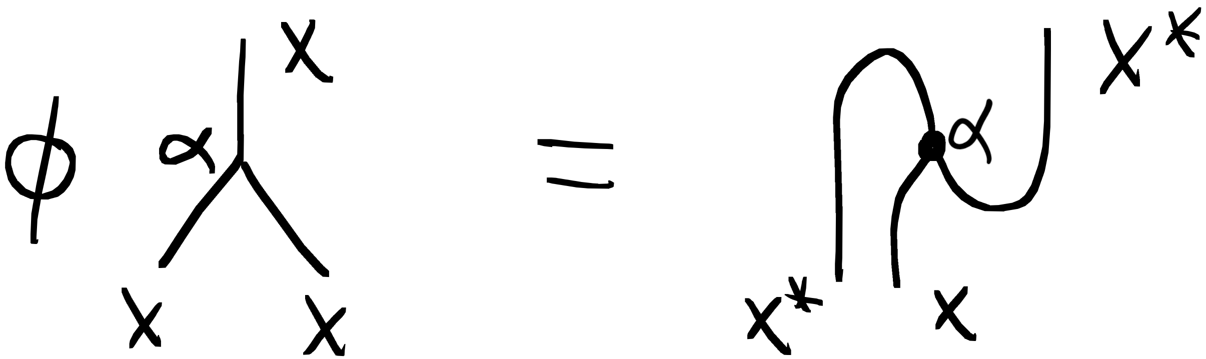
2. Denote the space of intertwiners between $X \otimes X$ and X by $\mathrm{Hom}(X \otimes X, X)$. Define a linear map $\phi \colon \mathrm{Hom}(X \otimes X, X) \to \mathrm{Hom}(X^* \otimes X, X^*)$ by



Why is ϕ invertible? If in addition $X = X^*$, what are the possible eigenvalues of ϕ?

Exercises

3. Suppose $X = \mathbb{C}^2$, and the set of all intertwiners $X \to X$ is

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\}$$

   and the trace is equal to the matrix trace. Which diagrams $X \to X$ are equivalent to zero?

4. Prove that

$$\mathrm{Tr}(\mathrm{id}_X) = \mathrm{Tr}(\mathrm{id}_{X^*}).$$

   This number is called the *quantum dimension* of X, and is written $\dim_q X$.

Spin Network References

- Penrose: Angular momentum: an approach to combinatorial space-time
- Moussouris: PhD thesis, Oxford University 1983
- Major: A spin network primer
- Kauffman: Spin networks and the bracket polynomial