The Spin Foam Lectures
1b: Spherical categories

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Categories

A category $\mathcal{C}$ is a set of 'objects' $\mathcal{C}_0$ and a set of 'morphisms' $\mathcal{C}_1$. There are two maps $s, t \colon \mathcal{C}_1 \to \mathcal{C}_0$ giving the source and target objects $A = s\phi$, $B = t\phi$ for each morphism ϕ,

$$A \xrightarrow{\phi} B.$$

There is a composition $\phi \cdot \psi$ on $\mathcal{C}_1$ defined if

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

This is associative and each object has a unit morphism,

$$1_A \cdot \phi = \phi, \qquad \phi \cdot 1_B = \phi.$$

Functors and natural transformations

A functor $\mathcal{C} \to \mathcal{D}$ is a pair of maps

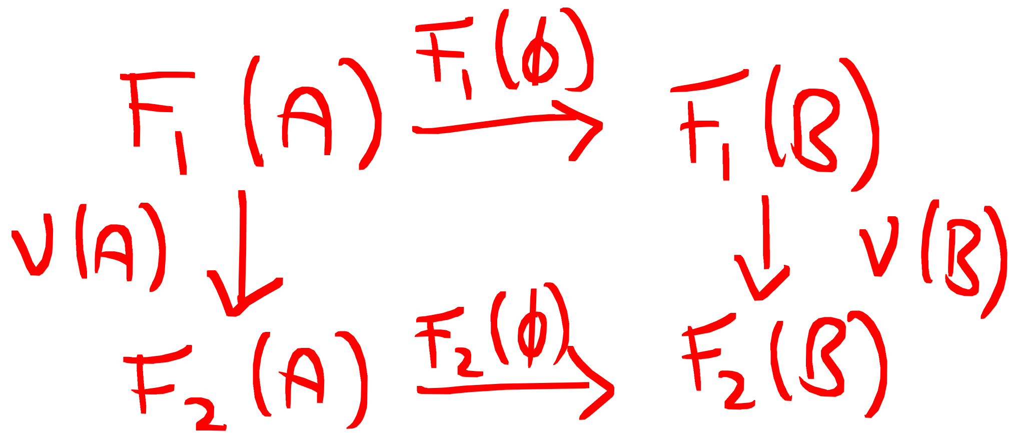
$$\mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad \mathcal{C}_1 \to \mathcal{D}_1$$

which commute with $s, t, \cdot, 1_\bullet$.

A natural transformation between two functors $F_1, F_2 \colon \mathcal{C} \to \mathcal{D}$ is a map

$$\nu \colon \mathcal{C}_0 \to \mathcal{D}_1$$

such that



commutes, for all $\phi \colon A \to B$ in $\mathcal{C}$. *cf.* Homotopy

Monoidal category with duals

A strict monoidal category with duals is a category $\mathcal{C}$ with functors

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

$$*\colon \mathcal{C}^{op} \to \mathcal{C}$$

$$e\colon \mathbb{I} \to \mathcal{C}$$

one obj, one morph.
i.e $e \in C_0$

satisfying axioms

$$(\phi \otimes \psi) \otimes \rho = \phi \otimes (\psi \otimes \rho)$$

i.e
$(\otimes \times id)\otimes = (id \times \otimes)\otimes$
$C \times C \times C \to C$

$$1_e \otimes \phi = \phi$$

$$\phi \otimes 1_e = \phi$$

$$\phi = \phi^{**}$$

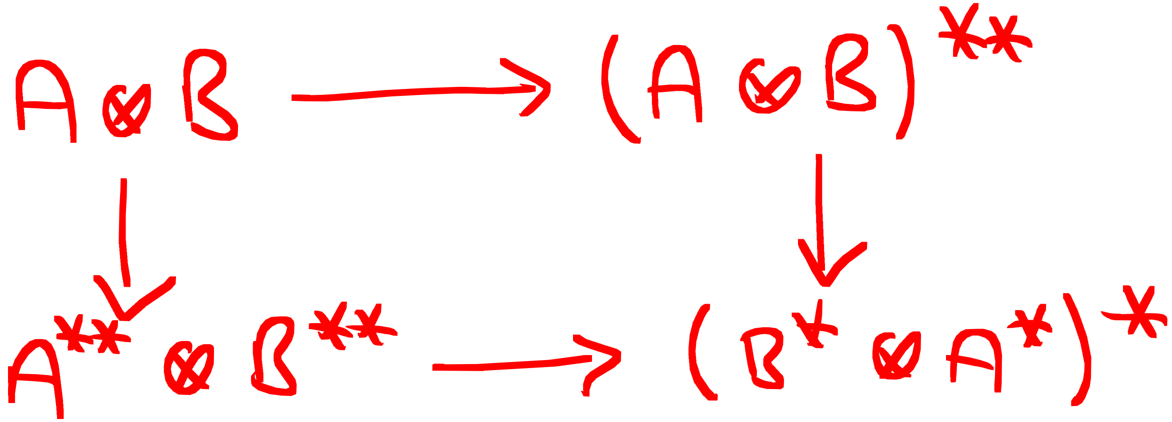
$$\phi^* \otimes \psi^* = (\psi \otimes \phi)^*$$

$$e^* = e$$

Non-strict version

A non-strict monoidal category with duals has natural transformations instead of $=$. These obey 9 axioms.

Example of an axiom:



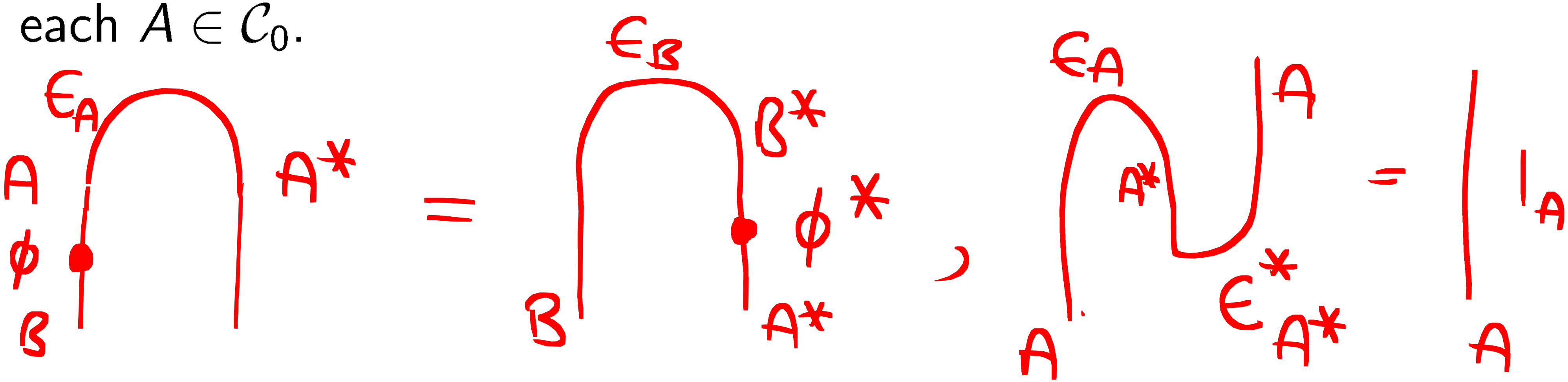
Theorem (Coherence)

All diagrams using the natural transformations commute.

The category is equivalent to a strict one. Canonical isomorphisms.

Pivotal category

A pivotal category also has a morphism $\epsilon_A \colon e \longrightarrow A \otimes A^*$ for each $A \in \mathcal{C}_0$.

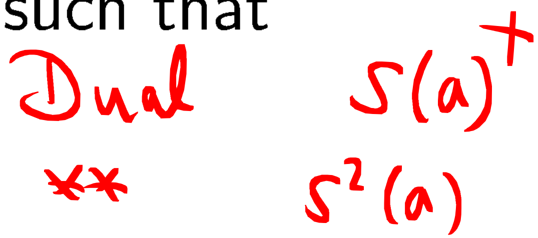


and compatibility with e and $\otimes$.

Example

A Hopf algebra H with a group-like element w such that

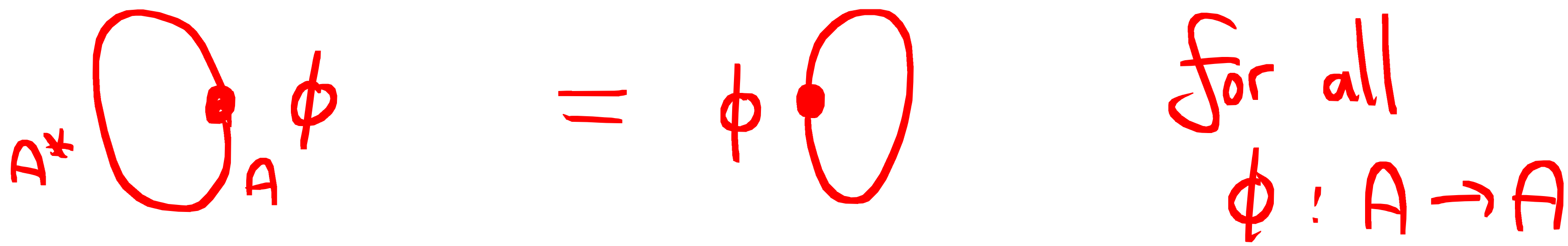
$$s^2(a) = waw^{-1}$$



determines a pivotal category. Define $\epsilon_A = \sum_i \xi_i \otimes \xi^i$, $\epsilon_{A^*} = \sum_i \xi^i \otimes w\xi_i$.

Spherical category

A spherical category is a pivotal category for which



Theorem

Closed diagrams are invariant under diffeomorphisms of S^2.

Examples

- Let $\mathrm{H}=\mathrm{U}_q\mathfrak{g}$ for q a root of unity. Take the category of tilting modules.
- $H = \mathbb{C}[G]$, $w \in Z(G)$, $w^2 = 1$.
- A ribbon category.

Spherical functors

$F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a spherical functor if it commutes with $\otimes$, e, $*$ and ϵ.

e.g

$$F(A \otimes B) = F(A) \otimes F(B)$$

$$F(\epsilon_A) = \epsilon_{F(A)}$$

Examples

- $(H, w) \longrightarrow (K, w')$ induces $\mathrm{Rep}(K) \longrightarrow \mathrm{Rep}(H)$.
- Quotient by equivalence $\alpha \mapsto [\alpha]$

Exercises

1. Let $\mathcal{G}$ be the category with one object and the morphisms forming a group. Show that a functor $\mathcal{G} \to \mathrm{Vect}$, the category of vector spaces, determines a representation of the group. What is the interpretation of a natural transformation between two such functors in terms of group representations?
2. Which of the following objects of a strict spherical category are equal?

$$e^* \otimes X^* \otimes Y \qquad Y^* \otimes e \otimes X \qquad (Y^* \otimes X)^* \qquad (Y^* \otimes X)^*$$

3. Prove that in a spherical category, $\mathrm{Tr}\, f = \mathrm{Tr}\, f^*$ for all morphisms $f \colon X \to X$.

Exercises

4. Explain why the pivotal condition (Lecture 1) holds in a spherical category.
5. Let X be an object of a strict spherical category. If there is an isomorphism $\mu \colon X \to X^*$, what is the coherence condition for μ which guarantees the consistency for identifying $X = X^*$?
6. Show that a spherical functor preserves quantum dimension, i.e., if $F \colon \mathcal{C} \to \mathcal{D}$ is a spherical functor, then for all $X \in \mathcal{C}_0$, $\mathrm{Tr} 1_X = \mathrm{Tr} 1_{F(X)}$.

References for spherical categories

- MacLane: Categories for the Working Mathematician (book)
- JWB and Westbury: Spherical categories
- Andersen and Paradowski: Fusion categories arising from semisimple Lie algebras
- Selinger: A survey of graphical languages for monoidal categories (book)